

Some Remarks on a Distortion Theorem

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Abstract. The object of the present paper is to derive the boundary value of $\left| \frac{z}{f(z)} - 1 \right|$ for the class of starlike functions of order α in the open unit disk.

Let A denote the class of functions of the form

$$(1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in the open unit disk $U = \{z : |z| < 1\}$. Let S denote the class of normalized analytic and univalent functions in U . We denote by $S^*(\alpha)$ and C the subclasses of S of starlike functions of order α and of close-to-convex functions, respectively. In particular for $f(z) \in C$, a function $f(z)$ is said to be close-to-convex if it satisfies

$$\operatorname{Re} f'(z) > 0 \quad (z \in U).$$

In [1], Causey, Krzyz and Merkes obtained the following

Theorem A. If f is in S and $|z| = r < 1$, then

$$(2) \quad \left| \frac{z}{f(z)} - 1 \right| \leq \left\{ A^2(t_0) - 2A(t_0)\cos\psi(t_0) + 1 \right\}^{\frac{1}{2}}$$

where

$$(3) \quad A(t) = A_r(t) = (1-r^2) \left(\frac{1+r}{1-r} \right)^{\cos t}, \quad \psi(t) = \psi_r(t) = \sin t \log \frac{1+r}{1-r}$$

and $t_0 = t_0(r)$ is a suitable zero of the function

$$(4) \quad D_r(t) = \sin(t + \psi_r(t)) - A_r(t) \sin t.$$

For each $r \in (0,1)$ there is a function in S such that the equality holds in (2).

The proof of Theorem A was shown by using Lemma B.

Lemma B[3]. For each $z, |z| = r < 1$, the region $\{ \log \frac{f(z)}{z} : f \in S \}$ is the disk

$$(5) \quad \{ \zeta : |\zeta + \log(1-r^2)| \leq \log \frac{1+r}{1-r} \}.$$

This lemma was discovered by Grunsky in 1932.

For analytic functions $g(z)$ and $h(z)$ in U with $g(0) = h(0)$, $g(z)$ is said to be subordinate to $h(z)$ if there exists an analytic function $\omega(z)$ so that $\omega(0) = 0, |\omega(z)| < 1$ ($z \in U$) and $g(z) = h(\omega(z))$. We denote this subordination by

$$g(z) \prec h(z).$$

Lemma C[4]. Let $f(z) \in A$, and let $g(z) \in A$ be convex in U . If $f(z) \prec g(z)$, then

$$(6) \quad \int_0^z \frac{f(t)}{t} dt \prec \int_0^z \frac{g(t)}{t} dt.$$

We have the following lemma on $S^*(\alpha)$.

Lemma. For each $z, |z|=r<1$, the region $\{\log \frac{f(z)}{z} : f \in S^*(\alpha)\}$ is the disk

$$(7) \quad \left\{ \zeta; \left| \zeta + (1-\alpha) \log(1-r^2) \right| \leq (1-\alpha) \log \frac{1+r}{1-r} \right\}.$$

where $0 \leq \alpha < 1$.

Proof. We define the function $p(z)$ by

$$(8) \quad p(z) = \frac{zf'(z)}{f(z)}.$$

Then $p(z)$ is regular in U with $p(0)=1$ and $\operatorname{Re} p(z) > \alpha$ in U . Hence

$$(9) \quad \frac{zf'(z)}{f(z)} < \frac{1+(1-2\alpha)z}{1-z}.$$

It follows from (9) that

$$\frac{zf'(z)}{f(z)} - 1 < \frac{2(1-\alpha)z}{1-z}.$$

Since $\frac{2(1-\alpha)z}{1-z}$ is convex in U , an application of lemma C gives that

$$(10) \quad \int_0^z \left(\frac{f'(t)}{f(t)} - \frac{1}{t} \right) dt < 2(1-\alpha) \int_0^z \frac{1}{1-t} dt$$

which is equivalent to

$$(11) \quad \log \frac{f(z)}{z} < -2(1-\alpha) \log(1-z).$$

Hence,

$$(12) \quad \frac{1}{1-\alpha} \log \frac{f(z)}{z} < -2 \log(1-z).$$

The region $\{-2\log(1-z); z \in U\}$ is contained in the closed disk with center

$w_0 = -\log(1-r^2)$ and radius $R = \log \frac{1+r}{1-r}$. Therefore by properties for

subordinations, we have

$$(13) \quad \left| \frac{1}{1-\alpha} \log \frac{f(z)}{z} + \log(1-r^2) \right| \leq \log \frac{1+r}{1-r}.$$

It follows from (13) and $0 \leq \alpha < 1$ that

$$\left| \log \frac{f(z)}{z} + (1-\alpha) \log(1-r^2) \right| \leq (1-\alpha) \log \frac{1+r}{1-r}.$$

This completes the proof of the lemma. From this lemma, we can obtain a

result which is similar to theorem A for all $f \in S^*(\alpha)$.

Indeed, the boundary of the range of $\frac{z}{f(z)}$, for $f \in S^*(\alpha)$, $|z|=r$, can be by (7)

$$(14) \quad -\log \frac{f(z)}{z} - (1-\alpha) \log(1-r^2) = e^{it} (1-\alpha) \log \frac{1+r}{1-r}.$$

From (14), it holds that

$$\begin{aligned} \log \frac{z}{f(z)} &= (1-\alpha) \log \frac{1+r}{1-r} (\cos t + i \sin t) + (1-\alpha) \log(1-r^2) \\ &= (1-\alpha) \log(1-r^2) \left(\frac{1+r}{1-r} \right)^{\cos t} + i(1-\alpha) \sin t \log \left(\frac{1+r}{1-r} \right) \\ &= (1-\alpha) \log(1-r^2) \left(\frac{1+r}{1-r} \right)^{\cos t} + (1-\alpha) \log e^{i \sin t \log \frac{1+r}{1-r}} \\ &= (1-\alpha) \left\{ \log(1-r^2) \left(\frac{1+r}{1-r} \right)^{\cos t} + e^{i \sin t \log \frac{1+r}{1-r}} \right\}. \end{aligned}$$

Consequently,

$$(15) \quad \frac{z}{f(z)} = \{(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}\}^{1-\alpha} \cdot e^{i(1-\alpha)\sin t \log \frac{1+r}{1-r}}$$

The boundary of the range of $\frac{z}{f(z)}$ can be parametrized as

$$(16) \quad \frac{z}{f(z)} = A(t, \alpha)(\cos \psi(t, \alpha) + i \sin \psi(t, \alpha)),$$

where

$$(17) \quad A(t, \alpha) = \{(1-r^2)\left(\frac{1+r}{1-r}\right)^{\cos t}\}^{1-\alpha}, \quad \psi(t, \alpha) = (1-\alpha)\sin t \cdot \log \frac{1+r}{1-r}.$$

Furthermore, by a simple computation, we obtain the following

Theorem. If f is in $S^*(\alpha)$ and $|z|=r<1$, then

$$(18) \quad \left| \frac{z}{f(z)} - 1 \right| \leq \{A^2(t, \alpha) - 2A(t, \alpha)\cos \psi(t, \alpha) + 1\}^{\frac{1}{2}},$$

where $A(t, \alpha), \psi(t, \alpha)$ are defined by (17).

Putting $\alpha=0, t=0$ in theorem, we have

Corollary. If f is in S^* and $|z|=r<1$, then

$$(19) \quad \left| \frac{z}{f(z)} - 1 \right| \leq 2r + r^2.$$

Equality holds in (19) if and only if $f(z) = \frac{z}{(1+z)^2}$.

In [2], P. Pawłowski obtained the following theorem.

Theorem D. Let $|z|=r<1$ and $f \in C$. Then

$$(20) \quad \left| \frac{z}{f(z)} - 1 \right| \leq 2r + r^2.$$

As stated above, We have the same result for S^* and C . Consequently, the expression on the left in (20) is sharp.

References

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