

Extension theorems concerned with results by Ponnusamy and Karunakaran

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1 Introduction

Let $\mathcal{A}(n, k)$ be the class of functions $f(z)$ of the form

$$(1.1) \quad f(z) = z^n + \sum_{m=n+k}^{\infty} a_m z^m \quad (n \geq 1, k \geq 1)$$

which are analytic in the open unit disk $\mathbf{U} = \{z \in \mathbb{C} : |z| < 1\}$. For two functions $f(z)$ and $g(z)$ belonging to the class $\mathcal{A}(1, 1)$, Sakaguchi [5] has proved the following result.

Theorem A *Let $f(z) \in \mathcal{A}(1, 1)$ and $g(z) \in \mathcal{A}(1, 1)$ be starlike in $\mathbf{U}$. If $f(z)$ and $g(z)$ satisfy*

$$(1.2) \quad \operatorname{Re} \left(\frac{f'(z)}{g'(z)} \right) > 0 \quad (z \in \mathbf{U}),$$

then

$$(1.3) \quad \operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > 0 \quad (z \in \mathbf{U}).$$

After Theorem A, many mathematicians studying this field have applied this theorem to get some results. In 1989, Ponnusamy and Karunakaran [4] have improved Theorem A as following.

Theorem B *Let α be a complex number with $\operatorname{Re} \alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n, k)$ and $g(z) \in \mathcal{A}(n, j)$ ($j \geq 1$) satisfies*

$$(1.4) \quad \operatorname{Re} \left(\frac{\alpha g(z)}{z g'(z)} \right) > \delta \quad (z \in \mathbf{U})$$

2000 Mathematics Subject Classification : Primary 30C45.

Keywords and Phrases : Analytic, starlike, Jack's lemma.

with $0 \leq \delta < \frac{\operatorname{Re} \alpha}{n}$. If $f(z)$ and $g(z)$ satisfy

$$(1.5) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} \right\} > \beta \quad (z \in \mathbb{U}),$$

then

$$(1.6) \quad \operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > \frac{2\beta + \delta k}{2 + \delta k} \quad (z \in \mathbb{U}).$$

It is the purpose of the present paper is to discuss Theorem B applying the lemma due to Fukui and Sakaguchi [1]. To discuss our problems, we need the following lemmas.

Lemma 1 Let $w(z) = \sum_{n=k}^{\infty} a_n z^n$ ($a_k \neq 0, k \geq 1$) be analytic in $\mathbb{U}$. If the maximum value of $|w(z)|$ on the circle $|z| = r < 1$ is attained at $z = z_0$, then we have

$$(1.7) \quad \frac{z_0 w'(z_0)}{w(z_0)} = \ell \geq k,$$

which shows that $\frac{z_0 w'(z_0)}{w(z_0)}$ is a positive real number.

The proof of Lemma 1 can be found in [1] and we see that Lemma 1 is a generalization of Jack's lemma given by Jack [2]. Applying Lemma 1, we derive

Lemma 2 Let $p(z) = 1 + \sum_{n=k}^{\infty} c_n z^n$ ($c_k \neq 0, k \geq 1$) be analytic in $\mathbb{U}$ with $p(z) \neq 0$ ($z \in \mathbb{U}$). If there exists a point $z_0 \in \mathbb{U}$ such that

$$\operatorname{Re} p(z) > 0 \quad (|z| < |z_0|)$$

and

$$\operatorname{Re} p(z_0) = 0,$$

then we have

$$(1.8) \quad -z_0 p'(z_0) \geq \frac{\ell}{2} (1 + |p(z_0)|^2)$$

and so

$$(1.9) \quad \frac{z_0 p'(z_0)}{p(z_0)} = i\ell,$$

where

$$(1.10) \quad k \leq \frac{k}{2} \left(a + \frac{1}{a} \right) \leq \ell \quad \left(\arg p(z_0) = \frac{\pi}{2} \right)$$

and

$$(1.11) \quad -k \geq -\frac{k}{2} \left(a + \frac{1}{a} \right) \geq \ell \quad \left(\arg p(z_0) = -\frac{\pi}{2} \right)$$

with $p(z_0) = \pm ia$ ($a > 0$).

Proof. Let us consider

$$(1.12) \quad \phi(z) = \frac{1 - p(z)}{1 + p(z)} = \frac{c_k}{2} z^k + \dots$$

for $p(z)$. Then, it follows that $\phi(0) = \phi'(0) = \dots = \phi^{(k-1)}(0) = 0$, $|\phi(z)| < 1$ ($|z| < |z_0|$) and $|\phi(z_0)| = 1$. Therefore, applying Lemma 1, we have that

$$(1.13) \quad \frac{z_0 \phi'(z_0)}{\phi(z_0)} = \frac{-2z_0 p'(z_0)}{1 - (p(z_0))^2} = \frac{-2z_0 p'(z_0)}{1 + |p(z_0)|^2} = \ell \geq k.$$

This implies that $z_0 p'(z_0)$ is a negative real number and

$$(1.14) \quad -z_0 p'(z_0) \geq \frac{k}{2} (1 + |p(z_0)|^2).$$

Let us use the same method by Nunokawa [3]. If $\arg p(z_0) = \frac{\pi}{2}$, then we write $p(z_0) = ia$ ($a > 0$). This gives us that

$$\operatorname{Im} \left(\frac{z_0 p'(z_0)}{p(z_0)} \right) = \operatorname{Im} \left(-\frac{iz_0 p'(z_0)}{a} \right) \geq \frac{k}{2} \left(a + \frac{1}{a} \right).$$

If $\arg p(z_0) = -\frac{\pi}{2}$, then we write $p(z_0) = -ia$ ($a > 0$). Thus we have that

$$\operatorname{Im} \left(\frac{z_0 p'(z_0)}{p(z_0)} \right) = \operatorname{Im} \left(\frac{iz_0 p'(z_0)}{a} \right) \leq -\frac{k}{2} \left(a + \frac{1}{a} \right).$$

This completes the proof of Lemma 2. □

2 Main results

With the help of Lemma 2, we derive the following theorem.

Theorem 1 *Let α be a complex number with $\operatorname{Re} \alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n, k)$ and $g(z) \in \mathcal{A}(n, j)$ ($j \geq 1$) satisfies*

$$(2.1) \quad \operatorname{Re} \left(\frac{\alpha g(z)}{z g'(z)} \right) > \delta \quad (z \in \mathbf{U})$$

with $0 \leq \delta < \frac{\operatorname{Re} \alpha}{n}$. If $f(z)$ and $g(z)$ satisfy

$$(2.2) \quad \operatorname{Re} \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} \right\} + \frac{\delta k}{2(1 - \beta_1)} \left| \frac{f(z)}{g(z)} - \beta_1 \right|^2 > \beta \quad (z \in \mathbf{U})$$

then

$$(2.3) \quad \operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > \beta_1 \quad (z \in \mathbf{U}),$$

where $\beta_1 = \frac{2\beta + \delta k}{2 + \delta k}$.

Proof. Defining the function $p(z)$ by

$$(2.4) \quad p(z) = \frac{\frac{f(z)}{g(z)} - \beta_1}{1 - \beta_1},$$

we see that $p(0) = 1$ and

$$(2.5) \quad \begin{aligned} (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} - \beta \\ = (\beta_1 - \beta) + (1 - \beta_1) \left(p(z) + \frac{\alpha g(z)}{z g'(z)} z p'(z) \right) \\ > -\frac{\delta k}{2(1 - \beta_1)} \left| \frac{f(z)}{g(z)} - \beta_1 \right|^2 \end{aligned}$$

for all $z \in \mathbb{U}$. Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg p(z_0)| < \frac{\pi}{2} \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2}.$$

Then, by means of Lemma 2, we have that

$$(2.6) \quad -z_0 p'(z_0) \geq \frac{k}{2} (1 + |p(z_0)|^2).$$

It follows from the above that

$$\begin{aligned} & \operatorname{Re} \left\{ (1 - \alpha) \frac{f(z_0)}{g(z_0)} + \alpha \frac{f'(z_0)}{g'(z_0)} - \beta \right\} \\ &= (\beta_1 - \beta) + (1 - \beta_1) \operatorname{Re} \left\{ p(z_0) + \frac{\alpha g(z_0)}{z_0 g'(z_0)} z_0 p'(z_0) \right\} \\ &= (\beta_1 - \beta) - (1 - \beta_1) \operatorname{Re} \left\{ \frac{\alpha g(z_0)}{z_0 g'(z_0)} (-z_0 p'(z_0)) \right\} \\ &\leq (\beta_1 - \beta) - (1 - \beta_1) \frac{\delta k}{2} (1 + |p(z_0)|^2) \\ &= -\frac{\delta k}{2(1 - \beta_1)} \left| \frac{f(z_0)}{g(z_0)} - \beta_1 \right|^2 \end{aligned}$$

which contradicts (2.5). This completes the proof of the theorem. $\square$

Remark 1 If $f(z)$ and $g(z)$ satisfy $f(z_0) = \beta_1 g(z_0)$ in Theorem 1, then Theorem 1 becomes Theorem B given by Ponnusamy and Karunakaran [4]. We also have

Theorem 2 Let α be a complex number with $\operatorname{Re} \alpha > 0$ and $\beta < 1$. Further, let $f(z) \in \mathcal{A}(n, k)$ and $g(z) \in \mathcal{A}(n, j)$ ($j \geq 1$) satisfies the condition (2.1) with $0 \leq \delta < \frac{\operatorname{Re} \alpha}{n}$. If $f(z)$ and $g(z)$ satisfy

$$(2.7) \quad \left| \arg \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2} + \tan^{-1} \left(\frac{\delta k |p(z)|}{2 \left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right)$$

for $|z| = r < 1$, then

$$(2.8) \quad \left| \arg \left(\frac{f(z)}{g(z)} - \beta_1 \right) \right| < \frac{\pi}{2} \quad (z \in \mathbf{U})$$

or

$$(2.9) \quad \operatorname{Re} \left(\frac{f(z)}{g(z)} \right) > \beta_1 \quad (z \in \mathbf{U}),$$

where $\beta_1 = \frac{2\beta + \delta k}{2 + \delta k}$ and

$$p(z) = \frac{\frac{f(z)}{g(z)} - \beta_1}{1 - \beta_1}.$$

Proof. Note that the function $p(z)$ is analytic in $\mathbf{U}$ and $p(0) = 1$. It follows that

$$\begin{aligned} \left| \arg \left\{ (1 - \alpha) \frac{f(z)}{g(z)} + \alpha \frac{f'(z)}{g'(z)} - \beta \right\} \right| &= \left| \arg \left\{ (\beta_1 - \beta) + (1 - \beta_1) \left(p(z) + \frac{\alpha g(z)}{z g'(z)} z p'(z) \right) \right\} \right| \\ &< \frac{\pi}{2} + \tan^{-1} \left(\frac{\delta k |p(z)|}{2 \left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right) \end{aligned}$$

for $|z| = r < 1$. If there exists a point $z_0 \in \mathbf{U}$ such that

$$|\arg p(z_0)| < \frac{\pi}{2} \quad (|z| < |z_0|)$$

and

$$|\arg p(z_0)| = \frac{\pi}{2},$$

then, by Lemma 2, we have that

$$\frac{z_0 p'(z_0)}{p(z_0)} = i\ell,$$

where

$$\frac{k}{2} \left(a + \frac{1}{a} \right) \leq \ell \quad \left(\arg p(z_0) = \frac{\pi}{2} \right)$$

and

$$-\frac{k}{2} \left(a + \frac{1}{a} \right) \geq \ell \quad \left(\arg p(z_0) = -\frac{\pi}{2} \right)$$

with $p(z_0) = \pm ia$ ($a > 0$). If $\arg p(z_0) = \frac{\pi}{2}$, then it follows that

$$\begin{aligned} & \arg \left\{ (1 - \alpha) \frac{f(z_0)}{g(z_0)} + \alpha \frac{f'(z_0)}{g'(z_0)} - \beta \right\} \\ &= \arg p(z_0) \left(\frac{\beta_1 - \beta}{p(z_0)} + (1 - \beta_1) \left(1 + \frac{\alpha g(z_0)}{z_0 g'(z_0)} \frac{z_0 p'(z_0)}{p(z_0)} \right) \right) \\ &= \frac{\pi}{2} + \arg \left\{ - \left(\frac{\beta_1 - \beta}{a} \right) i + (1 - \beta_1) \left(1 + i\ell \frac{\alpha g(z_0)}{z_0 g'(z_0)} \right) \right\} \\ &= \frac{\pi}{2} + \arg I(z_0), \end{aligned}$$

where

$$(2.9) \quad I(z_0) = - \left(\frac{\beta_1 - \beta}{a} \right) i + (1 - \beta_1) \left(1 + i\ell \frac{\alpha g(z_0)}{z_0 g'(z_0)} \right).$$

Note that

$$\begin{aligned} (2.10) \quad \operatorname{Im} I(z_0) &= \frac{\beta - \beta_1}{a} + (1 - \beta_1) \ell \operatorname{Re} \frac{\alpha g(z_0)}{z_0 g'(z_0)} \\ &> (1 - \beta_1) \delta \ell + \frac{\beta - \beta_1}{a} \\ &\geq \frac{\delta k}{2} (1 - \beta_1) \left(a + \frac{1}{a} \right) + \frac{\beta - \beta_1}{a} \\ &= \frac{\delta k}{2} (1 - \beta_1) a > 0 \end{aligned}$$

and

$$(2.11) \quad \operatorname{Re} I(z_0) = (1 - \beta_1) \left(1 - \ell \operatorname{Im} \left(\frac{\alpha g(z_0)}{z_0 g'(z_0)} \right) \right) \leq (1 - \beta_1) \left(1 + \ell \left| \operatorname{Im} \left(\frac{\alpha g(z_0)}{z_0 g'(z_0)} \right) \right| \right).$$

Letting

$$(2.12) \quad q(z) = \frac{\alpha g(z)}{zg'(z)} + 1 - \frac{\alpha}{n},$$

we know that $q(z)$ is analytic in U with $q(0) = 1$. This gives us that

$$(2.13) \quad |\operatorname{Im} q(z)| = \left| \operatorname{Im} \left(\frac{\alpha g(z)}{zg'(z)} + 1 - \frac{\alpha}{n} \right) \right| \leq \frac{2r}{1-r^2}$$

for $|z| = r < 1$. Thus we have that

$$(2.14) \quad \left| \operatorname{Im} \left(\frac{\alpha g(z_0)}{z_0 g'(z_0)} \right) \right| \leq \frac{2r}{1-r^2} + \frac{|\operatorname{Im} \alpha|}{n} \quad (|z| = r < 1).$$

Using (2.11) and (2.14), we obtain that

$$\arg I(z_0) = \operatorname{Tan}^{-1} \left(\frac{\operatorname{Im} I(z_0)}{\operatorname{Re} I(z_0)} \right) \geq \operatorname{Tan}^{-1} \left(\frac{\delta ka}{2 \left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right),$$

which contradicts our condition (2.7).

If $\arg p(z_0) = -\frac{\pi}{2}$, using the same way, we also have that

$$\arg \left\{ (1-\alpha) \frac{f(z_0)}{g(z_0)} + \alpha \frac{f'(z_0)}{g'(z_0)} - \beta \right\} \leq - \left\{ \frac{\pi}{2} + \operatorname{Tan}^{-1} \left(\frac{\delta ka}{2 \left(\frac{2r}{1-r^2} + \frac{|\operatorname{Im} \alpha|}{n} + 1 \right)} \right) \right\},$$

which contradicts (2.7). □

Remark 2 If $f(z)$ satisfies the conditions in Theorem B, then $f(z)$ satisfies the conditions of Theorem 2. In this case, we see that Theorem 2 becomes Theorem B.

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