

Solutions to The Nonhomogeneous Associated Laguerre's Equation by Means of N-Fractional Calculus Operator

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Abstract

In this article, solutions to the nonhomogeneous associated Laguerre's equations

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \quad (z \neq 0)$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0, \varphi_0 = \varphi = \varphi(z), f = f(z))$$

are discussed by means of N-fractional calculus operator (NFCO- Method).

By our method, some particular solutions to the above equations are given as below for example, in fractional differintegrated forms.

Group I.

$$(i) \quad \varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \varphi_{[1](\alpha, \beta)}^* \quad (\text{denote})$$

and

$$(ii) \quad \varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \varphi_{[2](\alpha, \beta)}^*$$

where

$$X_{[1]} = (f_\beta z^{\alpha+\beta} e^{-z})_{-1}, \quad Y_{[1]} = e^z z^{-(\alpha+\beta+1)}.$$

Group II.

$$(i) \quad \varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi_{[3](\alpha, \beta)}^*$$

and

$$(ii) \quad \varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi_{[4](\alpha, \beta)}^*$$

where

$$X_{[2]} = ((f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1}, \quad Y_{[2]} = e^{-z} z^\beta.$$

§0. Introduction (Definition of Fractional Calculus) and

§1. Preliminary

are omitted, then refer to the previous paper [35] .

§ 2. Solutions to The Nonhomogeneous Associated Laguerre's Equation by NFCO-Method

Theorem 1. Let $\varphi \in F$ and $f \in F$, then nonhomogeneous associated Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = f \quad (z \neq 0) \quad (1)$$

$$(\varphi_\nu = d^\nu \varphi / dz^\nu \text{ for } \nu > 0, \varphi_0 = \varphi = \varphi(z), f = f(z))$$

has particular solutions of the forms (fractional differintegrated form);

Group I.

$$(i) \quad \varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \varphi_{[1](\alpha, \beta)}^* \quad (\text{denote}) \quad (2)$$

$$(ii) \quad \varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \varphi_{[2](\alpha, \beta)}^* \quad (3)$$

where

$$X_{[1]} = (f_\beta z^{\alpha+\beta} e^{-z})_{-1}, \quad Y_{[1]} = e^z z^{-(\alpha+\beta+1)} \quad (4)$$

Group II.

$$(i) \quad \varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi_{[3](\alpha, \beta)}^* \quad (5)$$

$$(ii) \quad \varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi_{[4](\alpha, \beta)}^* \quad (6)$$

where

$$X_{[2]} = ((f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1}, \quad Y_{[2]} = e^{-z} z^\beta \quad (7)$$

Group III.

$$(i) \quad \varphi = z^{-\alpha} (X_{[3]} \cdot Y_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi_{[5](\alpha, \beta)}^* \quad (8)$$

$$(ii) \quad \varphi = z^{-\alpha} (Y_{[3]} \cdot X_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi_{[6](\alpha, \beta)}^* \quad (9)$$

where

$$X_{[3]} = ((f z^\alpha)_{\alpha+\beta} z^\beta e^{-z})_{-1}, \quad Y_{[3]} = e^z z^{-(\beta+1)} \quad (10)$$

Group IV.

$$(i) \quad \varphi = z^{-\alpha} e^z (X_{[4]} \cdot Y_{[4]})_\beta \equiv \varphi_{[7](\alpha, \beta)}^* \quad (11)$$

$$(ii) \quad \varphi = z^{-\alpha} e^z (Y_{[4]} \cdot X_{[4]})_\beta \equiv \varphi_{[8](\alpha, \beta)}^* \quad (12)$$

where

$$X_{[4]} = ((f z^\alpha e^{-z})_{-(1+\beta)} e^z z^{-(1+\alpha+\beta)})_{-1}, \quad Y_{[4]} = e^{-z} z^{\alpha+\beta} \quad (13)$$

Proof of Group I.

Operate N-fractional calculus (NFC) operator N^ν to the both sides of equation (1), we have then

$$(\varphi_2 \cdot z)_\nu + (\varphi_1 \cdot (-z + \alpha + 1))_\nu + (\varphi \cdot \beta)_\nu = f_\nu, \quad (f \neq 0). \quad (14)$$

Now we have

$$(\varphi_2 \cdot z)_\nu = \sum_{k=0}^1 \frac{\Gamma(\nu+1)}{k! \Gamma(\nu+1-k)} (\varphi_2)_{\nu-k}(z)_k \quad (15)$$

$$= \varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot \nu, \quad (16)$$

$$(\varphi_1 \cdot (-z + \alpha + 1))_\nu = \varphi_{1+\nu} \cdot (-z + \alpha + 1) - \varphi_\nu \cdot \nu \quad (17)$$

and

$$(\varphi \cdot \beta)_\nu = \varphi_\nu \cdot \beta, \quad (18)$$

respectively, by Lemmas (i) and (i v).

Therefore, we have

$$\varphi_{2+\nu} \cdot z + \varphi_{1+\nu} \cdot (-z + \alpha + 1 + \nu) + \varphi_\nu \cdot (\beta - \nu) = f_\nu \quad (19)$$

from (14), applying (16), (17) and (18).

Choosing ν such that

$$\nu = \beta \quad (20)$$

we obtain

$$\varphi_{2+\beta} \cdot z + \varphi_{1+\beta} \cdot (-z + \alpha + \beta + 1) = f_\beta. \quad (21)$$

Set

$$\varphi_{1+\beta} = \phi = \phi(z) \quad (\varphi = \phi_{-(1+\beta)}), \quad (22)$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\alpha + \beta + 1}{z} - 1 \right) = f_\beta \cdot z^{-1} \quad (23)$$

from (21). A particular solution to this linear first order equation is given by

$$\phi = X_{[1]} Y_{[1]}, \quad (24)$$

Where $X_{[1]}$ and $Y_{[1]}$ are the ones shown by (4), respectively.

Therefore, we obtain

$$\varphi = (X_{[1]} \cdot Y_{[1]})_{-(1+\beta)} \equiv \varphi_{[1](\alpha, \beta)}^* \quad (2)$$

from (24) and (22).

Inversely (24) satisfies equation (23). then (2) satisfies equation (1).

Next, changing the order

$X_{[1]}$ and $Y_{[1]}$ in parenthesis ()_{-(1+β)}

we obtain other solution $\varphi_{[2](\alpha, \beta)}^*$ which is different from (2) for $-(1 + \beta) \notin \mathbb{Z}_0^+$, that is,

$$\varphi = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \equiv \varphi_{[2](\alpha, \beta)}^* . \quad (3)$$

(Refer to Theorem D in the previous paper [35].)

Proof of Group II.

Set

$$\varphi = e^{\gamma z} \psi \quad (\psi = \psi(z)) , \quad (25)$$

we have then

$$\psi_2 \cdot z + \psi_1 \cdot \{z(2\gamma - 1) + \alpha + 1\} + \psi \cdot \{z\gamma(\gamma - 1) + \gamma(\alpha + 1) + \beta\} = f e^{-\gamma z} \quad (26)$$

from (1).

Here we choose γ such that

$$\gamma(\gamma - 1) = 0 ,$$

that is,

$$\gamma = 0, 1 . \quad (27)$$

When $\gamma = 0$, (26) is reduced to (1), therefore, we have the same solutions as Group I.

When $\gamma = 1$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{z + \alpha + 1\} + \psi \cdot (\alpha + \beta + 1) = f e^{-z} \quad (28)$$

from (26)

Operate N^ν to the both sides of equation (28), we have then

$$(\psi_2 \cdot z)_\nu + (\psi_1 \cdot (z + \alpha + 1))_\nu + (\psi \cdot (\alpha + \beta + 1))_\nu = (f e^{-z})_\nu . \quad (29)$$

Hence, using Lemma (i v), we obtain

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (z + \alpha + 1 + \nu) + \psi_\nu \cdot (\nu + \alpha + \beta + 1) = (f e^{-z})_\nu . \quad (30)$$

Choosing ν such that

$$\nu = -(\alpha + \beta + 1) \quad (31)$$

we obtain

$$\psi_{1-(\alpha+\beta)} \cdot z + \psi_{-(\alpha+\beta)} \cdot (z - \beta) = (f e^{-z})_{-(\alpha+\beta+1)} . \quad (32)$$

from (30).

Set

$$\psi_{-(\alpha+\beta)} = \phi = \phi(z) \quad (\psi = \phi_{\alpha+\beta}) , \quad (33)$$

we have then

$$\phi_1 + \phi \cdot \left(1 - \frac{\beta}{z}\right) = (f e^{-z})_{-(\alpha+\beta+1)} \cdot z^{-1} \quad (34)$$

from (32). A particular solution to this equation is given by

$$\phi = X_{[2]} Y_{[2]} . \quad (35)$$

Hence we obtain

$$\psi = (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \quad (36)$$

from (35) and (33).

Therefore, we obtain

$$\varphi = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \equiv \varphi_{[3](\alpha,\beta)}^* \quad (5)$$

from (25) and (36), having $\gamma = 1$.

Inversely , (35) satisfies (34), then (36) satisfies equation (28).

Hence (5) satisfies equation (1).

Next, changing the order

$X_{[2]}$ and $Y_{[2]}$ in parenthesis () _{$\alpha+\beta$} in (5)

we obtain other solution

$$\varphi = e^z (Y_{[2]} \cdot X_{[2]})_{\alpha+\beta} \equiv \varphi_{[4](\alpha,\beta)}^* , \quad (6)$$

which is different from (5) for $(\alpha + \beta) \notin Z_0^+$,

Proof of Group III.

Set

$$\varphi = z^\lambda \psi \quad (\psi = \psi(z)) , \quad (37)$$

we have then

Hence we obtain

$$\begin{aligned} \psi_2 \cdot z^{\lambda+1} + \psi_1 \cdot \{-z^{\lambda+1} + z^\lambda(2\lambda + \alpha + 1)\} \\ + \psi \cdot \{z^\lambda(\beta - \lambda) + z^{\lambda-1}\lambda(\lambda + \alpha)\} = f \end{aligned} \quad (38)$$

from (1).

Here we choose λ such that

$$\lambda(\lambda + \alpha) = 0 ,$$

that is,

$$\lambda = 0, -\alpha. \quad (39)$$

When $\lambda = 0$, (38) is reduced to (1), therefore, we have the same solutions as Group I.

When $\lambda = -\alpha$ we have

$$\psi_2 \cdot z + \psi_1 \cdot \{-z + 1 - \alpha\} + \psi \cdot (\alpha + \beta) = f z^\alpha \quad (40)$$

from (38)

Operate N^ν to the both sides of equation (40), we have then

$$\psi_{2+\nu} \cdot z + \psi_{1+\nu} \cdot (-z + 1 - \alpha + \nu) + \psi_\nu \cdot (\alpha + \beta - \nu) = (f z^\alpha)_\nu. \quad (41)$$

Choosing ν such that

$$\nu = \alpha + \beta \quad (42)$$

we obtain

$$\psi_{2+\alpha+\beta} \cdot z + \psi_{1+\alpha+\beta} \cdot (-z + \beta + 1) = (f z^\alpha)_{\alpha+\beta}. \quad (43)$$

from (41), applying (42).

Set

$$\psi_{1+\alpha+\beta} = \phi = \phi(z) \quad (\psi = \phi_{-(1+\alpha+\beta)}), \quad (44)$$

we have then

$$\phi_1 + \phi \cdot \left(\frac{\beta + 1}{z} - 1 \right) = (f z^\alpha)_{\alpha+\beta} z^{-1} \quad (45)$$

from (43).

A particular solution to this equation is given by

$$\phi = X_{[3]} Y_{[3]}. \quad (46)$$

Where $X_{[3]}$ and $Y_{[3]}$ are the ones given by (10).

Hence we obtain

$$\psi = (X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \quad (47)$$

from (44) and (46).

Therefore, we obtain

$$\varphi = z^{-\alpha} (X_{[3]} Y_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi_{[5]}^*(\alpha, \beta) \quad (8)$$

from (37) and (47), having $\lambda = -\alpha$.

Inversely , (46) satisfies (equation (45), then (47) satisfies equation (43).
Therefore, (8) satisfies equation (1)

Next, changing the order

$X_{[3]}$ and $Y_{[3]}$ in parenthesis ()_{-(1+α+β)} in (8)

we obtain other solution

$$\varphi = z^{-\alpha} (Y_{[3]} X_{[3]})_{-(1+\alpha+\beta)} \equiv \varphi_{[6]}^*(\alpha, \beta) \quad (9)$$

which is different from (8) for $-(1 + \alpha + \beta) \notin Z_0^+$,

Proof of Group IV.

First set

$$\varphi = z^\lambda \psi \quad (\psi = \psi(z)) , \quad (37)$$

and substitute (37) into equation (1), we have then (38).

We have then (40) from (38), having

$$\lambda = -\alpha .$$

Next set

$$\psi = e^{\delta z} \phi \quad (\phi = \phi(z)) , \quad (48)$$

We have then

$$\begin{aligned} \phi_2 \cdot z + \phi_1 \cdot \{z(2\delta - 1) + 1 - \alpha\} \\ + \phi \cdot \{z(\delta^2 - \delta) + \delta(1 - \alpha) + \alpha + \beta\} = f z^\alpha e^{-\delta z} \end{aligned} \quad (49)$$

from (40), applying (48).

Choose δ such that

$$\delta^2 - \delta = 0 ,$$

that is,

$$\delta = 0, 1 . \quad (50)$$

When $\delta = 0$, we obtain (40) from (49). Then we have the same solutions as Group III .

When $\delta = 1$ we have

$$\phi_2 \cdot z + \phi_1 \cdot (z + 1 - \alpha) + \phi \cdot (1 + \beta) = f z^\alpha e^{-z} \quad (51)$$

from (49).

Operate N^ν to the both sides of equation (51), we have then

$$\phi_{2+\nu} \cdot z + \phi_{1+\nu} \cdot (z + 1 - \alpha + \nu) + \phi_\nu \cdot (\nu + 1 + \beta) = (f z^\alpha e^{-z})_\nu . \quad (52)$$

Choose ν such that

$$\nu = -(1 + \beta) \quad (53)$$

we obtain

$$\phi_{1-\beta} \cdot z + \phi_{-\beta} \cdot (z - \alpha - \beta) = (f z^\alpha e^{-z})_{-(1+\beta)} \quad (54)$$

from (52).

Therefore, setting

$$\phi_{-\beta} = u = u(z) \quad (\phi = u_\beta), \quad (55)$$

we have

$$u_1 + u \cdot \left(1 - \frac{\alpha + \beta}{z} \right) = (f z^\alpha e^{-z})_{-(1+\beta)} z^{-1} \quad (56)$$

from (54). A particular solution to this equation is given by

$$u = X_{[4]} Y_{[4]}, \quad (57)$$

where $X_{[4]}$ and $Y_{[4]}$ are the ones shown by (13).

Hence we obtain

$$\phi = (X_{[4]} \cdot Y_{[4]})_\beta \quad (58)$$

from (55) and (57).

Therefore, we have

$$\psi = e^z (X_{[4]} \cdot Y_{[4]})_\beta \quad (59)$$

from (58) and (48), having $\delta = 1$.

We have then

$$\varphi = z^{-\alpha} e^z (X_{[4]} \cdot Y_{[4]})_\beta \equiv \varphi_{[7](\alpha, \beta)}^* \quad (11)$$

from (59) and (37), having $\lambda = -\alpha$.

Inversely, the function shown by (57) satisfies equation (56), then (58) satisfies equation (54), and hence (59) satisfies (40).

Therefore, the function given by (11) satisfies equation (1), by (37) where $\lambda = -\alpha$.

Next, changing the order

$X_{[4]}$ and $Y_{[4]}$ in parenthesis () _{β} in (11)

we obtain other solution

$$\varphi = z^{-\alpha} e^z (Y_{[4]} X_{[4]})_\beta \equiv \varphi_{[8](\alpha, \beta)}^* \quad (12)$$

which is different from (11) for $\beta \notin \mathbb{Z}_0^+$,

§3. Some Illustrative Example

(I) Let

$$f(z) = e^z$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = e^z \quad (z \neq 0) \quad (1)$$

from §2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi_{[1](\alpha, \beta)}^* = (Y_{[1]} \cdot X_{[1]})_{-(1+\beta)} \quad (2)$$

$$= \left(((e^z)_\beta z^{\alpha+\beta} e^{-z})_{-1} (e^z z^{-(\alpha+\beta+1)}) \right)_{-(1+\beta)} \quad (3)$$

$$= \frac{1}{\alpha + \beta + 1} e^z, \quad (4)$$

since

$$(e^z)_\beta = e^z \quad (5)$$

and

$$(z^{\alpha+\beta})_{-1} = \frac{1}{\alpha + \beta + 1} z^{\alpha+\beta+1}. \quad (6)$$

Indeed we have

$$\varphi_1 = \frac{1}{\alpha + \beta + 1} e^z \quad \text{and} \quad \varphi_2 = \frac{1}{\alpha + \beta + 1} e^z \quad (7)$$

from (4). thence applying (4) and (7) we obtain

$$\text{LHS of (1)} = \frac{1}{\alpha + \beta + 1} e^z (z - z + \alpha + 1 + \beta) = e^z. \quad (8)$$

(II) Let

$$\alpha = 0, \quad \beta = -1 \quad \text{and} \quad f(z) = e^z$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + 1) - \varphi = e^z \quad (z \neq 0) \quad (9)$$

from § 2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi_{[5](0, -1)}^* = (X_{[3]} \cdot Y_{[3]})_{-(1+\beta)} = (f_{-1} z^{-1} e^{-z})_{-1} e^z \quad (10)$$

$$= e^z \log z . \quad (11)$$

Hence we obtain

$$\varphi_1 = e^z \log z + e^z \frac{1}{z} \quad (12)$$

and

$$\varphi_2 = e^z \log z + 2e^z \frac{1}{z} - e^z \frac{1}{z^2} . \quad (13)$$

from (11), respectively.

Therefore, we have

$$\begin{aligned} \text{LHS of (9)} &= ze^z \log z + 2e^z - e^z \frac{1}{z} \\ &\quad - ze^z \log z - e^z + e^z \log z + e^z \frac{1}{z} - e^z \log z \\ &= e^z \end{aligned} \quad (14)$$

applying (11). (12) and (13).

(III) Let

$$f(z) = z^{-\alpha} e^z$$

we have then the nonhomogeneous Laguerre's equation

$$\varphi_2 \cdot z + \varphi_1 \cdot (-z + \alpha + 1) + \varphi \cdot \beta = z^{-\alpha} e^z \quad (z \neq 0) \quad (15)$$

from §2. (1)

A particular solution to this equation is given by

$$\varphi = \varphi_{[3](\alpha, \beta)}^* = e^z (X_{[2]} \cdot Y_{[2]})_{\alpha+\beta} \quad (16)$$

$$= e^z \left(((f e^{-z})_{-(\alpha+\beta+1)} e^z z^{-(1+\beta)})_{-1} e^{-z} z^\beta \right)_{\alpha+\beta} \quad (17)$$

$$= \frac{1}{\beta+1} e^z z^{-\alpha} \quad (18)$$

since we have

$$(f e^{-z})_{-(\alpha+\beta+1)} = (z^{-\alpha})_{-(\alpha+\beta+1)} \quad (19)$$

$$= e^{i\pi(\alpha+\beta+1)} \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} z^{\beta+1} \quad \left(\left| \frac{\Gamma(-\beta-1)}{\Gamma(\alpha)} \right| < \infty \right) \quad (20)$$

and

$$(z^\beta)_{\alpha+\beta} = e^{-i\pi(\alpha+\beta)} \frac{\Gamma(\alpha)}{\Gamma(-\beta)} z^{-\alpha} \quad \left(\left| \frac{\Gamma(\alpha)}{\Gamma(-\beta)} \right| < \infty \right). \quad (21)$$

by Lemma (i), respectively.

Indeed we have

$$\varphi_1 = \frac{1}{\beta+1} e^z (z^{-\alpha} - \alpha z^{-\alpha-1}) \quad (22)$$

and

$$\varphi_2 = \frac{1}{\beta+1} e^z [z^{-\alpha} - 2\alpha z^{-\alpha-1} + \alpha(\alpha+1)z^{-\alpha-2}] \quad (23)$$

from (18), respectively.

Therefore, we obtain

$$\begin{aligned} \text{LHS of (15)} &= \frac{1}{\beta+1} e^z [z^{1-\alpha} - 2\alpha z^{-\alpha} + \alpha(\alpha+1)z^{-\alpha-1} \\ &\quad - z^{1-\alpha} + \alpha z^{-\alpha} + \alpha z^{-\alpha} - \alpha^2 z^{-\alpha-1} + z^{-\alpha} - \alpha z^{-\alpha-1} + \beta z^{-\alpha}] \end{aligned} \quad (24)$$

$$= \frac{1}{\beta+1} e^z [z^{-\alpha} + \beta z^{-\alpha}] \quad (25)$$

$$= e^z z^{-\alpha}, \quad (26)$$

applying (18). (22) and (23).

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