

Topics on Kac-Moody Lie algebras

Yōhei Tanaka (Nagoya Univ.)

In this note we introduce Kac-Moody Lie algebras, their representation theory, and relation with theory of differential equations.

1. Kac-Moody Lie algebras ([6], [14], [15])

Let $C=(c_{ij})$ be a $n \times n$ -matrix satisfying following conditions;

$$c_{ij} \in \mathbb{Z}, \quad c_{ii}=2, \quad c_{ij} \leq 0 \quad (i \neq j), \quad c_{ij}=0 \iff c_{ji}=0,$$

and define Lie algebra $\mathcal{L}(C)$ over $\mathbb{C}$ with generators $\{h_i, e_i, f_i \mid i=1, \dots, n\}$ and relations;

$$\begin{cases} [h_i, h_j]=0, & [h_i, e_j]=c_{ij}e_j, & [h_i, f_j]=-c_{ij}f_j, & [e_i, f_j]=\delta_{ij}h_i, \\ (\text{ad } e_i)^{-c_{ij}+1}e_j=0, & (\text{ad } f_i)^{-c_{ij}+1}f_j=0. \end{cases}$$

$\mathcal{L}=\mathcal{L}(C)$ is called Kac-Moody Lie algebra.

$\mathcal{L}$ has $\Gamma=\mathbb{Z}^n$ grading with $\deg h_i=(0, \dots, 0)$, $\deg e_i=(0, \dots, 1, \dots, 0)=a_i$, and $\deg f_i=-a_i$.

Put $\mathcal{L}_a=\{x \in \mathcal{L} \mid \deg x=a, (a \in \Gamma)\}$, $m_a=\dim \mathcal{L}_a$, $\Delta=\{a \in \Gamma \setminus \{0\} \mid m_a \neq 0\}$.

From defining relations $\mathcal{L}$ has vector space decomposition

$$\mathcal{L} = \mathcal{L}_0 \oplus \mathcal{L}_+ \oplus \mathcal{L}_-, \quad \text{where } \mathcal{L}_0 = \sum_i \mathbb{C} h_i, \quad \mathcal{L}_+ = \langle e_1, \dots, e_n \rangle, \quad \mathcal{L}_- = \langle f_1, \dots, f_n \rangle.$$

So Δ is disjoint union of $\Delta_+ = \Gamma_+ \cap \Delta$ and $\Delta_- = (-\Gamma_+) \cap \Delta$

$$(\Gamma_+ = \mathbb{Z}_+ a_1 + \dots + \mathbb{Z}_+ a_n)$$

For each i , define $s_i \in \text{GL}(\Gamma)$ by

$$s_i(a_j) = a_j - c_{ij}a_i$$

and $W = \langle s_i \mid i=1, \dots, n \rangle$.

2

If C is decomposable, i.e. has a permutation matrix P such that

$$PCP^{-1} = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix}, \text{ then } \mathcal{L}(C) \cong \mathcal{L}(C_1) + \mathcal{L}(C_2), \quad W(C) \cong W(C_1) \times W(C_2).$$

So we assume C is indecomposable.

A finite-dimensional complex simple Lie algebra is isomorphic to a $\mathcal{L}(C)$ whose $W(C)$ is finite group. Equivalent condition on C to finiteness of $W(C)$ is that there exist positive numbers $d_1, \dots, d_n$ such that $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} C$ is a positive definite symmetric matrix.

If we replace "positive definite" by "positive semi-definite" in this condition, we get a new class of infinite-dimensional Lie algebras. They are called Euclidean (or affine) Lie algebras. We deal with Euclidean case hereafter.

2. Character formula and denominator formula

Put $V = \mathbb{C} \otimes_{\mathbb{Z}} \Gamma$, and for $\lambda \in \mathfrak{p}^*$ let $c_\lambda: W \rightarrow V$ be the 1-cocycle, i.e. satisfies

$$c_\lambda(w_1 w_2) = w_1 c_\lambda(w_2) + c_\lambda(w_1), \text{ such that } c_\lambda(s_i) = \lambda(h_i) a_i \quad (i=1, \dots, n).$$

If $\lambda(h_i) \in \mathbb{Z}$ for all i , λ is said integral, and moreover if $\lambda(h_i) \geq 0$, λ is said dominant integral. It is easy to see

$$\begin{cases} \lambda : \text{integral} & \Rightarrow c_\lambda(W) \subset \Gamma \\ \lambda : \text{dominant integral} & \Rightarrow c_\lambda(W) \subset \Gamma_+ \end{cases}$$

To each dominant integral, there exists $\mathcal{L}$ -module V^λ , unique up to isomorphism, with the following property;

$$0 \neq v \in V \text{ s.t. } \begin{cases} e_i v = 0, & h_i v = \lambda(h_i) v, & f_i^{\lambda(h_i)} v = 0 \quad (i=1, \dots, n) \\ v = \sum \mathbb{C} f_{i_1} f_{i_2} \dots f_{i_k} v \end{cases}$$

V^λ is called standard module with highest weight λ .

For $a \in \Gamma_+$, let $v_a^\lambda = \sum \mathbb{C} f_{i_1} \dots f_{i_k} v$, and put

$$\text{ch } V^\lambda = \sum_{a \in \Gamma_+} \dim V_a^\lambda e^{(-a)} \quad (a_{i_1} + \dots + a_{i_k} = a) \quad (\text{ch } V^\lambda \text{ is in } \mathbb{Z}[[e^{(-a_1)}, \dots, e^{(-a_n)}]]).$$

Let $\rho \in \mathfrak{f}^*$ be defined by $\rho(h_i) = 1$ for all i . We have

character formula $([5], [8], [10])$

$$\text{ch } V^\lambda = \frac{\sum_{w \in W} \det(w) e^{-c_{\lambda+\rho}(w)}}{\sum_{w \in W} \det(w) e^{-c_\rho(w)}}$$

and

denominator formula

$$\sum_{w \in W} \det(w) e^{-c(w)} = \prod_{a \in \Delta_+} (1 - e^{-a})^{m_a}$$

If we make specialization $e(-a_i) \mapsto q^{m_i}$ ($m_1, \dots, m_n \geq 0$) to both sides of denominator (or character) formula, we have various combinatorial identities.

([7], [8], [12], [16])

3. K-dV equation and construction of representation

Let $u(x), V_{n1}(x)$ ($n=1, 2, 3, \dots, 1 \leq n \leq 2n-2$) be functions in infinite many variables $x = (x_1, x_3, x_5, \dots)$. From the compatibility conditions of linear partial differential equations

$$\left. \begin{aligned} \left(\frac{\partial}{\partial x_1}\right)^2 u(x) \psi(x) &= \lambda \psi(x) \\ \frac{\partial \psi}{\partial x_{2n-1}} &= \left(\frac{\partial}{\partial x_1}\right)^{2n-1} u + V_{n1} \left(\frac{\partial}{\partial x_1}\right)^{2n-3} u + \dots + V_{n, 2n-2} \psi \end{aligned} \right\} \quad (n=1, 2, \dots)$$

we obtain non-linear differential equations on u , and V_{n1} are expressed by differential polynomials of u . These differential equations are called K-dV equations. A function $\tau(x)$ is called τ -function, if $u = \left(\frac{\partial}{\partial x_1}\right)^2 \log \tau$ is a solution of K-dV equations. The space of infinitesimal transformations of τ -functions forms a Lie algebra $\mathcal{L}$ of linear transformations on $C[x_1, x_3, \dots]$. Date, Jimbo, Kashiwara and Miwa find ([I]) that $\mathcal{L} \cong \mathcal{L}\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ and $\mathcal{L}$ -module $C[x_1, x_3, \dots]$ coincides with the standard module constructed by Lepowsky and Wilson. ([II]). Kac, Kazhdan, Lepowsky and Wilson construct standard modules for other Euclidean Lie algebras in analogous way to L-W, and D-J-K-M show

(I) About classification and realizations of Euclidean Lie algebras, c.f. [6], [15].

(2) Let $\mathcal{R} \subset \mathcal{L}$ be the maximum homogenous ideal with $\mathcal{R} \cap \mathcal{f} = 0$. Strictly speaking, $\mathcal{G}(C) = \mathcal{L}(C)/\mathcal{R}$ is Kac-Moody Lie algebra. But finite or Euclidean case $\mathcal{R} = 0$, and we conjecture that $\mathcal{R} = 0$ for any case.

(3) Character formula and denominator formula are analogy of finite case, and proved for $\mathcal{G}(C)$ with symmetrizable C , i.e. there exist positive numbers $d_1, \dots, d_n$ such that $\begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_n \end{pmatrix} C$ is symmetric matrix.

(4) Original K-dV equation is obtained by to put $x_5, x_7, \dots$ constant.

(5) Frenkel and Kac ([4]) construct standard modules in different way from K-K-L-W. It will be interesting to describe isomorphism explicitly between K-K-L-W's module and F-K's.

[I] E.Date,M.Kashiwara and T.Miwa,Transformation groups for soliton equations
II, RIMS preprint 357.

[2] E.Date,M.Jimbo,M.Kashiwara and T.Miwa, " IV,RIMS preprint 359 .

[3] " " " ", RIMS preprint 362.

[4] I.B.Frenkel and V.G.Kac, Basic representations of affine Lie algebras and dual resonance models, Inv.Math., 62(1980), 23-66.

[5] H.Garland and J.Lepowsky, Lie algebra homology and the Macdonald-Kac formulas, Inv.Math., 34 (1976), 37-76.

[6] V.G.Kac, Simple irreducible graded Lie algebras of finite growth, Math.USSR Izvestija, 2(1968), I27I-I3II.

[7] V.G.Kac, Infinite-dimensional Lie algebras and Dedekind's η -function,

Func. Anal. and its App., 8(1974), 68-70.

- [8] V.G.Kac, Infinite-dimensional algebras, Dedekind's η -function, classical Mobius function and the very strange formula, Adv. in Math., 30(1978), 85-136.
- [9] V.G.Kac, D.A.Kazhdan, J.Lepowsky and R.L.Wilson, Realization of the basic representations of the Euclidean Lie algebra, Adv. in Math., to appear.
- [10] J.Lepowsky, Lectures on Kac-Moody Lie algebras, Université Paris VI, 1978.
- [11] J.Lepowsky and R.L.Wilson, Construction of the affine Lie algebra $A_I^{(1)}$, Comm. Math. Physics, 62(1978), 43-53.
- [12] J.Lepowsky and R.L.Wilson, A Lie theoretic interpretation and proof of Rogers-Ramanujan identities, preprint.
- [13] I.G.Macdonald, Affine root systems and Dedekind's η -function, Inv. Math., 15(1972), 91-143.
- [14] R.V.Moody, A new class of Lie algebra, J. of Alg., 10(1968), 211-230.
- [15] " , Euclidean Lie algebra, Canad. J. Math., 21(1969), 1432-1454
- [16] " , Macdonald identities and Euclidean Lie algebras, Proc. A.M.S. 48(1975), 43-52.