

Farey series and the Riemann hypothesis

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The aim of this paper is to consider the equivalent conditions to the Riemann hypothesis in terms of Farey series.

Farey series F_x of order x is the sequence of all irreducible fractions in $(0, 1]$ with denominator not bigger than x , arranged in increasing order of magnitude;

$$F_x = F_{[x]} = \left\{ \rho_\nu = \frac{b_\nu}{c_\nu} \mid (b_\nu, c_\nu) = 1, 0 < b_\nu \leq c_\nu \leq x \right\}$$

and the cardinality of F_x is the summatory function of Euler's function

$$\#F_x = \Phi(x) = \sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x) \quad (\text{Mertens}).$$

$$\varphi(n) = \sum_{\substack{m \leq n \\ (m, n) = 1}} 1: \text{Euler's function.}$$

This asymptotic formula is due to Mertens.

For example, from F_2 we form F_3 :

$$F_2 = \left\{ \frac{1}{2}, \frac{1}{1} \right\} \rightarrow F_3 = \left\{ \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1} \right\},$$

and so on.

And the Riemann hypothesis RH states that the Riemann zeta function does not vanish for real part σ of s bigger than $\frac{1}{2}$. It is well known that the RH is equivalent to each of the following asymptotic formulas, forms of the prime number theorem:

$$\begin{aligned} \text{RH} &\iff \zeta(s) \neq 0 \text{ for } \sigma := \Re s > \frac{1}{2} \\ &\iff M(x) := \sum_{n \leq x} \mu(n) = O\left(x^{\frac{1}{2}+\epsilon}\right) \\ &\quad \mu(n): \text{Möbius' function} \\ &\iff \psi(x) := \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log p \\ &\quad = x + O\left(x^{\frac{1}{2}+\epsilon}\right), \\ &\quad \Lambda(n): \text{von Mangoldt's function} \\ &\quad \psi(x): \text{Chebyshev's function} \end{aligned}$$

Here the weak Riemann hypothesis $\text{RH}(\alpha)$ states that $\zeta(s)$ does not vanish for $\sigma > \alpha$:

$$\text{RH}(\alpha) \iff \zeta(s) \neq 0 \text{ for } \sigma > \alpha.$$

In this paper I'd like to state main results of Part V [9] and Part VI [6] of this series of papers. In Part V we aim at the implications of the $\text{RH}(\alpha)$ on the estimates of error terms associated to Farey series, with occasional acquisition of equivalent conditions.

Principle.

Suppose f has a bounded derivative and consider the error term $E_f(x)$ defined by

$$E_f(x) := \sum_{\nu=1}^{\Phi(x)} f(\rho_\nu) - \Phi(x) \int_0^1 f(u) du.$$

Suppose the RH implies the estimate:

$$\text{RH} \implies E_f(x) = O\left(x^{\frac{1}{2}+\epsilon}\right),$$

and that the Mellin transform $F(s)$ defined by

$$F(s) = s\zeta(s) \int_1^\infty E_f(x) x^{-s-1} dx \quad \text{for } \sigma > 1$$

satisfies following conditions:

- (i) $F(s)$ is regular for $\sigma > \frac{1}{2}$, $s \neq 1$,
- (ii) $F(s) \neq 0$ for $\frac{1}{2} < \sigma < 1$.

Then

$$\text{RH} \iff E_f(x) = O\left(x^{\frac{1}{2}+\epsilon}\right).$$

We note that if we define the arithmetic function $a(n)$ by

$$(1) \quad a(n) = \sum_{k=1}^n f\left(\frac{k}{n}\right) - n \int_0^1 f(u) du,$$

then $E_f(x)$ can be written as

$$E_f(x) = \sum_{n \leq x} (\mu * a)(n) = \sum_{n \leq x} M\left(\frac{x}{n}\right) a(n),$$

where $*$ denotes the Dirichlet convolution, and $F(s)$ becomes the generating function of $a(n)$:

$$(2) \quad F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}.$$

Theorem 1 (Part V, Theorem 2). Suppose f is an integrable function on $(0, 1)$. Let $a(n)$ be the arithmetic function defined by (1), let $F(s)$ be the generating function (2) of $a(n)$, and we suppose that $F(s)$ and $a(n)$ satisfy following conditions (i)–(v):

- (i) $F(s)$ is absolutely convergent for $\sigma > \sigma_a$ with $\sigma_a \leq 1$,
- (ii) $F(s)$ is continued to an analytic function with finitely many singularities in the half-plane $\sigma > \alpha$,
- (iii) $F(s) \ll |t|^{\kappa+\varepsilon}$ for some $\kappa \geq 0$ and every $\varepsilon > 0$ uniformly in the region $\alpha < \sigma \leq 1$, $|t| \geq t_0 > 0$,
- (iv) $(\mu * a)(n) \ll n^{\beta+\varepsilon}$ for some β , $0 \leq \beta (\leq \sigma_a)$,
- (v) there exists a non-negative number θ satisfying

$$\sum_{n=1}^{\infty} \frac{|(\mu * a)(n)|}{n^{\sigma}} \ll (\sigma - 1)^{-\theta} \quad \text{as } \sigma \rightarrow 1.$$

Then, on the $RH(\alpha)$, we have the (asymptotic) formula:

$$\sum_{n \leq x} (\mu * a)(n) = \frac{1}{2\pi i} \int_C \frac{F(s)}{s\zeta(s)} x^s ds + O(x^{\omega+\varepsilon}),$$

where

$$\omega = \min_{0 \leq \xi \leq 1} \{ \max\{ \beta + 1 - \xi, 1 + (\kappa - 1)\xi, \alpha + \kappa\xi \} \},$$

and the contour C encircles all singularities of $F(s)/\zeta(s)$ in the strip $\alpha < \sigma < 1$.

In particular, in the special cases of $\kappa = 0$ and $\beta = 0$ we have $\omega = \max\{\alpha, \beta\}$, and $\omega = \alpha + \kappa(1 - \alpha)$, respectively.

Corollary 1 (Codecà-Perelli [2], Theorem 1). (i) Let $f(u)$ be absolutely continuous and let $f' \in L^p[0, 1]$ for some $p \in (1, 2]$. Then, on the $RH(\eta)$, we have

$$E_f(x) = O\left(x^{\max\{\eta, \frac{1}{p}\} + \varepsilon}\right).$$

(This covers the Main result of Codecà-Perelli, Theorem 1.)

(ii) Moreover, if $F(s)$ satisfies the conditions (i) – (iii) of Theorem 1 and $0 \leq \kappa \leq \frac{2}{p} - 1$. Then, on the $RH(\alpha)$,

$$E_f(x) = O\left(x^{\alpha + \kappa(1 - \alpha) + \varepsilon}\right).$$

Corollary 2. For any rational number $\frac{r}{q} \in (0, 1)$ other than $\frac{1}{2}$, the RH implies

$$E\left(\frac{r}{q}; x\right) := \sum_{\rho_\nu \leq \frac{r}{q}} 1 - \frac{r}{q} \Phi(x) = O\left(x^{\frac{1}{2} + \frac{35}{432} + \varepsilon}\right),$$

by the result of Kolesnik.

Moreover, if we assume the GRH (on some Dirichlet L -functions mod q) and the RH , we have the estimate:

$$E\left(\frac{r}{q}; x\right) = O\left(x^{\frac{1}{2} + \varepsilon}\right). \quad (\text{Codecà [1]})$$

We can not only cover the strong result of Codecà-Perelli [2] (Corollary 1), some results of Codecà [1] and their developments as above, but also we can widen the width of validity of the parameter by $\frac{1}{2}$ of some theorems proved earlier.

In particular, on the GRH, the RH is equivalent to each of estimates

$$E\left(\frac{1}{3}; x\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right)$$

and

$$E\left(\frac{1}{4}; x\right) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Theorem 2 (Part VI). *If $f(u)$ is the gap-Fourier series;*

$$f(u) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^{mk_1}(2n+1)^{k_2}} \cos 2\pi 2^{ml_1}(2n+1)^{l_2} u,$$

with $k_1, k_2 \in \mathbb{C}$, $l_1, l_2 \in \mathbb{N}$, $2\Re k_1 \geq l_1 + 1$ and $2\Re k_2 \geq l_2 + 2$, then we have the equivalence:

$$RH \iff E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

Corollary 3. $k_1 = l_1 = l_2 = 1$, $k_2 = 2 \implies$
 $f(u)$ is Takagi's function, and

$$F(s) = \frac{3}{2} \frac{1 - 2^{-s-1}}{1 - 2^{-s}} \zeta(s) \zeta(s+1) \neq 0 \text{ for } \sigma > \frac{1}{2}.$$

Hence

$$RH \iff E_f(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

($k_1 = k_2 = l_1 = l_2 = 2 \implies f$: Riemann's function)

Recall that if $E_f(x) = (M * a)(x)$ with suitable $a(n)$, then

$$F(s) = s\zeta(s) \int_1^{\infty} E_f(x) x^{-s-1} dx = \sum_{n=1}^{\infty} \frac{a(n)}{n^s},$$

and vice versa.

Hence, when $f(u)$ has a Fourier expansion

$$f(u) = \sum_{n=1}^{\infty} c(n) \cos 2\pi nu$$

satisfying the condition;

$$\sum_{n=1}^{\infty} |c(n)| d(n) < \infty,$$

then, with $a(n) = n \sum_{m=1}^{\infty} c(mn)$, we have the Ramanujan expansion:

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} = \sum_{n=1}^{\infty} c(n) \sigma_{1-s}(n), \quad \sigma > 1$$

($E_f = M * a$ also holds).

Conversely, if $F(s)$ is the generating Dirichlet series;

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \quad \sigma \geq 1,$$

the Fourier coefficient of the corresponding f is given by

$$c(n) = \frac{1}{n} \sum_{k=1}^{\infty} \frac{\mu(k)}{k} a(kn).$$

This is a Hecke-like correspondence.

$f(\tau) = \sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$	$\longleftrightarrow$	$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$
dynamical system		generating Dirichlet series
generating Fourier series		$\approx$ zeta-function with
		Euler product if a is multiplicative

Example.

$f(\tau)$ $\sum_{n=1}^{\infty} c(n) e^{2\pi i n \tau}$	$\longleftrightarrow$	$F(s)$ $\sum_{n=1}^{\infty} c(n) \sigma_{1-s}(n)$ $= \sum_{n=1}^{\infty} \frac{a(n)}{n^s}$	$\longleftrightarrow$	$E_f(x)$ $(M * a)(x)$
$\cos 2\pi\tau$ $\log 2 \sin \pi x $ $B_{2n}(\tau) \quad (n \in \mathbb{N})$		1 $\zeta'(s)$ $\zeta(s + 2n - 1)$		$M(x)$ $\psi(x)$ $\sum_{m \leq x} M\left(\frac{x}{m}\right) \frac{1}{m^{2n-1}}$

Here $B_k(x)$ is the k -th Bernoulli polynomial.

Theorem 3 (Part VI). (i) Let $f_{k,l}(u)$ be a gap Fourier series

$$f_{k,l}(u) := \sum_{n=1}^{\infty} \frac{1}{n^k} \cos 2\pi n^l u \quad \text{for } \Re k > 1, l \in \mathbb{N}.$$

Then $F_{k,l}$ can be decomposed with $G_{k,l}$ having an Euler product as follows:

$$F_{k,l}(s) = \zeta(k) \zeta(ls + k - l) G_{k,l}(s),$$

$$G_{k,l}(s) = \prod_p \left(1 + p^{-k} \sum_{n=1}^{l-1} p^{n(1-s)} \right).$$

(ii) If $2\Re k \geq l + 2$, we have

$$RH \iff E_{f_{k,l}}(x) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

(iii) For $k = 2$, $l = 3$

$$RH \iff E_{f_{2,3}}(x) = \frac{\zeta(2)G_{2,3}\left(\frac{2}{3}\right)}{2\zeta\left(\frac{2}{3}\right)}x^{\frac{2}{3}} + O\left(x^{\frac{1}{2}+\varepsilon}\right),$$

$$G_{2,3}\left(\frac{2}{3}\right) = \prod_p \left(1 + p^{-\frac{4}{3}} + p^{-\frac{5}{3}}\right).$$

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