

NON EXISTENCE OF TWISTED WAVE EQUATIONS

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ABSTRACT. Let X be a complex manifold, V an involutive submanifold of its cotangent bundle, and Σ a bicharacteristic leaf of V . A ring of twisted differential operators $\mathcal{A}$ on X has a characteristic class in $H^1(X; d\mathcal{O}_X^\times)$. To such a class we associate a class in $H^2(\Sigma; \mathbb{C}_\Sigma^\times)$ whose vanishing is necessary for the existence of an $\mathcal{A}$ -module globally simple along V . As an application, we show that there are no generalized massless field equations with non trivial twist on Grassmann manifolds.

1. STATEMENT OF THE PROBLEM

The grassmannian $\mathbb{G}$ of 2-dimensional planes in a 4-dimensional complex vector space $\mathbb{T}$ is a homogeneous space G/H , where $G = SL(4; \mathbb{C})$ and H is the stabilizer of a point. Let $M = (\mathbb{R}^4, \|\cdot\|)$ be the Minkowski space, where $\|(x_0, x_1, x_2, x_3)\| = x_0^2 - x_1^2 - x_2^2 - x_3^2$. According to Penrose, consider the embedding

$$\iota: M \rightarrow \mathbb{G}$$

$$(x_0, x_1, x_2, x_3) \mapsto \langle (x_0 + x_3, x_1 + ix_2, 1, 0), (x_1 - ix_2, x_0 - x_3, 0, 1) \rangle_{\mathbb{C}}$$

where $\langle v, w \rangle_{\mathbb{C}}$ denotes the vector space spanned by v and w . The closure of $\iota(M)$ is an orbit of the totally real form $SU(2, 2) \subset SL(4; \mathbb{C})$, whose action corresponds to that of the Poincaré group on M . Thus $\mathbb{G}$ is a conformal compactification of the complexified Minkowski space. In particular, the usual wave equation, as well as the other massless field equations, extend as differential operators acting between homogeneous bundles on $\mathbb{G}$. Let us denote by $\mathcal{M}_{(m)}$ the quasi-equivariant $\mathcal{D}_{\mathbb{G}}$ -module corresponding to the massless field equation of helicity $m \in \mathbb{Z}/2$. As we will recall, the modules $\mathcal{M}_{(m)}$ are “globally simple” along the characteristic variety V of the wave equation.

Denote by $\mathfrak{g} = \mathfrak{sl}(4; \mathbb{C})$ the Lie algebra of $G = SL(4; \mathbb{C})$. By the Beilinson-Bernstein-Kashiwara correspondence, quasi-equivariant $\mathcal{D}_{\mathbb{G}}$ -module are associated with $(\mathfrak{g}, H)$ -modules with trivial twist. In order to deal with arbitrary twists, one has to consider quasi-equivariant modules over rings of twisted differential operators on $\mathbb{G}$.

It thus arises a natural question: given a ring $\mathcal{A}$ of twisted differential operators on $\mathbb{G}$, are there $\mathcal{A}$ -modules globally simple along V ?

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2. TWISTED SHEAVES AND DIFFERENTIAL OPERATORS

Let us briefly review the notions of twisted sheaves and twisted differential operators from [7, 1] (see also [2] for an exposition).

Let X be a complex manifold, $\mathbb{C}_X$ the constant sheaf with stalk $\mathbb{C}$ on X , $\mathcal{O}_X$ the structure sheaf on X , and $\mathcal{D}_X$ the ring of finite order differential operators on X .

- A ring of twisted differential operators (a t.d.o. ring for short) is an $\mathcal{O}_X$ -ring locally isomorphic to the ring $\mathcal{D}_X$. They are classified by $H^1(X; d\mathcal{O}_X)$, up to isomorphisms.

A basic example of t.d.o. ring is the ring

$$\mathcal{D}_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{L}^{-1}$$

of differential operators acting on a line bundle $\mathcal{L}$. For $\lambda \in \mathbb{C}$ one can also consider the t.d.o ring $\mathcal{D}_{\mathcal{L}^\lambda}$ defined as follows. If s is a nowhere vanishing local section of $\mathcal{L}$, sections of $\mathcal{D}_{\mathcal{L}^\lambda}$ are written as $s^\lambda \otimes P \otimes s^{-\lambda}$, for $P \in \mathcal{D}_X$. If t is another nowhere vanishing local section of $\mathcal{L}$, then $s^\lambda \otimes P \otimes s^{-\lambda} = t^\lambda \otimes Q \otimes t^{-\lambda}$ in $\mathcal{D}_{\mathcal{L}^\lambda}$ if and only if $Q = (s/t)^\lambda \cdot P \cdot (s/t)^{-\lambda}$ in $\mathcal{D}_X$. This is independent from the choice of a branch for the ramified function $(s/t)^\lambda$. It is also possible to give a meaning to $\mathcal{L}^\lambda$ as a twisted sheaf, as follows.

Denote by $\text{Mod}(\mathbb{C}_X)$ the category of sheaves of $\mathbb{C}$ -vector spaces on X , and by $\mathfrak{Mod}(\mathbb{C}_X)$ the corresponding $\mathbb{C}$ -stack, $U \mapsto \text{Mod}(\mathbb{C}_U)$.

- A stack of twisted sheaves is a $\mathbb{C}$ -stack $\mathfrak{S}$ on X locally $\mathbb{C}$ -equivalent to the stack of sheaves $\mathfrak{Mod}(\mathbb{C}_X)$. They are classified by $H^2(X; \mathbb{C}_X^\times)$, up to $\mathbb{C}$ -equivalences. Twisted sheaves are objects of $\mathfrak{S}(X)$.

For an open covering $X = \bigcup_i U_i$, let $c_{ijk} \in \mathbb{C}_X^\times(U_{ijk})$ be a Čech cocycle for the characteristic class of $\mathfrak{S}$ in $H^2(X; \mathbb{C}_X^\times)$. Twisted sheaves in $\mathfrak{S}(X)$ are described by a family of sheaves F_i on U_i , and a family of isomorphisms $\theta_{ij}: F_j|_{U_{ij}} \rightarrow F_i|_{U_{ij}}$, satisfying $\theta_{ij} \circ \theta_{jk} = c_{ijk} \theta_{ik}$ on U_{ijk} .

For $\mathcal{B}$ a sheaf of $\mathbb{C}$ -algebras, let $\text{Mod}(\mathcal{B}; \mathfrak{S})$ be the category of $\mathcal{B}$ -modules in $\mathfrak{S}$.

- Twisted line bundles are object of $\text{Mod}(\mathcal{O}_X; \mathfrak{S})$ locally isomorphic to $\mathcal{O}_X$.

The twisted sheaf $\mathcal{L}^\lambda$ is an example of a twisted line bundle. Its twist is described as follows. Let s_i be non vanishing sections of $\mathcal{L}$ on U_i . Then $\mathcal{L}^\lambda$ belongs to a stack of twisted sheaves whose cocycle c_{ijk} describes the difference of determinations between the ramified functions $(s_i/s_j)^\lambda (s_j/s_k)^\lambda$ and $(s_i/s_k)^\lambda$.

To any t.d.o. ring $\mathcal{A}$ one associates a stack of twisted sheaves $\mathfrak{S}_{\mathcal{A}}$ and a twisted line bundle $\mathcal{O}_{\mathcal{A}} \in \text{Mod}(\mathcal{O}_X; \mathfrak{S}_{\mathcal{A}})$, such that

$$\mathcal{A} \simeq \mathcal{O}_{\mathcal{A}} \otimes_{\mathcal{O}} \mathcal{D}_X \otimes_{\mathcal{O}} \mathcal{O}_{\mathcal{A}}^{-1}.$$

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The Riemann-Hilbert correspondence associates flat connections of rank 1 in $\text{Mod}(\mathcal{A})$ with local systems of rank 1 in $\mathfrak{S}_{\mathcal{A}}(X)$, by $\mathcal{M} \mapsto \text{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{O}_{\mathcal{A}})$. Recalling that $\mathfrak{S}_{\mathcal{A}}$ is globally $\mathbb{C}$ -equivalent to the stack of sheaves $\mathfrak{Mod}(\mathbb{C}_X)$ if and only if there is a local systems of rank 1 in $\mathfrak{S}_{\mathcal{A}}(X)$, it follows

Proposition 2.1. *Let $\mathcal{A}$ be a t.d.o. ring on X , and $\mathcal{M}$ a flat connection of rank 1 in $\text{Mod}(\mathcal{A})$. Then $\mathfrak{S}_{\mathcal{A}}$ is globally $\mathbb{C}$ -equivalent to $\mathfrak{Mod}(\mathbb{C}_X)$.*

We will make use of the exact sequence

$$(2.1) \quad H^1(X; \mathcal{O}_X^\times) \xrightarrow{\gamma} H^1(X; d\mathcal{O}_X) \xrightarrow{\delta} H^2(X; \mathbb{C}_X^\times),$$

induced by the short exact sequence

$$1 \rightarrow \mathbb{C}_X^\times \rightarrow \mathcal{O}_X^\times \xrightarrow{d \log} d\mathcal{O}_X \rightarrow 0.$$

If $\mathcal{L}$ is a line bundle, and $\mathcal{A}$ a t.d.o. ring, one has $\gamma([\mathcal{L}]) = [\mathcal{D}_{\mathcal{L}}]$, $\delta([\mathcal{A}]) = [\mathfrak{S}_{\mathcal{A}}]^{-1}$.

3. SYSTEMS WITH SIMPLE CHARACTERISTICS

Let us now recall some definitions and results on microdifferential operators, due to [11, 10]. See also [6, 8] for an exposition.

Let X be a complex manifold, and $\pi: T^*X \rightarrow X$ its cotangent bundle. Denote by $\mathcal{E}_X$ the ring of microdifferential operators on T^*X , and by $\mathbb{F}_m\mathcal{E}_X$ its subsheaf of microdifferential operators of order at most m .

Let (x) be a system of local coordinates in X , and denote by $(x; \xi)$ the associated system of symplectic coordinates in T^*X . With this choice of coordinates, a microdifferential operator $P \in \mathbb{F}_m\mathcal{E}$ is described by its total symbol $\{p_k(x; \xi)\}_{k \leq m}$, where $p_k \in \mathcal{O}_{T^*X}(k)$ is a function homogeneous of degree k . The principal symbol of order m , independent from the choice of coordinates, is given by

$$\begin{aligned} \sigma_m: \mathbb{F}_m\mathcal{E} &\rightarrow \mathcal{O}_{T^*X}(m) \\ P &\mapsto p_m. \end{aligned}$$

Denoting by $a: T^*X \rightarrow T^*X$ the antipodal map, the formal adjoint of P is the operator $P^* \in a^{-1}\mathbb{F}_m\mathcal{E}$ whose principal symbol $\{p_k^*(x; \xi)\}_{k \leq m}$ is given by

$$p_k^*(x; \xi) = \sum_{k=l-|\alpha|} \frac{(-1)^{|\alpha|}}{\alpha!} (\partial_\xi^\alpha \partial_x^\alpha p_l)(x; -\xi).$$

For $P \in \mathbb{F}_m\mathcal{E}$, one has $P - (-1)^m P^* \in \mathbb{F}_{m-1}\mathcal{E}$. The subprincipal symbol of P in the coordinate system (x) is defined by

$$\begin{aligned} \sigma'_{m-1}(P) &= \frac{1}{2} \sigma_{m-1}(P - (-1)^m P^*) \\ &= p_{m-1} - \frac{1}{2} \sum_i \partial_{x_i} \partial_{\xi_i} p_m. \end{aligned}$$

Identifying X with the zero-section of T^*X , we set $\dot{T}^*X = T^*X \setminus X$. In this paper, by submanifold of T^*X we mean a smooth locally closed submanifold. A submanifold V of T^*X is conic if it is locally $\mathbb{C}^\times$ -invariant. The canonical 1-form induces a homogeneous symplectic structure on T^*X . A submanifold V of T^*X is involutive if for any pair f, g , of holomorphic functions vanishing on V , the Poisson bracket $\{f, g\}$ vanishes on V .

Let V be a conic involutive submanifold of $\dot{T}^*X$. The ring $\mathcal{E}_V$ on V is the subring of $\mathcal{E}_X$ generated by

$$\mathcal{J}_V = \{P \in F_1\mathcal{E}_X : \sigma_1(P)|_V = 0\}.$$

The V -filtration is defined by $F_k^V\mathcal{E}_X = F_k\mathcal{E}_X \cdot \mathcal{E}_V = \mathcal{E}_V \cdot F_k\mathcal{E}_X$.

If $\mathcal{M}$ is an $\mathcal{E}_X$ -module and $\mathcal{M}_0$ is an $F_0\mathcal{E}_X$ -submodule, we set

$$\mathcal{M}_k = (F_k\mathcal{E}_X)\mathcal{M}_0.$$

Definition 3.1. One says that a coherent $\mathcal{E}_X$ -module $\mathcal{M}$ is globally simple along V if it admits a coherent sub- $F_0\mathcal{E}_X$ -module $\mathcal{M}_0$ such that $\mathcal{M} = \mathcal{E}_X\mathcal{M}_0$, $\mathcal{E}_V\mathcal{M}_0 \subset \mathcal{M}_0$, and $\mathcal{M}_0/\mathcal{M}_{-1}$ is a locally isomorphic to $\mathcal{O}_V(0)$. Such an $\mathcal{M}_0$ is called a V -lattice.

To $P \in \mathcal{J}_V$ one associates the operator $\mathcal{L}_V(P) \in F_1\mathcal{D}_{\Omega_{V/X}^{1/2}}$ defined as follows. For $\omega \in \Omega_V$ one sets

$$\mathcal{L}_V(P)(\omega^{1/2}/\sqrt{dx}) = \left(L_{H_{\sigma_1(P)}}(\omega^{1/2}) + \sigma'_0(P) \cdot \omega^{1/2} \right) / \sqrt{dx},$$

where dx is the volume form associated with a chosen local coordinate system. The Hamiltonian vector field $H_{\sigma_1(P)}$ is tangent to V since $\sigma_1(P)|_V = 0$. Recall that for $v \in \Theta_V$, one sets $L_v(\omega^{1/2}) = \frac{1}{2} \frac{L_v(\omega)}{\omega} \omega^{1/2}$. The operator $\mathcal{L}_V(\cdot)$ does not depend on the choice of coordinates, and extends as a ring morphism

$$(3.1) \quad \mathcal{L}_V : \mathcal{E}_V \rightarrow \mathcal{D}_{\Omega_{V/X}^{1/2}}.$$

Denote by $\mathcal{D}_V(0)$ the subring of $\mathcal{D}_V$ consisting of differential operators which commute with the Euler vector field on V , by $\mathcal{D}_V^{bic}$ the subring of $\mathcal{D}_V$ consisting of differential operators which commute with the functions $a \in \mathcal{O}_V$ constant along the bicharacteristic leaves of V , and set $\mathcal{D}_V^{bic}(0) = \mathcal{D}_V^{bic} \cap \mathcal{D}_V(0)$.

The following result is essentially due to [10, 9].

Theorem 3.2. *The ring morphism (3.1) induces a ring isomorphism*

$$(3.2) \quad \mathcal{L}_V : \mathcal{E}_V / F_{-1}\mathcal{E}_V \xrightarrow{\sim} \mathcal{D}_{\Omega_{V/X}^{1/2}}^{bic}(0).$$

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4. STATEMENT OF THE MAIN RESULT

Let $\mathcal{A}$ be a t.d.o. ring on X , and V a conic involutive submanifold of T^*X . All definitions and constructions of §3 extend to the twisted case. In particular, setting

$$\begin{aligned}\mathcal{E}_{\mathcal{A}} &= \pi^{-1}\mathcal{O}_{\mathcal{A}} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_{\mathcal{A}}^{-1}, \\ \mathcal{E}_{V,\mathcal{A}} &= \pi^{-1}\mathcal{O}_{\mathcal{A}} \otimes_{\pi^{-1}\mathcal{O}} \mathcal{E}_V \otimes_{\pi^{-1}\mathcal{O}} \pi^{-1}\mathcal{O}_{\mathcal{A}}^{-1},\end{aligned}$$

Theorem 3.2 gives an isomorphism

$$(4.1) \quad \mathcal{L}_V: \mathcal{E}_{V,\mathcal{A}}/\mathbb{F}_{-1}^V \mathcal{E}_{\mathcal{A}} \xrightarrow{\sim} \mathcal{D}_{\Omega_{V/X}^{1/2} \otimes_{\mathcal{O}} \pi_V^* \mathcal{O}_{\mathcal{A}}}^{bic}(0),$$

where π_V is the restriction to V of $\pi: T^*X \rightarrow X$.

Let $j_{\Sigma}: \Sigma \hookrightarrow V$ be the embedding of a smooth bicharacteristic leaf of V . For $\mathcal{G}$ a twisted line bundle on V , we consider the restriction functor

$$j_{\Sigma}^*: \text{Mod}(\mathcal{D}_{\mathcal{G}}^{bic}(0)) \rightarrow \text{Mod}(\mathcal{D}_{j_{\Sigma}^* \mathcal{G}}(0)),$$

and the pull-back

$$(4.2) \quad \pi_{\Sigma}^{\sharp}: H^2(X; \mathbb{C}_X^{\times}) \rightarrow H^2(\Sigma; \mathbb{C}_{\Sigma}^{\times}).$$

Recall the maps γ and δ in (2.1).

Theorem 4.1. *Let V be a conic involutive submanifold of T^*X , and let Σ be a smooth bicharacteristic leaf of V . Let $\mathcal{A}$ be a t.d.o. ring on V , and let $\mathcal{M}$ be a globally simple $\mathcal{E}_{\mathcal{A}}$ -module along V . Then*

$$\delta(\pi_{\Sigma}^{\sharp}[\mathcal{A}]) = \delta\left(-\frac{1}{2} \cdot \gamma([\Omega_{\Sigma/X}])\right) \quad \text{in } H^2(\Sigma; \mathbb{C}_{\Sigma}^{\times}).$$

Sketch of proof. The proof follows the same lines as in [9, §I.5.2]. Let $\mathcal{M}_0$ be a V -lattice in $\mathcal{M}$. By (4.1), $\mathcal{M}_0/\mathcal{M}_{-1}$ is locally isomorphic to $\mathcal{O}_V(0)$ as $\mathcal{D}_{\Omega_{V/X}^{1/2} \otimes_{\mathcal{O}} \pi_V^* \mathcal{O}_{\mathcal{A}}}^{bic}(0)$ -modules. Note that $j_{\Sigma}^* \Omega_V \simeq \Omega_{\Sigma}$. Then $j_{\Sigma}^*(\mathcal{M}_0/\mathcal{M}_{-1})$ is a $\mathcal{D}_{\Omega_{\Sigma/X}^{1/2} \otimes_{\mathcal{O}} \pi_{\Sigma}^* \mathcal{O}_{\mathcal{A}}}(0)$ -module which is locally isomorphic to $\mathcal{O}_{\Sigma}(0)$. The statement follows by Proposition 2.1. $\square$

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Let $\mathbb{T} \simeq \mathbb{C}^4$ be a complex vector space, $\mathbb{P}$ the projective space of lines in $\mathbb{T}$, and $\mathbb{G}$ the Grassmannian of 2-dimensional subspaces in $\mathbb{T}$. The Penrose correspondence (see [5]) is associated with the double fibration

$$(5.1) \quad \mathbb{P} \xleftarrow{f} \mathbb{F} \xrightarrow{g} \mathbb{G}$$

where $\mathbb{F} = \{(y, x) \in \mathbb{P} \times \mathbb{G}; y \subset x\}$ is the incidence relation, and f, g are the natural projections. The double fibration (5.1) induces the maps

$$T^*\mathbb{P} \xleftarrow{p} T^*\mathbb{F}(\mathbb{P} \times \mathbb{G}) \xrightarrow{q} T^*\mathbb{G},$$

where $T_{\mathbb{F}}^*(\mathbb{P} \times \mathbb{G}) \subset T^*(\mathbb{P} \times \mathbb{G})$ denotes the conormal bundle to $\mathbb{F}$, and p and q are the natural projections. Note that p is smooth surjective, and q is a closed embedding. Set

$$V = q(\dot{T}_{\mathbb{F}}^*(\mathbb{P} \times \mathbb{G})).$$

Then V is a closed conic regular involutive submanifold of $\dot{T}^*\mathbb{G}$, and q identifies the fibers of p with the bicharacteristic leaves of V .

For $m \in \mathbb{Z}$ consider the line bundles $\mathcal{O}_{\mathbb{P}}(m)$, and set

$$\mathcal{M}_{(1+m/2)} = H^0(\mathbb{D}g_* \mathbb{D}f^*(\mathcal{D}_{\mathbb{P}} \otimes_{\mathcal{O}} \mathcal{O}_{\mathbb{P}}(-m))),$$

where $\mathbb{D}g_*$ and $\mathbb{D}f^*$ denote the direct and inverse image in the derived categories of $\mathcal{D}$ -module. As we recalled in the Introduction, according to Penrose $\mathbb{G}$ is a conformal compactification of the complexified Minkowski space, and the $\mathcal{D}_{\mathbb{G}}$ -module $\mathcal{M}_{(1+m/2)}$ corresponds to the massless field equation of helicity $1 + m/2$.

By [3], $\mathcal{M}_{(1+m/2)}$ is globally simple along V .

Theorem 5.1. *Let $\mathcal{A}$ be a t.d.o. ring on $\mathbb{G}$, and $\mathcal{M}$ an $\mathcal{A}$ -module globally simple along V . Then $\delta[\mathcal{A}] = 1$ in $H^2(\mathbb{G}; \mathbb{C}_{\mathbb{G}}^{\times})$. In particular, $\text{Mod}(\mathcal{A})$ and $\text{Mod}(\mathcal{D}_{\mathbb{G}})$ are $\mathbb{C}$ -equivalent.*

Proof. By Theorem 4.1 it is enough to show that for a bicharacteristic leaf $\Sigma \subset V$, one has an isomorphism $\pi_{\Sigma}^{\sharp}: H^1(\mathbb{G}; d\mathcal{O}_{\mathbb{G}}) \xrightarrow{\sim} H^1(\Sigma; d\mathcal{O}_{\Sigma})$, and moreover $\delta(-\gamma([\Omega_{\Sigma/\mathbb{G}}])/2) = 1 \in H^2(\Sigma; \mathbb{C}_{\Sigma}^{\times})$.

With the identification

$$T^*\mathbb{G} = \{(x; \xi); x \subset \mathbb{T}, \xi \in \text{Hom}(\mathbb{T}/x, x)\},$$

one has

$$V = \{(x; \xi); \text{rk}(\xi) = 1\},$$

where $\text{rk}(\xi)$ denotes the rank of the linear map ξ . There is an isomorphism

$$\dot{T}^*\mathbb{P} = \{(y, z; \theta); y \subset z \subset \mathbb{T}, \theta: \mathbb{T}/z \xrightarrow{\sim} y\},$$

where y is a line and z is a hyperplane. The projection $q: V \rightarrow \dot{T}^*\mathbb{P}$ is given by $q(x; \xi) = (\text{im } \xi, x + \ker \xi; \tilde{\xi})$, where $\tilde{\xi}$ satisfies $\tilde{\xi} \circ \ell = \xi$ for $\ell: \mathbb{T}/x \rightarrow \mathbb{T}/x + \ker \xi$ the natural map.

Recall that the bicharacteristic leaves of V are the fibers of q . For $(y, z, \theta) \in \dot{T}^*\mathbb{P}$, the bicharacteristic leaf $q^{-1}(y, z, \theta)$ of V is given by

$$\Sigma = \{(x; \xi); y \subset x \subset z, \xi = \theta \circ \ell\},$$

where $\ell: \mathbb{T}/x \rightarrow \mathbb{T}/z$ is the natural map. Thus, Σ is the projective space of lines in z/y . Hence the sequence

$$H^1(\Sigma; \mathcal{O}_{\Sigma}^{\times}) \xrightarrow{\gamma} H^1(\Sigma; d\mathcal{O}_{\Sigma}) \xrightarrow{\delta} H^2(\Sigma; \mathbb{C}_{\Sigma}^{\times})$$

is isomorphic to the sequence of additive abelian groups

$$\mathbb{Z} \xrightarrow{\gamma} \mathbb{C} \xrightarrow{\delta} \mathbb{C}/\mathbb{Z},$$

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with $1 \in \mathbb{Z}$ corresponding to $[\mathcal{O}_\Sigma(-1)]$.

Denote by $\mathcal{O}_\mathbb{G}(-1)$ the determinant of the tautological bundle on $\mathbb{G}$. Then $H^1(\mathbb{G}; d\mathcal{O}_\mathbb{G}) \simeq \mathbb{C}$ with generator $\mathcal{D}_{\mathcal{O}_\mathbb{G}(-1)}$. Since $\pi_\Sigma^* \mathcal{O}_\mathbb{G}(-1) \simeq \mathcal{O}_\Sigma(-1)$, it follows that

$$\pi_\Sigma^\# : H^1(\mathbb{G}; d\mathcal{O}_\mathbb{G}) \xrightarrow{\sim} H^1(\Sigma; d\mathcal{O}_\Sigma).$$

There are isomorphisms $\Omega_\mathbb{G} \simeq \mathcal{O}_\mathbb{G}(-4)$, and $\Omega_\Sigma \simeq \mathcal{O}_\Sigma(-2)$, so that $\pi_\Sigma^* \Omega_\mathbb{G} \simeq \pi_\Sigma^* \mathcal{O}_\mathbb{G}(-4) \simeq \mathcal{O}_\Sigma(-4)$, and $\Omega_{\Sigma/\mathbb{G}} \simeq \mathcal{O}_\Sigma(2)$. It follows that

$$[\Omega_{\Sigma/\mathbb{G}}] = 2 \quad \text{in } \mathbb{Z} \simeq H^1(\Sigma; \mathcal{O}_\Sigma^\times),$$

and therefore

$$\delta(-\gamma([\Omega_{\Sigma/\mathbb{G}}])/2) = 0 \quad \text{in } \mathbb{C}/\mathbb{Z} \simeq H^2(\Sigma; \mathbb{C}_\Sigma^\times).$$

By Theorem 4.1, it follows that $\delta(\pi_\Sigma^\# [\mathcal{A}]) = 1$. Hence $\delta([\mathcal{A}]) = 1$, so that $\mathfrak{S}_\mathcal{A}$ is globally $\mathbb{C}$ -equivalent to $\mathfrak{Mod}(\mathbb{C}_\mathbb{G})$. In particular, $\mathcal{O}_\mathcal{A}$ is an untwisted sheaf. The equivalence

$$\text{Mod}(\mathcal{D}_\mathbb{G}) \xrightarrow{\sim} \text{Mod}(\mathcal{A})$$

is given by $\mathcal{M} \mapsto \mathcal{O}_\mathcal{A} \otimes_{\mathcal{O}} \mathcal{M}$. □

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