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ON MULTIPLY TRANSITIVE GROUPS

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1. We treat a classification of 4-fold transitive groups. Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$, and set $H = G_{1\ 2\ 3\ 4}$.

The first step of the classification:

Jordan proved that if $H=1$ then $G=S_4$, S_5 , A_6 or M_{11} . By this theorem we have that $|I(H)|=4, 5, 6$ or 11 and $N_G(H)^{I(H)}=S_4, S_5, A_6$ or M_{11} respectively. Except the first case the classification is completed.

Theorem 1. [2]

If $N_G(H)^{I(H)}=S_5, A_6$ or M_{11} , then $G=S_5, A_6$ or M_{11} respectively.

The second step of the classification:

Let P be a Sylow 2-subgroup of H . The Jordan's theorem was extended by M. Hall in the following way: If H is of odd order then $G=S_4, S_5, A_6, A_7$ or M_{11} . By this theorem we have that $|I(P)|=4, 5, 6, 7$ or 11 and $N_G(P)^{I(P)}=S_4, S_5, A_6, A_7$ or M_{11} respectively.

Here we give a classification of the special cases in which $|I(P)|=6$ or 11 or $|I(P)|=4, 5$ or 7 and P satisfies some assumptions.

Definition and Notation. Let G be a permutation group on Ω . The stabilizer of points $i, j, \dots, k$ in G is denoted by $G_{i\ j\ \dots\ k}$. If X is a subset of G fixing a subset Δ of Ω , then X induces a set of permutation on Δ , which we denote by X^Δ . For a subset X of G , $I(X)$ denotes the set of all the fixed points of X . A G -orbit of minimal length ($\neq 1$) is called a minimal G -orbit.

2. Let G be a 4-fold transitive group and assume that a Sylow 2-subgroup P of $G_{1\ 2\ 3\ 4}$ is not the identity. For a point t of a minimal

P -orbit set $N_G(P_t)^{I(P_t)} = N$. Then N is a permutation group on $I(P_t)$ and satisfies the following conditions:

For any four points i, j, k and l of $I(P_t)$ let R be a Sylow 2-subgroup of $N_{i j k l}$. Then

- (1) R is nonidentity semi-regular,
- (2) $I(R) = I(P)$.

First we determine the structure of the group N .

Theorem 2. [6,7,8]

Let G be a permutation group on $\Omega = \{1, 2, \dots, n\}$ where $n > 4$. Assume that a Sylow 2-subgroup P of the stabilizer of any four points in G satisfies the following two conditions:

- (i) P is a nonidentity semi-regular group.
- (ii) P fixes exactly r points.

Then

- (I) If $r=4$, then $|\Omega| = 6, 8$ or 12 and $G = S_6, A_8$ or M_{12} respectively.
- (II) If $r=5$, then $|\Omega| = 7, 9$ or 13 . In particular, if $|\Omega|=9$, then $G \leq A_9$, and if $|\Omega|=13$, then $G = S_1 \times M_{12}$.
- (III) If $r=7$ and $N_G(P)^{I(P)} \leq A_7$, then $G = M_{23}$.
- (IV) It is impossible that $r=6$ and $N_G(P)^{I(P)} \leq A_6$ or $r=11$ and $N_G(P)^{I(P)} \leq M_{11}$.

By Theorem 2 we have the following

Theorem 3. [6,7,8]

Let G be a 4-fold transitive group on Ω and P be a Sylow 2-subgroup of $G_{1 2 3 4}$.

- (I) If $|I(P)| = 6$ or 11 then $G = A_6$ or M_{11} respectively.

(2)

(II) Assume that P is not the identity, and for a point t of $\Omega - I(P)$ a Sylow 2-subgroup R of the stabilizer of any four points in $N_G(P_t)^{I(P_t)}$ satisfies the following two conditions:

- (i) R is a nonidentity semi-regular group.
- (ii) $|I(R)| = |I(P)|$.

Then one of the conclusions (I), (II) and (III) in Theorem 2 holds for $N_G(P_t)^{I(P_t)}$. In particular, if t is a point of a minimal P -orbit, then $N_G(P_t)^{I(P_t)}$ satisfies the conditions (i) and (ii).

To prove Theorem 2 we need the following

Theorem 4. [5]

Let G be a 4-fold transitive group on $\Omega = \{1, 2, \dots, n\}$. If a Sylow 2-subgroup of $G_{1\ 2\ 3\ 4}$ is semi-regular and not identity, then $G = S_6, S_7, A_8, A_9, M_{12}$ or M_{23} .

In the proofs of these theorems we use frequently the combinatrial argument. For instance the case (II) of Theorem 2 will be proved in the following way.

Assume $|\Omega| > 9$. Let a be an involution of P and $I(P) = \{1, 2, 3, 4, 5\}$. We may assume a is of the form

$$a = (1)(2) \dots (5)(6\ 7)(8\ 9)(10\ 11) \dots$$

Since $a \in N_G(G_{6\ 7\ 8\ 9})$, there is an involution b of $G_{6\ 7\ 8\ 9}$ commuting with a . Since $|I(b)| = 5$, we may assume

$$b = (1)(2\ 3)(4\ 5)(6)(7)(8)(9) \dots$$

Since $\langle a, b \rangle < N_G(G_{2\ 3\ 6\ 7})$, there is an involution c of $G_{2\ 3\ 6\ 7}$ commuting with a and b , c is of the form

$$c = (1)(2)(3)(4\ 5)(6)(7)(8\ 9) \dots$$

Then $I(ac) = \{1, 2, 3, 8, 9\}$. Hence $\langle a, c \rangle$ is semi-regular on $\{10, 11, \dots, n\}$, and so we may assume

$$a = (1)(2) \dots (5)(6 \ 7)(8 \ 9)(10 \ 11)(12 \ 13) \dots,$$

$$c = (1)(2)(3)(4 \ 5)(6)(7)(8 \ 9)(10 \ 12)(11 \ 13) \dots$$

Since $\langle a, c \rangle < N_G(G_{10 \ 11 \ 12 \ 13})$, there is an involution d of $G_{10 \ 11 \ 12 \ 13}$ commuting with a and c . We may assume

$$d = (1)(2 \ 3)(4 \ 5)(6 \ 7)(8 \ 9)(10)(11)(12)(13) \dots$$

Since $\langle a, d \rangle < N_G(G_{2 \ 3 \ 10 \ 11})$, there is an involution f of $G_{2 \ 3 \ 10 \ 11}$ commuting with a and d . f is one of the following forms:

$$(i) \ f = (1)(2)(3)(4 \ 5)(6 \ 7)(8 \ 9)(10)(11)(12 \ 13) \dots,$$

$$(ii) \ f = (1)(2)(3)(4 \ 5)(6 \ 8)(7 \ 9)(10)(11)(12 \ 13) \dots$$

If f is of the form (i), then

$$af = (1)(2)(3)(4 \ 5)(6)(7)(8)(9) \dots$$

Thus $|I(af)| > 5$, which contradicts the assumption. Hence f is of the form (ii). Then

$$cf = (1)(2)(3)(4)(5)(6 \ 8 \ 7 \ 9) \dots$$

Thus 6, 7, 8 and 9 are contained in the same $G_{I(a)}$ -orbit. Since we took 2-cycles (6 7) and (8 9) as arbitrary 2-cycles of a , $G_{I(a)}$ is transitive on $\Omega - I(a)$. Hence for any involution x fixing five points $G_{I(x)}$ is also transitive on $\Omega - I(x)$.

By using this result repeatedly, we can prove that for some point i G_i is 4-fold transitive on $\Omega - \{i\}$. Hence by Theorem 4 $G = S_1 \times M_{12}$.

For $|\Omega| \leq 9$ the proof is similar.

3. By Theorem 3 if G is 4-fold transitive on $\Omega = \{1, 2, \dots, n\}$ and a Sylow 2-subgroup P of $G_{1 \ 2 \ 3 \ 4}$ is not the identity, then $|I(P)| = 4$,

5 or 7. In these cases the classification of G is not completed. If P is abelian or transitive on $\Omega - I(P)$ and normal in $G_{1\ 2\ 3\ 4}$, then G is determined:

Theorem 5. [4,8]

If P is a nonidentity abelian group, then $G = S_6, S_7, A_8, A_9$ or M_{23} .

Theorem 6. [1]

If P is a nonidentity normal subgroup of $G_{1\ 2\ 3\ 4}$ and transitive on $\Omega - I(P)$, then $G = S_6, A_8, M_{12}$ or M_{23} .

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