

6. QUATERNIONIC KÄHLER MANIFOLDS

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A quaternionic Kähler manifold is a riemannian manifold of dimension $4n$ whose holonomy group is contained in $Sp_1 \cdot Sp_n$. A better description of such manifolds is the following. Let M be a riemannian $4n$ -manifold ($n > 1$). Then an almost quaternionic Kähler structure is a 3-dimensional subbundle $\mathcal{Q} \subset \text{SkewEnd}(TM, TM)$ which at each point x has an orthonormal basis J_1, J_2, J_3 such that $J_j^2 = -1$, $J_j J_k = -J_k J_j$ for $j \neq k$, and $J_1 J_2 = J_3$. There is a natural bundle isometry

$$\text{SkewEnd}(TM, TM) \xrightarrow{\cong} \wedge^2 TM$$

which associates to each J_k a non-degenerate 2-form ω_k where

$$\omega_k(V, W) = \langle J_k V, W \rangle, \quad k = 1, 2, 3.$$

The 4-form

$$\Omega \equiv \omega_1^2 + \omega_2^2 + \omega_3^2$$

is independent of the choice of local frame field $\omega_1, \omega_2, \omega_3$ and is globally defined on M . The subgroup $G_x \equiv \{L \in SO(T_x M) : L^* \Omega = \Omega\}$ is isometric to $Sp_1 \cdot Sp_n = Sp_1 \times Sp_n / \mathbb{Z}_2$. The manifold M is quaternionic Kähler if and only if

$$(*) \quad \nabla \Omega \equiv 0$$

The Riemann curvature tensor is a symmetric map $R: \wedge^2 TM \rightarrow \wedge^2 TM$, and condition $(*)$ implies that

$$R|_{\mathcal{Q}} = \lambda \text{Id}_{\mathcal{Q}}$$

where λ is a universal positive multiple of the scalar curvature of M . In other words, with respect to the decomposition $\wedge^2 TM = \mathcal{Q} \oplus \mathcal{Q}^\perp$, we have

$$(**) \quad R = \begin{pmatrix} \lambda & 0 \\ 0 & * \end{pmatrix}$$

When $\dim(M) = 4$, condition $(*)$ is trivial and $\mathcal{Q} = \Lambda_+^2$. However, we shall define a 4-manifold M to be quaternionic Kähler if $(**)$ holds, i.e., if M is Einstein and anti-self-dual.

The main point of this paper is to define a momentum mapping for quaternionic Kähler manifolds and to describe a process of "quaternionic reduction" which produces new quaternionic Kähler manifolds from given ones. More specifically, let M be a quaternionic Kähler manifold of dimension $4n > 4$, and suppose $S^1 \subset \text{Aut}(M)$ is an S^1 -subgroup generated by a Killing vector field V which satisfies $L_V \Omega = 0$. Let $\Theta_V = i_V \Omega$ be the 3-form obtained by contraction with V . With respect to a local frame field $\omega_1, \omega_2, \omega_3$ as above, we have $\Theta_V = \sum_j (i_V \omega_j) \wedge \omega_j \cong \sum_j (i_V \omega_j) \otimes \omega_j$, and so Θ_V can be considered as a 1-form with values in $\mathcal{Q}$ (an element of $\Omega^1(\mathcal{Q})$).

Theorem 1. If $\lambda \neq 0$, then there exists a unique section $\mu_V \in \Omega^0(\mathcal{Q})$ such that

$$\nabla \mu_V = \Theta_V$$

Let $\mathfrak{g} \subset \mathfrak{X}_M$ be the Lie algebra of a compact subgroup G of the automorphism group of M . Then Theorem 1 gives a momentum mapping

$$\mu \in \Omega^0(\mathfrak{g}^* \otimes \mathcal{Q})$$

with the equivariance property

$$g_*(\mu_V(x)) = \mu_{\text{Ad}_g(V)}(gx)$$

(provided that $\lambda \neq 0$). The automatic nature of the existence and equivariance of μ is strikingly different from the "abelian" case where $\lambda = 0$. Let $Z_G \equiv \{x \in M : \mu(x) = 0\}$.

Theorem 2. At all regular points, the manifold Z_G/G with its naturally induced metric, is quaternionic Kähler.

Corollary 3. Let $G \cong S^1 \subset \text{Aut}(M)$ be generated by the vector field V and suppose that $V \neq 0$ along Z_G . Then Z_G/G is a quaternionic Kähler orbifold of dimension $4n-4$.

Starting with $\mathbb{P}_{\mathbb{H}}^n$ we obtain a large number of examples. In particular, we prove the following. For relatively prime integers a, b, c let $\mathbb{P}_{a,b,c}^2$ be the weighted projective plane defined as $(\mathbb{C}^3 - \{0\})/\mathbb{C}^\times$ where $\mathbb{C}^\times$ acts by $\varphi_t(x, y, z) = (t^a x, t^b y, t^c z)$. Each $\mathbb{P}_{a,b,c}^2$ is a compact simply-connected orbifold.

Theorem 4. Let $p, q \in \mathbb{Z}^+$ be relatively prime integers with $q < p$. Then each of the weighted projective planes

$$\begin{array}{ll} \mathbb{P}_{p+q, p+q, 2p}^2 & \text{if } p+q \text{ is odd} \\ \mathbb{P}_{\frac{p+q}{2}, \frac{p+q}{2}, p}^2 & \text{if } p+q \text{ is even} \end{array}$$

carries a (non-locally symmetric) riemannian orbifold metric which is Einstein, anti-self-dual, and of positive scalar curvature.

Note. N. Hitchin has proved that any simply-connected riemannian 4-manifold which is quaternionic Kähler with $\lambda > 0$ is S^4 or $\mathbb{P}_{\mathbb{C}}^2$, with the standard symmetric metrics. Hence, the appearance of singularities in dimension 4 is necessary.

Similar examples in all dimensions $4n$ are constructed.

This represents work done in collaboration with K. Galicki at I.T.P. in Stony Brook who pioneered the construction. The first suggestion that such a reduction process might exist came from M. Roček, also at I.T.P..