

Smallest complex nilpotent orbits with real points

Takayuki Okuda*

Abstract

In this paper, we show that there uniquely exists a real minimal nilpotent orbit in a non-compact simple Lie algebra $\mathfrak{g}$ if $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type. For the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$, the complexification $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ of such the real minimal nilpotent orbit in $\mathfrak{g}$ is not the complex minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. For such cases, we also determine $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of it.

1 Introduction and main results

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. In this paper, an adjoint nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ will be simply called a complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. It is well-known that there exists a unique non-zero complex nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$, which is called a complex minimal nilpotent orbit, with the following property: The closure of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ is just $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \sqcup \{0\}$. By the uniqueness of such $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$, for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$, the closure of $\mathcal{O}$ contains $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$. In other words, $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ is minimum in $\mathcal{N}/G_{\mathbb{C}}$ without the zero-orbit, where $\mathcal{N}/G_{\mathbb{C}}$ denotes the set of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ with the closure ordering.

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Namely, $\mathfrak{g}$ is a non-compact real simple Lie algebra without complex structures and $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$. Our concern in this paper is in real minimal nilpotent orbits in $\mathfrak{g}$. Here, we say that a non-zero real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if the closure

*Graduate School of Mathematical Science, The University of Tokyo

of $\mathcal{O}^G$ in $\mathfrak{g}$ is just $\mathcal{O}^G \sqcup \{0\}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

If the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ meets $\mathfrak{g}$, then the intersection $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\mathfrak{su}^*(2k)$ ($k \geq 2$), $\mathfrak{so}(n-1, 1)$ ($n \geq 5$), $\mathfrak{sp}(p, q)$ ($p \geq q \geq 1$), $\mathfrak{f}_{4(-20)}$ nor $\mathfrak{e}_{6(-26)}$ (see Brylinski [3, Theorem 4.1]). In particular, if $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, then $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Furthermore, for the cases where $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ meets $\mathfrak{g}$, the number of real minimal nilpotent orbits (i.e. the number of adjoint orbits in $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g}$) is two if $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type; one if $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type.

In this paper, we study real minimal nilpotent orbits in $\mathfrak{g}$ including the cases where $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meet $\mathfrak{g}$. For any real non-compact simple Lie algebra $\mathfrak{g}$ without complex structures, we put

$$\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}} := \{ \text{Complex nilpotent orbits in } \mathfrak{g}_{\mathbb{C}} \text{ meeting } \mathfrak{g} \}$$

and consider the closure ordering on it. Our first main result is here:

Theorem 1.1. *There uniquely exists a complex nilpotent orbit $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ in $\mathfrak{g}_{\mathbb{C}}$ which is minimum in $\mathcal{N}_{\mathfrak{g}}/G_{\mathbb{C}}$ without the zero-orbit (i.e. for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, if $\mathcal{O} \cap \mathfrak{g} \neq \emptyset$, then the closure of $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$ contains $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$). Furthermore, the intersection $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$.*

We will construct such $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ as the complex adjoint orbit through a non-zero longest restricted root vector in $\mathfrak{g}$. By the definition of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$, the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ is not our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ if and only if $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meet $\mathfrak{g}$ (namely, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ ($k \geq 2$), $\mathfrak{so}(n-1, 1)$ ($n \geq 5$), $\mathfrak{sp}(p, q)$ ($p \geq q \geq 1$), $\mathfrak{f}_{4(-20)}$ or $\mathfrak{e}_{6(-26)}$). This means that for such cases, a non-zero longest restricted root vector in $\mathfrak{g}$ is not a longest root vector in $\mathfrak{g}_{\mathbb{C}}$.

Theorem 1.1 claims that $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. Our second main result is here:

Theorem 1.2. *For the cases where the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ does not meet $\mathfrak{g}$, there exists a unique real minimal nilpotent orbit in $\mathfrak{g}$. In particular, the complex nilpotent orbit $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ in Theorem 1.1 (which is not $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ in these cases) is the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$.*

Theorem 1.4. *For the cases where $\mathcal{O}_{\min, \mathfrak{g}}^{G_c} \neq \mathcal{O}_{\min}^{G_c}$, the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_c}$ are the following:*

This work is motivated by recent works [7], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an L^2 -model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [8], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with a $(\mathfrak{g}, K)$ -module which is discretely decomposable as an $(\mathfrak{h}, H \cap K)$ -module.

2 Preliminary results for weighted Dynkin diagrams of complex minimal nilpotent orbits

In this section, we recall weighted Dynkin diagrams of complex minimal nilpotent orbits in complex simple Lie algebras.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra, and denote by $G_{\mathbb{C}}$ the inner automorphism group of $\mathfrak{g}_{\mathbb{C}}$. Fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. We denote by $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ can be regarded as a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{ H \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(H) \in \mathbb{R} \ (\forall \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})) \}.$$

We write $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ for the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{ H \in \mathfrak{h} \mid \alpha(H) \geq 0 \ (\forall \alpha \in \Delta_+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})) \}$$

is a fundamental domain of $\mathfrak{h}$ under the action of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

Let Π be the simple system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \rightarrow \mathbb{R}, \alpha \mapsto \alpha(H).$$

We call Ψ_H the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R}), H \mapsto \Psi_H$$

is a linear isomorphism (between vector spaces). Furthermore,

$$\mathfrak{h}_+ \rightarrow \text{Map}(\Pi, \mathbb{R}_{\geq 0}), \quad H \mapsto \Psi_H$$

is also bijective.

A triple (H, X, Y) is said to be an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}_{\mathbb{C}}$ if

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H \quad (H, X, Y \in \mathfrak{g}_{\mathbb{C}}).$$

For any $\mathfrak{sl}_2$ -triple (H, X, Y) in $\mathfrak{g}_{\mathbb{C}}$, the elements X and Y are nilpotent in $\mathfrak{g}_{\mathbb{C}}$, and H is hyperbolic in $\mathfrak{g}_{\mathbb{C}}$ (i.e. $\text{ad}_{\mathfrak{g}_{\mathbb{C}}} H \in \text{End}(\mathfrak{g}_{\mathbb{C}})$ is diagonalizable with only real eigenvalues).

Combining the Jacobson–Morozov theorem with Kostant [9], for any complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$, there uniquely exists an element $H_{\mathcal{O}}$ of $\mathfrak{h}_+$ with the following property: There exists $X, Y \in \mathcal{O}^{G_{\mathbb{C}}}$ such that $(H_{\mathcal{O}}, X, Y)$ is an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}_{\mathbb{C}}$. Furthermore, by Malcev [10], the following map is injective:

$$\{ \text{Complex nilpotent orbits in } \mathfrak{g}_{\mathbb{C}} \} \hookrightarrow \mathfrak{h}_+, \quad \mathcal{O}^{G_{\mathbb{C}}} \mapsto H_{\mathcal{O}}.$$

The weighted Dynkin diagram corresponding to $H_{\mathcal{O}}$ is called the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$. Dynkin [6] proved that for any complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}$, any weight of the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}$ is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (More precisely, Dynkin [6] classified $\mathfrak{sl}_2$ -triples in $\mathfrak{g}_{\mathbb{C}}$. See Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple. Let ϕ be the highest root of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, the complex minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ can be written by

$$\mathcal{O}_{\min}^{G_{\mathbb{C}}} = G_{\mathbb{C}} \cdot \mathfrak{g}_{\phi} \setminus \{0\}.$$

We define the element $H_{\phi^{\vee}}$ of $\mathfrak{h}$ by

$$\alpha(H_{\phi^{\vee}}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle}$$

for any $\alpha \in \mathfrak{h}^*$ (where $\langle \cdot, \cdot \rangle$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}_{\mathbb{C}}$). Namely, $H_{\phi^{\vee}}$ is the element of $\mathfrak{h}$ corresponding to the coroot $\phi^{\vee}$ of ϕ . Since ϕ is dominant, $H_{\phi^{\vee}}$ is in $\mathfrak{h}_+$. Furthermore, $H_{\phi^{\vee}}$ is the hyperbolic element corresponding to $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ since we can find $X_{\phi} \in \mathfrak{g}_{\phi}$, $Y_{\phi} \in \mathfrak{g}_{-\phi}$ such that $(H_{\phi^{\vee}}, X_{\phi}, Y_{\phi})$ is an $\mathfrak{sl}_2$ -triple. The list of weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ for all simple $\mathfrak{g}_{\mathbb{C}}$ can be found in Collingwood–McGovern [4, Ch.5.4 and 8.4].

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$. The complexifications of such algebras are $\mathfrak{sl}(2k, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(p+q, \mathbb{C})$, $\mathfrak{e}_{6, \mathbb{C}}$ and $\mathfrak{f}_{4, \mathbb{C}}$, respectively. For the convenience of the reader, we give a list of weighted Dynkin diagrams of complex minimal nilpotent orbits in such complex simple Lie algebras.

$\mathfrak{g}_{\mathbb{C}}$	$\dim_{\mathbb{C}} \mathcal{O}_{\min}^{G_{\mathbb{C}}}$	Weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$
$\mathfrak{sl}(n, \mathbb{C})$	$2n$	$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 \\ \circ & \circ & \circ & \circ & \cdots & \circ & \circ & \circ & \circ \end{array} \quad (n \geq 2)$
$\mathfrak{so}(n, \mathbb{C})$	$2n - 6$	$\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & 0 & \\ \circ & \circ & \circ & \cdots & \circ & \circ & \circ \end{array} \quad (n \text{ is odd, } n \geq 7)$ $\begin{array}{cc} 0 & 1 \\ \circ & \circ \end{array} \quad (n = 5)$ $\begin{array}{ccccccc} 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \circ & \circ & \circ & \cdots & \circ & \circ & \circ \end{array} \quad (n \text{ is even, } n \geq 6)$
$\mathfrak{sp}(n, \mathbb{C})$	$2n$	$\begin{array}{ccccccc} 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \circ & \circ & \circ & \circ & \cdots & \circ & \circ \end{array} \quad (n \geq 2)$
$\mathfrak{e}_{6, \mathbb{C}}$	22	$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \circ & \circ & \circ & \circ & \circ \\ & & & & \circ \\ & & & & 1 \end{array}$
$\mathfrak{f}_{4, \mathbb{C}}$	16	$\begin{array}{cccc} 1 & 0 & 0 & 0 \\ \circ & \circ & \circ & \circ \end{array}$

Table 2: List of weighted Dynkin diagrams of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ for $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{so}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, $\mathfrak{e}_{6, \mathbb{C}}$ and $\mathfrak{f}_{4, \mathbb{C}}$.

3 Outline of a proof of Theorem 1.1

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In this section, we describe an idea of the proof of Theorem 1.1.

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ (such $\mathfrak{a}$ is called a maximally split abelian subspace of $\mathfrak{g}$) and write $\Sigma(\mathfrak{g}, \mathfrak{a})$ for the restricted root system

for $(\mathfrak{g}, \mathfrak{a})$. For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$, we define $A_{\xi^\vee} \in \mathfrak{a}$ by

$$\eta(A_{\xi^\vee}) = \frac{2(\xi, \eta)}{(\xi, \xi)} \quad (\forall \eta \in \mathfrak{a}^*)$$

(where $(\cdot, \cdot)$ is the inner product on $\mathfrak{a}^*$ induced by the Killing form on $\mathfrak{g}$). Namley, $A_{\xi^\vee}$ is the element of $\mathfrak{a}$ corresponding to the coroot $\xi^\vee$ of ξ . Then, the fact below holds:

Fact 3.1. *For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$ and any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists $Y_\xi \in \mathfrak{g}_{-\xi}$ such that $(A_{\xi^\vee}, X_\xi, Y_\xi)$ is an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}$.*

We fix an ordering on $\mathfrak{a}$ and write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$ corresponding to the ordering on $\mathfrak{a}$. We denote by λ the highest root of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ with respect to the ordering on $\mathfrak{a}$. Next two lemmas give characterizations of the highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (we omit proofs of the two lemmas in this paper):

Lemma 3.2. *The highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ is a unique dominant longest root of $\Sigma(\mathfrak{g}, \mathfrak{a})$.*

Lemma 3.3. *Let ξ be a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$. If ξ is not the highest root λ , then for any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists a positive root η in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and a root vector $X_\eta \in \mathfrak{g}_\eta$ such that $[X_\xi, X_\eta] \neq 0$. In particular, $\xi = \lambda$ if and only if $\xi + \eta \in \mathfrak{a}^*$ is not a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ for any $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$.*

We write $G_{\mathbb{C}}$ for the inner automorphism group of $\mathfrak{g}_{\mathbb{C}}$. Then, the following two propositions hold:

Proposition 3.4. *For any non-zero real nilpotent orbit $\mathcal{O}'_0$ in $\mathfrak{g}$. Then, there exists a non-zero highest root vector X_λ in $\mathfrak{g}_\lambda$ such that X_λ is in the closure of $\mathcal{O}'_0$ in $\mathfrak{g}$.*

Proposition 3.5. *For any two highest root vectors X_λ, X'_λ in $\mathfrak{g}_\lambda$, there exists $g_{\mathbb{C}} \in G_{\mathbb{C}}$ such that $g_{\mathbb{C}}X_\lambda = X'_\lambda$.*

Proof of Proposition 3.4. There is no loss of generality in assuming that the ordering on $\mathfrak{a}$ is lexicographic. Let us put $\mathfrak{m} = Z_{\mathfrak{t}}(\mathfrak{a})$. Then, $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\xi.$$

For any $X' \in \mathfrak{g}$, we denote by

$$X' = X'_m + X'_a + \sum_{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})} X'_\xi \quad (X'_m \in \mathfrak{m}, X'_a \in \mathfrak{a}, X'_\xi \in \mathfrak{g}_\xi).$$

We put $\overline{\mathcal{O}'_0}$ to the closure of $\mathcal{O}'_0$ in $\mathfrak{g}$ and fix an element X' in $\overline{\mathcal{O}'_0}$. Let us denote by λ' the highest one of

$$\Sigma_{X'} := \{ \xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid X'_\xi \neq 0 \}$$

with respect to the ordering on $\mathfrak{a}$ (if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since X' is nilpotent element in $\mathfrak{g}$). As a first step of the proof, we shall prove that the root vector $X'_{\lambda'}$ is also in $\overline{\mathcal{O}'_0}$. We take $A' \in \mathfrak{a}$ satisfying that

$$\xi(A') < \lambda'(A') \quad (\forall \xi \in \Sigma_{X'} \setminus \{\lambda'\}).$$

(such A' exists since λ' is highest in $\Sigma_{X'}$ with respect to the lexicographic ordering on $\mathfrak{a}$). Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(\text{ad}_{\mathfrak{g}} kA') X' \quad (\text{for } k \in \mathbb{N})$$

Then, X'_k is in $\overline{\mathcal{O}'_0}$ for any k since $\overline{\mathcal{O}'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k \rightarrow \infty} X'_k = \lim_{k \rightarrow \infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A') - \lambda'(A'))} X'_\xi = X'_{\lambda'}.$$

This means that $X'_{\lambda'}$ is in $\overline{\mathcal{O}'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{\mathcal{O}'_0}$ such that $\lambda' = \lambda$ (where λ' is the highest one of $\Sigma_{X'}$). Let λ_0 be the highest one of

$$\Sigma_{\overline{\mathcal{O}'_0}} := \{ \xi \in \Sigma(\mathfrak{g}, \mathfrak{a}) \mid \exists X' \in \overline{\mathcal{O}'_0} \text{ such that } X'_\xi \neq 0 \}$$

(namely, $\Sigma_{\overline{\mathcal{O}'_0}} = \bigcup_{X' \in \overline{\mathcal{O}'_0}} \Sigma_{X'}$) with respect to the ordering on $\mathfrak{a}$. Then, we can find a root vector X'_{λ_0} in $\mathfrak{g}_{\lambda_0} \cap \overline{\mathcal{O}'_0}$ by the argument above. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 3.3, there exists $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$ and $X_\eta \in \mathfrak{g}_\eta$ such that $[X_\eta, X'_{\lambda_0}] \neq 0$. In particular, for the element $X'' := \exp(\text{ad}_{\mathfrak{g}}(X_\eta)) X'_{\lambda_0}$ in $\overline{\mathcal{O}'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{\mathcal{O}'_0}}.$$

This contradicts the definition of λ_0 . Thus, $\lambda_0 = \lambda$. □

Proof of Proposition 3.5. Let $A_{\lambda^\vee}$ be the element in $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root λ . We put

$$(\mathfrak{g}_\mathbb{C})_2 = \{ X \in \mathfrak{g}_\mathbb{C} \mid [A_{\lambda^\vee}, X] = 2X \}.$$

Then, $\mathfrak{g}_\lambda$ is included in $(\mathfrak{g}_\mathbb{C})_2$. We note that there exists $X, Y \in \mathfrak{g}_\mathbb{C}$ such that $(A_{\lambda^\vee}, X, Y)$ is an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}_\mathbb{C}$ (in fact, we can find such X, Y in $\mathfrak{g}_\lambda$ by Fact 3.1). Therefore, we can use Malcev's theorem. Namely, for any two non-zero vectors X and X' in $(\mathfrak{g}_\mathbb{C})_2$, there exists $g_\mathbb{C} \in G_\mathbb{C}$ such that $g_\mathbb{C}X = X'$. Since $\mathfrak{g}_\lambda \subset (\mathfrak{g}_\mathbb{C})_2$, the proof is completed. $\square$

By using Proposition 3.4 and Proposition 3.5, Theorem 1.1 follows by taking $\mathcal{O}_{\min, \mathfrak{g}}^{G_\mathbb{C}}$ as

$$\mathcal{O}_{\min, \mathfrak{g}}^{G_\mathbb{C}} := G_\mathbb{C} \cdot \mathfrak{g}_\lambda \setminus \{0\}.$$

4 Outline of a proof of Theorem 1.2

Let us consider the same setting in §3. Recall that $\mathcal{O}_{\min, \mathfrak{g}}^{G_\mathbb{C}}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_\mathbb{C}}$ if and only if $\mathcal{O}_{\min}^{G_\mathbb{C}}$ does not meet $\mathfrak{g}$. The proposition below give a characterization of $\mathfrak{g}$ for which $\mathcal{O}_{\min}^{G_\mathbb{C}}$ is not $\mathcal{O}_{\min, \mathfrak{g}}^{G_\mathbb{C}}$ (see Proposition 5.6 for another characterizations of it).

Proposition 4.1. *The following conditions on $\mathfrak{g}$ are equivalent:*

1. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.
2. $\mathcal{O}_{\min}^{G_\mathbb{C}} \cap \mathfrak{g} = \emptyset$.

We can prove the proposition without any classification, but we omit it in this paper.

Here, we put $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$ and denote by M_0, A to the analytic subgroups of G corresponding to $\mathfrak{m}, \mathfrak{a}$, respectively. Then, the connected Lie group M_0A (which is the analytic subgroup of G corresponding to $\mathfrak{m} \oplus \mathfrak{a}$) acts on $\mathfrak{a}$. Furthermore, the following proposition holds:

Proposition 4.2. *If $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, then $\mathfrak{g}_\lambda \setminus \{0\}$ is a single M_0A -orbit.*

Combining Proposition 3.4, Proposition 4.1 with Proposition 4.2, we obtain Theorem 1.2.

We will use the next lemma to prove Proposition 4.2.

Lemma 4.3. *Suppose that $\mathfrak{g}$ has real rank one (i.e. $\dim_{\mathbb{R}} \mathfrak{a} = 1$) and $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$. Then, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single M_0A -orbit.*

Proof of Lemma 4.3. Let $A_{\lambda^{\vee}}$ be the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^{\vee}$ of the highest root λ in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (see §3). Since $\mathfrak{g}$ has real rank one, we have $\mathfrak{a} = \mathbb{R}A_{\lambda^{\vee}}$, and $\mathfrak{g}$ can be written by

$$\mathfrak{g} = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-\frac{\lambda}{2}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\frac{\lambda}{2}} \oplus \mathfrak{g}_{\lambda}$$

($\mathfrak{g}_{\pm\frac{\lambda}{2}}$ can be zero). Let us denote by $\mathfrak{g}_{\mathbb{C}}, \mathfrak{m}_{\mathbb{C}}, \mathfrak{a}_{\mathbb{C}}, (\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}, (\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}$ the complexification of $\mathfrak{g}, \mathfrak{m}, \mathfrak{a}, \mathfrak{g}_{\pm\lambda}, \mathfrak{g}_{\pm\frac{\lambda}{2}}$, respectively. We set

$$(\mathfrak{g}_{\mathbb{C}})_i = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid [A_{\lambda^{\vee}}, X] = iX \} \quad (\text{for } i \in \mathbb{Z}).$$

Then,

$$(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}, \quad (\mathfrak{g}_{\mathbb{C}})_{\pm 1} = (\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}, \quad (\mathfrak{g}_{\mathbb{C}})_{\pm 2} = (\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}.$$

By Fact 3.1, for any non-zero highest root vector X_{λ} in $\mathfrak{g}_{\lambda}$, there exists $Y_{\lambda} \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda^{\vee}}, X_{\lambda}, Y_{\lambda})$ is an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}_{\mathbb{C}}$. By the theory of representations of $\mathfrak{sl}(2, \mathbb{C})$, we obtain that $[(\mathfrak{g}_{\mathbb{C}})_0, X_{\lambda}] = (\mathfrak{g}_{\mathbb{C}})_2$. In particular, we have

$$[\mathfrak{m} \oplus \mathfrak{a}, X_{\lambda}] = \mathfrak{g}_{\lambda}.$$

Therefore, for the M_0A -orbit $\mathcal{O}^{M_0A}(X_{\lambda})$ in $\mathfrak{g}_{\lambda}$ through X_{λ} , we obtain that

$$\dim_{\mathbb{R}} \mathcal{O}^{M_0A}(X_{\lambda}) = \dim_{\mathbb{R}} \mathfrak{g}_{\lambda}.$$

This means that the M_0A -orbit $\mathcal{O}^{M_0A}(X_{\lambda})$ is open in $\mathfrak{g}_{\lambda}$ for any non-zero root vector X_{λ} in $\mathfrak{g}_{\lambda}$. Recall that we are assuming that $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$. Hence, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is connected. Therefore, $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single M_0A -orbit. $\square$

We are ready to prove Proposition 4.2.

Sketch of a proof of Proposition 4.2. Let $\mathfrak{h}' := [\mathfrak{g}_{\lambda}, \mathfrak{g}_{-\lambda}] \subset \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{g}' := \mathfrak{g}_{-\lambda} \oplus \mathfrak{h}' \oplus \mathfrak{g}_{\lambda}$ becomes a subalgebra of $\mathfrak{g}$ (since $\pm 2\lambda$ is not a root). Furthermore, one can prove that $\mathfrak{g}'$ is a real rank one simple Lie algebra with a maximally split abelian subspace $\mathfrak{a}' := \mathbb{R}A_{\lambda^{\vee}}$, where $A_{\lambda^{\vee}}$ is the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^{\vee}$ of the highest root λ in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (see §3). We put $\mathfrak{m}' \oplus \mathfrak{a}' := Z_{\mathfrak{g}'}(\mathfrak{a}')$ and denote by M'_0A' the analytic subgroup of G corresponding to $\mathfrak{m}' \oplus \mathfrak{a}'$. Then, by Lemma 4.3, we obtain that $\mathfrak{g}_{\lambda} \setminus \{0\}$ is a single M'_0A' -orbit. Since M'_0A' is a subgroup of M_0A , the proof is completed. $\square$

5 Determination of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$

In this section, we determine $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ by describing the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$. Recall that Proposition 4.1 claims that $\mathcal{O}_{\min}^{G_{\mathbb{C}}} = \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ if and only if $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} = 1$. Thus, our concern is in the cases where $\dim_{\mathbb{R}} \mathfrak{g}_{\lambda} \geq 2$ (i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$).

5.1 Satake diagrams and weighted Dynkin diagrams

In order to determine the weighted Dynkin diagram of our $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$, we describe some lemmas of relationship between weighted Dynkin diagrams of $\mathfrak{g}_{\mathbb{C}}$ and Satake diagrams of $\mathfrak{g}$ in this subsection.

Let $\mathfrak{g}_{\mathbb{C}}$ be a semisimple Lie algebra and $\mathfrak{g}$ a real form of it through this subsection. First, we recall briefly the definition of Satake diagram of a real form $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}_{\mathbb{C}}$ (see also [1] for more details). Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and extend it to a maximal abelian subspace $\mathfrak{h} = \sqrt{-1}\mathfrak{k} \oplus \mathfrak{a}$ in $\sqrt{-1}\mathfrak{k} \oplus \mathfrak{p}$. Then, the complexification, denoted by $\mathfrak{h}_{\mathbb{C}}$, of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{C}}$, and $\mathfrak{h}$ coincide with the real form

$$\{X \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(X) \in \mathbb{R} \ (\forall \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}))\}$$

of $\mathfrak{h}_{\mathbb{C}}$, where $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let us denote by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\} \subset \mathfrak{a}^*$$

the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. We will denote by $W(\mathfrak{g}, \mathfrak{a})$, $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$, respectively. Fix an ordering on $\mathfrak{a}$ and extend it to an ordering on $\mathfrak{h}$. We write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ corresponding to the ordering on $\mathfrak{a}$, $\mathfrak{h}$, respectively. Then, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ can be written by

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})\} \setminus \{0\}.$$

We denote by Π the fundamental system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then,

$$\overline{\Pi} := \{\alpha|_{\mathfrak{a}} \mid \alpha \in \Pi\} \setminus \{0\}$$

is the simple system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let Π_0 be the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero. The Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ consists of the

following data: The Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π ; black nodes Π_0 in $S_{\mathfrak{g}}$; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in $S_{\mathfrak{g}}$ whose restrictions to $\mathfrak{a}$ are the same.

Second, we define that a weighted Dynkin diagram $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ “matches” the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 5.1. *Let $\Psi_H \in \text{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram (see §2) and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π . We say that Ψ_H matches $S_{\mathfrak{g}}$ if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.*

Remark 5.2. *The concept of “match” defined above is same as “weighted Satake diagrams” in Djocovic [5] and the condition described in Sekiguchi [11, Proposition 1.16].*

Recall that Ψ is a linear isomorphism from $\mathfrak{h}$ to $\text{Map}(\Pi, \mathbb{R})$ (see §2). Then, the next two lemmas hold (we omit proofs of the two lemmas in this paper):

Lemma 5.3. $\Psi : \mathfrak{h} \rightarrow \text{Map}(\Pi, \mathbb{R})$ induces a linear isomorphism below:

$$\mathfrak{a} \rightarrow \{ \Psi_H \in \text{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S_{\mathfrak{g}} \}.$$

Lemma 5.4. *For each simple root α of Π , we denote by $H_{\alpha^\vee}$ the element in $\mathfrak{h}$ corresponding to the coroot $\alpha^\vee$ of the simple root α . Then, the set*

$$\{ H_{\alpha^\vee} \mid \alpha \text{ is black in } S_{\mathfrak{g}} \} \cup \{ H_{\alpha^\vee} - H_{\beta^\vee} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S_{\mathfrak{g}} \}$$

is a basis of $\sqrt{-1}\mathfrak{t}$.

Lemma 5.3 and Lemma 5.4 will be used to compute the weighted Dynkin diagrams of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ for the cases where $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is not the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$.

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$. For the convenience of the reader, we give a list of Satake diagrams of such simple Lie algebras.

$\mathfrak{g}$	Satake diagrams of $\mathfrak{g}$
$\mathfrak{su}^*(2k)$	

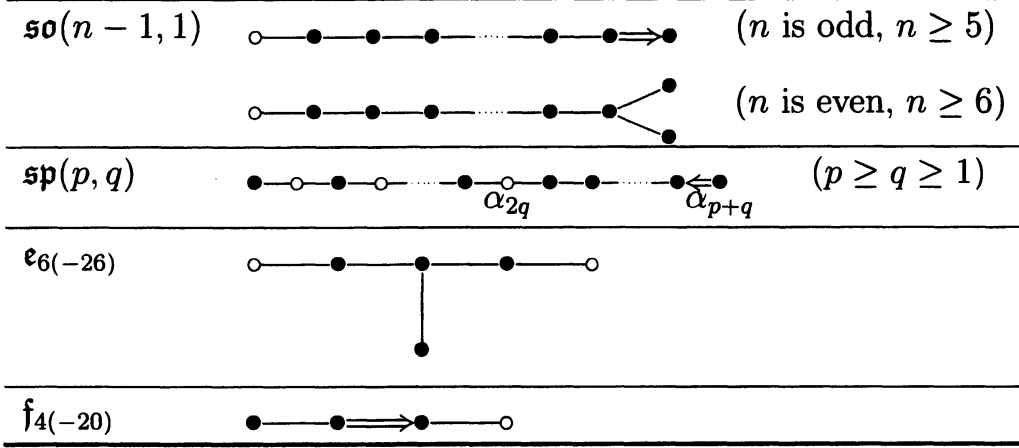


Table 3: List of Satake diagrams of $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_6(-26)$ and $\mathfrak{f}_4(-20)$.

5.2 Computation of weighted Dynkin diagrams of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$

We consider the same setting on §5.1 and suppose that $\mathfrak{g}_{\mathbb{C}}$ is simple and $\mathfrak{g}$ is non-compact. Let us denote by

$$\mathfrak{a}_+ := \{ A \in \mathfrak{a} \mid \xi(A) \geq 0 \ (\forall \xi \in \Sigma^+(\mathfrak{g}, \mathfrak{a})) \}.$$

Then $\mathfrak{a}_+$ is a fundamental domain of $\mathfrak{a}$ under the action of $W(\mathfrak{g}, \mathfrak{a})$. Since

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{ \alpha|_{\mathfrak{a}} \mid \alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}) \} \setminus \{0\},$$

the domain $\mathfrak{a}_+$ coincide with $\mathfrak{h}_+ \cap \mathfrak{a}$. Recall that λ is dominant (by Lemma 3.2) and $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ contains $\mathfrak{g}_{\lambda} \setminus \{0\}$ (by the proof of Theorem 1.1). Thus, $A_{\lambda^\vee}$ is the hyperbolic element in $\mathfrak{a}_+$ corresponding to $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ (see §2) since we can find $X_{\lambda} \in \mathfrak{g}_{\lambda}$, $Y_{\lambda} \in \mathfrak{g}_{-\lambda}$ such that the triple $(A_{\lambda^\vee}, X_{\lambda}, Y_{\lambda})$ is an $\mathfrak{sl}_2$ -triple in $\mathfrak{g}_{\mathbb{C}}$ by Lemma 3.1 (then, $X_{\lambda}, Y_{\lambda} \in \mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$). Therefore, to determine the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$, we shall compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$.

Let ϕ be the highest root of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Recall that the complex minimal nilpotent orbit $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ contains the root space $(\mathfrak{g}_{\mathbb{C}})_{\phi}$ without zero, and the weighted Dynkin diagram of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ is the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ (see §2). The next lemma gives a formula for $A_{\lambda^\vee}$ by $H_{\phi^\vee}$ (we omit a proof of the lemma):

Lemma 5.5. *We denote by τ the anti $\mathbb{C}$ -linear involution corresponding to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$ (i.e. τ is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$). Then, $H_{\phi^\vee}$ is in $\mathfrak{a}$ if and only if $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ and*

$$A_{\lambda^\vee} = \begin{cases} H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1), \\ H_{\phi^\vee} + \tau H_{\phi^\vee} & (\text{if } \dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2). \end{cases}$$

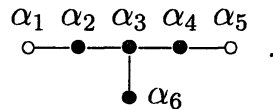
In particular, we have another characterizations of $\mathfrak{g}$ for which $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ is not $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ from Proposition 4.1.

Proposition 5.6. *The following conditions on $\mathfrak{g}$ are equivalent:*

1. $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}} \neq \mathcal{O}_{\min}^{G_{\mathbb{C}}}$.
2. $\mathcal{O}_{\min}^{G_{\mathbb{C}}} \cap \mathfrak{g} = \emptyset$.
3. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.
4. *The highest root ϕ in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is not a real root.*
5. *The weighted Dynkin diagram of $\mathcal{O}_{\min}^{G_{\mathbb{C}}}$ matches the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ (see Definition §5.1).*
6. *$\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$, where $k \geq 2$, $n \geq 5$ and $p \geq q \geq 1$.*

We now determine the weighted Dynkin diagram of $\mathcal{O}_{\min, \mathfrak{g}}^{G_{\mathbb{C}}}$ for the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1, 1)$, $\mathfrak{sp}(p, q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$. By Lemma 5.5, our purpose is to compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. We only give the computation for the case $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ below. For the other $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, we can compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ by the same way.

Example 5.7. *Let $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}) = (\mathfrak{e}_{6, \mathbb{C}}, \mathfrak{e}_{6(-26)})$. We denote the Satake diagram of $\mathfrak{e}_{6(-26)}$ by*



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