

Two-step Monte Carlo calculation method for sensitivity coefficients of reactor  
period with respect to kinetics parameters

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## Abstract

The calculation of the inverse reactor period ( $\alpha$ ), which is a fundamental mode eigenvalue of the  $\alpha$ -mode nonlinear Boltzmann eigenvalue equation, depends on the kinetics parameters (delayed neutron fractions, precursor decay constants, and delayed neutron spectra) used in the calculation. Recently, we developed a Monte Carlo method to calculate the derivatives of the  $k$ -eigenvalue with respect to  $\alpha$ . Here, the  $k$ -eigenvalue is not a critical eigenvalue; rather, it is a fictitious eigenvalue introduced to determine the  $\alpha$  value that satisfies the  $\alpha$ -mode nonlinear equation. The sensitivity coefficients of  $\alpha$  with respect to the kinetics parameters are expressed as the ratio of the two derivatives: the derivative of the  $k$ -eigenvalue with respect to the kinetics parameters and that with respect to  $\alpha$ . This study introduces a new step for calculating the derivatives of the  $k$ -eigenvalue with respect to kinetics parameters using the differential operator sampling method. The sensitivity coefficients obtained using the Monte Carlo method have been validated based on their close agreement with the reference solutions obtained using a deterministic method.

**Keywords:** Monte Carlo; reactor period; differential operator sampling; delayed neutron

## I. INTRODUCTION

In critical or near-critical reactors, reactor period measurements can be used to validate the calculation codes and nuclear data used for the calculation. By contrast, calculating the reactor period from the  $\alpha$ -mode Boltzmann transport eigenvalue equation is not necessarily straightforward due to the nonlinearity of the equations, which is particularly pronounced in Monte Carlo methods. In this eigenvalue equation,  $\alpha$  (the inverse reactor period) itself is not an eigenvalue; instead, a fictitious  $k$ -eigenvalue introduced artificially to satisfy the eigenvalue equation becomes the actual eigenvalue. When the  $\alpha$  value is identical to the inverse period, the  $k$ -eigenvalue of the equation becomes unity. Several Monte Carlo techniques have been developed to calculate  $\alpha$  by solving the  $\alpha$ -mode Boltzmann transport equation<sup>1</sup> such as the “weight-balancing” method<sup>2–4</sup>,  $\alpha$ - $k$  (or  $k$ - $\alpha$ ) method<sup>5</sup>, and the transition rate matrix method (TRMM).<sup>6–9</sup> TRMM is similar to the fission matrix method<sup>10</sup> for  $k$ -eigenvalues and requires discretization of space, direction, and energy, which consequently entails a large memory footprint for accurate results with minimized discretization error.

The Monte Carlo calculation method for the sensitivity coefficient of the *prompt* neutron decay constant  $\alpha$  with respect to the cross section used in the  $\alpha$ -mode eigenvalue equation, excluding delayed neutrons, has been proposed by us previously.<sup>11, 12</sup> The method in Ref. [12] employs a two-step approach wherein the derivative of the  $k$ -eigenvalue with respect to  $\alpha$  and that with respect to the cross section are calculated separately, and the sensitivity coefficient of  $\alpha$  with respect to the cross section is obtained from the ratio of the two derivatives. By adopting this method for the nonlinear  $\alpha$ -mode eigenvalue equation, which considers delayed neutrons, it is possible to determine the sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to nuclear data.

In Ref. [13], the authors proposed a new approach to calculate the inverse reactor period  $\alpha$ . In this method, the derivative of the  $k$ -eigenvalue with respect to  $\alpha$  is first calculated using the differential operator sampling (DOS) method.<sup>14–18</sup> This approach involves using the derivative to extrapolate the  $\alpha$  value so that the  $k$ -eigenvalue becomes unity, thereby determining the inverse period. Using this DOS method, the sensitivity coefficients of the  $k$ -eigenvalue with respect to  $\alpha$  and nuclear data can both be determined, which enables calculating the sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to nuclear data.

The method for calculating the sensitivity coefficient of the prompt neutron decay constant  $\alpha$  with respect to fission cross sections and other cross sections has already been presented in our previous work.<sup>11, 12</sup> The calculated reactor period depends not only on these cross sections, but also on kinetics parameters such as the delayed neutron fractions, precursor decay constants, and delayed neutron spectra. The aim of this study is to propose a new method for determining the sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to the kinetics parameters.

Sec. II revisits the fundamental equations of the  $\alpha$ -mode and the method for calculating the inverse reactor period using the Monte Carlo method, serving as the background of this study. Sec. III presents the calculation method for the sensitivity coefficients as a new feature of this study. Sec. IV presents examples of the applications of the proposed method. Sec. V presents concluding remarks.

## II. MONTE CARLO METHOD FOR REACTOR PERIOD

### II.A Fundamental Equations

In this section, we revisit the method of calculating the reactor period using the DOS presented in Ref. [13], which is essential for this study.

The neutron transport equation to be solved for determining the reactor period is as follows:

$$\frac{\alpha}{v_e} \phi_\alpha(\mathbf{r}, \mathbf{v}_e) + \mathbf{L} \phi_\alpha(\mathbf{r}, \mathbf{v}_e) = \frac{1}{k(\alpha)} \chi_\alpha(\mathbf{r}, \mathbf{v}_e, \alpha) \int v_\alpha(\mathbf{v}'_e, \alpha) \Sigma_f(\mathbf{r}, \mathbf{v}'_e) \phi_\alpha(\mathbf{r}, \mathbf{v}'_e) d\mathbf{v}'_e. \quad (1)$$

where

$$\mathbf{L} \phi_\alpha(\mathbf{r}, \mathbf{v}_e) = \boldsymbol{\Omega} \cdot \nabla \phi_\alpha(\mathbf{r}, \mathbf{v}_e) + \Sigma_t \phi_\alpha(\mathbf{r}, \mathbf{v}_e) - \int \Sigma_s(\mathbf{r}, \mathbf{v}'_e \rightarrow \mathbf{v}_e) \phi_\alpha(\mathbf{r}, \mathbf{v}'_e) d\mathbf{v}'_e, \quad (2)$$

$$v_\alpha(\mathbf{v}'_e, \alpha) = v(\mathbf{v}'_e) \left( 1 - \sum_i \beta_i \right) + \sum_i \frac{v(\mathbf{v}'_e) \beta_i \lambda_i}{\lambda_i + \alpha}, \quad (3)$$

$$\chi_\alpha(\mathbf{r}, \mathbf{v}_e, \alpha) = \chi_p(\mathbf{r}, \mathbf{v}_e) \left( 1 - \sum_i \frac{\beta_i \lambda_i}{\lambda_i + \alpha} \right) + \sum_i \chi_{d,i}(\mathbf{r}, \mathbf{v}_e) \frac{\beta_i \lambda_i}{\lambda_i + \alpha}. \quad (4)$$

The symbols used in this study are listed in the nomenclature section.  $k(\alpha)$ , which is the  $k$ -eigenvalue of Eq. (1), is exactly equal to unity if  $\alpha$  is identical to the inverse reactor period of the system and would apparently disappear from Eq. (1). Therefore,  $k(\alpha)$  is a fictitious eigenvalue introduced to balance the left and right sides of Eq. (1) when  $\alpha$  deviates from the true inverse reactor period.

Throughout this study, the kinetics parameters of delayed neutrons are treated as those of a single nuclide rather than being handled separately for each nuclide.

## II.B Inverse Reactor Period Calculation with DOS

The method for calculating the inverse reactor period  $\alpha$  using DOS, as outlined in Ref. [13], is as follows. Using the initial estimate of  $\alpha$ , denoted as  $\alpha_0$ , the eigenvalue  $k(\alpha_0)$  of Eq. (1) is calculated. This calculation is similar to that used to determine the criticality eigenvalue; however, the difference is that the first term on the left-hand side of Eq. (1) must be incorporated into the random walk process. For details of this calculation method, refer to Refs. [5] and [12].

Next, the derivative of  $k(\alpha)$  with respect to  $\alpha$  at  $\alpha_0$  is calculated using the DOS method. It is possible to calculate the derivatives of any order; however, the calculation is limited to the second order in Ref. [13]. Using the first- and second-order derivatives,  $\Delta\alpha$ , the change in  $\alpha$ , required for  $k(\alpha)$  to become unity can be estimated as follows:

$$\Delta\alpha = \frac{-k_{\alpha}^{(1)} - \sqrt{\left(k_{\alpha}^{(1)}\right)^2 - 2k_{\alpha}^{(2)}(k(\alpha_0) - 1)}}{k_{\alpha}^{(2)}}, \quad (5)$$

where

$$k_{\alpha}^{(n)} = \left. \frac{d^n k(\alpha)}{d\alpha^n} \right|_{\alpha=\alpha_0}, \quad n = 1 \text{ or } 2. \quad (6)$$

Using  $\Delta\alpha$  in Eq. (5), the inverse period that makes  $k(\alpha)$  equal to unity is estimated as follows:

$$\alpha' = \alpha_0 + \Delta\alpha. \quad (7)$$

To obtain a more accurate estimate of  $\alpha'$  using this method, it is desirable for the initial value of  $k(\alpha_0)$  to be as close to unity as possible and for the statistical uncertainty in  $(k(\alpha_0) - 1)$  to be sufficiently small. Refer to Ref. [13] for the method of calculating the derivatives of  $k_{\alpha}^{(1)}$  and  $k_{\alpha}^{(2)}$  using the DOS method.

## III. TWO-STEP METHOD FOR SENSITIVITY COEFFICIENT

### III.A. Derivative of $\alpha$ with respect to Nuclear Data

The two-step method for calculating the sensitivity coefficient of the prompt neutron decay constant  $\alpha$  with respect to the nuclear data is presented in Ref. [12]. Herein, we describe a method for determining the sensitivity coefficient of the inverse reactor period using the same two-step approach. Using the method described in the previous section, the inverse reactor

period  $\alpha$  is calculated by the DOS method. Then, the first-order derivative of  $k(\alpha)$ , as defined by Eq. (6), is calculated for this  $\alpha$ .

Next, by calculating the first-order derivative of  $k(\alpha)$  with respect to nuclear data  $x$  in Eq. (1) and using the DOS method presented in Ref. [12], the sensitivity coefficient of  $\alpha$  with respect to nuclear data  $x$  can be determined as follows:

$$\frac{d\alpha}{dx} = \frac{\frac{dk(\alpha)}{dx}}{\frac{dk(\alpha)}{d\alpha}}. \quad (8)$$

**Figure 1** illustrates the sequence of steps leading to Eq. (8). When delayed neutrons are neglected and  $k(\alpha)$  is set to unity, Eq. (1) can be regarded as an eigenvalue equation of the prompt neutron decay constant  $\alpha$ . Specifically, the equation takes the form  $\mathbf{A}\boldsymbol{\Phi} = \alpha/\mathbf{v}_e\boldsymbol{\Phi}$ , allowing the sensitivity coefficient  $d\alpha/dx$  to be determined directly without using the two-step method in Eq. (8).<sup>11</sup> The method that uses  $\alpha$  as the eigenvalue is referred to as the time source method. However, this method cannot be applied in this study because Eq. (1) is nonlinear in  $\alpha$ . In Ref. [12], the sensitivity coefficients were obtained using these two methods (two-step and time-source methods), and both showed consistent results.

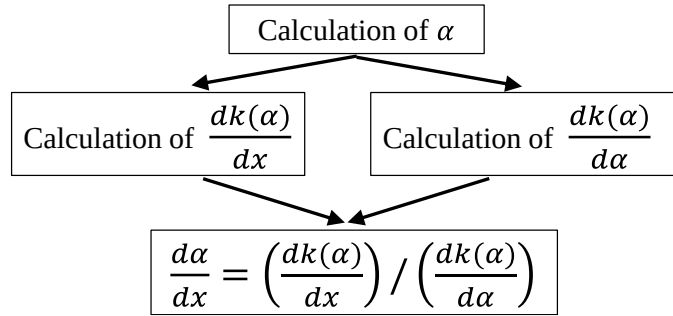


Figure 1 Flow of sensitivity coefficient of  $\alpha$  with respect to  $x$  (kinetics parameter)

### III.B. DOS method of Kinetics Parameters

In Refs. [12], a method for calculating the first-order derivative of the  $k$ -eigenvalue with respect to the cross section, such as the fission cross section, using the DOS method has already been presented. Because the focus of Ref. [12] was on the prompt neutron decay constant  $\alpha$ , the method did not include kinetics parameters related to delayed neutrons. This study presents the calculation method of the DOS method when  $x$  in  $dk(\alpha)/dx$  in Eq. (8) represents the kinetics parameter.

### III.B.1 Delayed Neutron Spectrum

In Monte Carlo eigenvalue calculations, the energy of the neutrons generated from the fission source is determined by the fission spectrum at the beginning of the particle history. In the  $\alpha$ -mode calculation of Eq. (1), the energy of the starting particle is determined using Eq. (4). In the DOS method, the following weighting coefficient, derived by differentiating Eq. (4) with respect to the delayed neutron spectrum of family  $i$ ,  $\chi_{d,i}$ , is assigned to the random walk particles from the beginning of the history until the particle history is terminated:

$$W_{\chi_{d,i}} = \frac{1}{\chi_{\alpha}(v_e)} \frac{d\chi_{\alpha}(v_e)}{d\chi_{d,i}} = \frac{1}{\chi_{\alpha}(v_e)} \frac{\beta_i \lambda_i}{\lambda_i + \alpha}, \quad (9)$$

where  $v_e$  is the velocity (energy) of neutrons emitted from the fission source. Hereafter, variables such as  $\mathbf{r}$  and  $\alpha$ , which are included in the kinetics parameters, are omitted.

At the end of the  $m$ -th particle history, the first derivative of  $k(\alpha)$  with respect to  $\chi_{d,i}$  is calculated as follows:

$$k_{\chi_{d,i},m} = \frac{dk(\alpha)}{d\chi_{d,i}} = \sum_k v_{\alpha} \frac{\Sigma_{f,k}}{\Sigma_{t,k}} w_k W_{\chi_{d,i}}, \quad (10)$$

where  $k$  denotes the  $k$ -th collision within the  $m$ -th history,  $w_k$  is the particle weight at the  $k$ -th collision, and the summation is performed for every collision until the end of the  $m$ -th history.

Once all histories in one cycle are calculated, the first-order derivative for that cycle is determined as follows:

$$k_{\chi_{d,i}} = \frac{1}{M} \sum_{m=1}^M k_{\chi_{d,i},m}, \quad (11)$$

where  $M$  is the number of fission sources in the cycle. The final result of  $k_{\chi_{d,i}}$  is obtained by averaging them over the active cycles after discarding the inactive cycles before convergence.

A change in the delayed neutron spectrum causes perturbations in the fission source distribution. This perturbation effect is not accounted for in Eq. (11). To estimate the derivative accurately, it is essential to consider the source perturbation effect. The method for incorporating the source perturbation effect into the DOS method is detailed in Refs. [11, 17, 19–21] and was applied in this study. Owing to the complexity of the method, details of the source perturbation effect are not provided here; however, readers can refer to Refs. [17, 19, 20] for further information.

### III.B.2 Delayed Neutron Fraction

As shown in Eqs. (3) and (4), the delayed neutron fraction  $\beta_i$  affects both the number of neutrons produced per fission and the neutron spectrum. The weighting coefficients, similar to those in Eq. (9) for calculating the derivative with respect to the delayed neutron fraction of family  $i$ ,  $\beta_i$ , are as follows:

$$W_{\chi, \beta_i} = \frac{1}{\chi_\alpha(v_e)} \frac{d\chi_\alpha(v_e)}{d\beta_i} = \frac{1}{\chi_\alpha(v_e)} (-\chi_p(v_e) + \chi_{d,i}(v_e)) \frac{\lambda_i}{\lambda_i + \alpha}, \quad (12)$$

$$W_{v, \beta_i} = \frac{1}{v_\alpha(v_e)} \frac{dv_\alpha(v_e)}{d\beta_i} = \frac{v(v_e)}{v_\alpha(v_e)} \left( -1 + \frac{\lambda_i}{\lambda_i + \alpha} \right). \quad (13)$$

Similar to the process in Sec. III.B.1,  $W_{\chi, \beta_i}$  in Eq. (12) is assigned to the particles with velocity  $v_e$  emitted from the fission source. Moreover,  $W_{v, \beta_i}$  in Eq. (13) is scored every time a particle collides with a fissionable material. At the end of the  $m$ -th particle history, the first derivative of  $k(\alpha)$  with respect to  $\beta_i$  is calculated as follows:

$$k_{\beta_i, m} = \frac{dk(\alpha)}{d\beta_i} = \sum_k v_\alpha \frac{\Sigma_{f,k}}{\Sigma_{t,k}} w_k (W_{\chi, \beta_i} + W_{v, \beta_i}). \quad (14)$$

Similar to Eq. (10),  $k$  denotes the  $k$ -th collision within the  $m$ -th history,  $w_k$  is the particle weight at the  $k$ -th collision, and a summation is performed for each collision until the end of the  $m$ -th history.

### III.B.3 Precursor Decay Constant

The decay constant of precursor  $\lambda_i$  also affects both the number of neutrons produced per fission and the neutron spectrum. The weighting coefficients for calculating the derivative with respect to the precursor decay constant of family  $i$ ,  $\lambda_i$ , are as follows:

$$W_{\chi, \lambda_i} = \frac{1}{\chi_\alpha(v_e)} \frac{d\chi_\alpha(v_e)}{d\lambda_i} = \frac{1}{\chi_\alpha(v_e)} (-\chi_p(v_e) + \chi_{d,i}(v_e)) \frac{\beta_i \alpha}{(\lambda_i + \alpha)^2}, \quad (15)$$

$$W_{v, \lambda_i} = \frac{1}{v_\alpha(v_e)} \frac{dv_\alpha(v_e)}{d\lambda_i} = \frac{v(v_e)}{v_\alpha(v_e)} \frac{\beta_i \alpha}{(\lambda_i + \alpha)^2}. \quad (16)$$

Similar to the process in Sec. III.B.2, the first derivative of  $k(\alpha)$  with respect to  $\lambda_i$  for the  $m$ -th history is also calculated as follows:

$$k_{\lambda_i, m} = \frac{dk(\alpha)}{d\lambda_i} = \sum_k v_\alpha \frac{\Sigma_{f,k}}{\Sigma_{t,k}} w_k (W_{\chi, \lambda_i} + W_{v, \lambda_i}). \quad (17)$$

The derivatives of the  $k$ -eigenvalue with respect to the delayed neutron fraction and precursor decay constant in one cycle can be obtained in the same manner as in Eq. (11).

## IV. NUMERICAL TESTS

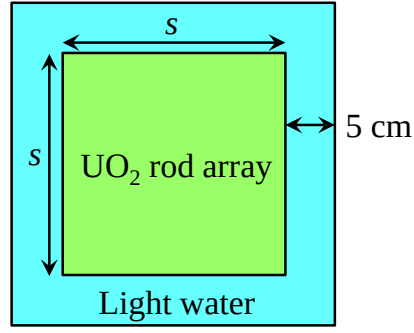
### IV.A. Overview of Test Problems

In this section, we present an example of Monte Carlo calculations for the sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to kinetics parameters and validate the results by comparing them with those obtained using deterministic methods. The test problem used in this study is shown below and is the same as that in Ref. [13].

The test problem comprised a homogenized  $\text{UO}_2$  fuel rod array with a side length of 23.05 cm, surrounded by a 5-cm thick light water reflector, as shown in **Figure 2**. Calculations were performed using three energy group constants and six family delayed neutron kinetics parameters. The group constants and kinetics parameters were prepared using the standard reactor analysis code.<sup>22</sup> Tables I and II list the group constants for the homogenized fuel rod array and light water reflector, respectively. The scattering was assumed to be isotropic. Tables III and IV present the spectra of the prompt and delayed neutrons and their kinetics parameters. The nuclear data used for this test problem were based on the nuclear data library JENDL-4.<sup>23</sup> The Monte Carlo calculations were performed using an in-house solver developed by the authors.

The test problem was slightly supercritical, with an effective multiplication factor of  $1.001769 \pm 0.000006$ . The inverse reactor period  $\alpha$ , calculated using the method from Ref. [13], is  $0.03729 \pm 0.00010 \text{ s}^{-1}$ . The  $k$ -eigenvalue,  $k(\alpha)$ , calculated using this  $\alpha$ , which is assumed to be exactly unity, was  $1.0000006 \pm 0.0000042$ .

To calculate the derivative of the inverse reactor period  $\alpha$  with respect to the kinetics parameter,  $dk(\alpha)/d\alpha$  is required, as shown in the denominator of Eq. (8). Moreover,  $dk(\alpha)/d\alpha$  was calculated using the DOS method presented in Ref. [13], yielding a value of  $-0.027659 \pm 0.000006 \text{ s}$ , which was used for the sensitivity coefficient calculation.



$$s = 23.05 \text{ cm}$$

Figure 2 Geometry of the test problem (not to scale).

Table I Three-group constants of the UO<sub>2</sub> rod array

|                                         | 1st group                | 2nd group                | 3rd group |
|-----------------------------------------|--------------------------|--------------------------|-----------|
| $\Sigma_t (cm^{-1})$                    | 0.298289                 | 0.833341                 | 1.63895   |
| $\Sigma_c (cm^{-1})$                    | $3.74414 \times 10^{-4}$ | $9.77075 \times 10^{-3}$ | 0.0301365 |
| $\Sigma_f (cm^{-1})$                    | $3.01089 \times 10^{-3}$ | $2.12425 \times 10^{-3}$ | 0.0560435 |
| $\nu$                                   | 2.438                    | 2.438                    | 2.438     |
| $\chi_p$                                | 0.878198                 | 0.121802                 | 0.0       |
| $\Sigma_{sg \rightarrow g} (cm^{-1})$   | 0.221061                 | 0.777643                 | 1.55277   |
| $\Sigma_{sg \rightarrow g+1} (cm^{-1})$ | 0.0738427                | 0.043803                 | 0.0       |

Table II Three-group constants of light water

|                                         | 1st group               | 2nd group               | 3rd group |
|-----------------------------------------|-------------------------|-------------------------|-----------|
| $\Sigma_t (cm^{-1})$                    | 0.332073                | 1.12647                 | 2.78124   |
| $\Sigma_c (cm^{-1})$                    | $3.0500 \times 10^{-4}$ | $3.6990 \times 10^{-4}$ | 0.018250  |
| $\Sigma_{sg \rightarrow g} (cm^{-1})$   | 0.227126                | 1.02814                 | 2.76278   |
| $\Sigma_{sg \rightarrow g+1} (cm^{-1})$ | 0.104642                | 0.0979597               | 0.0       |

Table III Neutron velocity and prompt and delayed neutron spectra

|              | 1st group             | 2nd group             | 3rd group             |
|--------------|-----------------------|-----------------------|-----------------------|
| $v_e (cm/s)$ | $1.66743 \times 10^9$ | $1.73734 \times 10^7$ | $2.78124 \times 10^5$ |
| $\chi_p$     | 0.878198              | 0.121802              | 0.0                   |

|              |          |          |     |
|--------------|----------|----------|-----|
| $\chi_{d,1}$ | 0.302845 | 0.697155 | 0.0 |
| $\chi_{d,2}$ | 0.390439 | 0.609561 | 0.0 |
| $\chi_{d,3}$ | 0.374032 | 0.625968 | 0.0 |
| $\chi_{d,4}$ | 0.457453 | 0.542547 | 0.0 |
| $\chi_{d,5}$ | 0.396728 | 0.603272 | 0.0 |
| $\chi_{d,6}$ | 0.411100 | 0.588900 | 0.0 |

Table IV Delayed neutron fractions and decay constants

| Family | $\beta_i$                 | $\lambda_i(s^{-1})$        |
|--------|---------------------------|----------------------------|
| 1      | $2.158803 \times 10^{-4}$ | $1.2485124 \times 10^{-2}$ |
| 2      | $1.466645 \times 10^{-3}$ | $3.0661454 \times 10^{-2}$ |
| 3      | $1.359992 \times 10^{-3}$ | $1.139355 \times 10^{-1}$  |
| 4      | $2.771604 \times 10^{-3}$ | $3.074786 \times 10^{-1}$  |
| 5      | $9.151014 \times 10^{-4}$ | 1.188787                   |
| 6      | $3.247986 \times 10^{-4}$ | 3.176650                   |

#### IV.B. Sensitivity Coefficients with respect to Delayed Neutron Spectrum

By using the method presented in Sec. III.B.1, the sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to the delayed neutron spectrum was determined using the Monte Carlo method. The sensitivity coefficient was defined as follows:

$$S_{\alpha, \chi_{d,i,g}} = \frac{\chi_{d,i,g}}{\alpha} \cdot \frac{d\alpha}{d\chi_{d,i,g}} = \frac{\chi_{d,i,g}}{\alpha} \cdot \frac{\frac{dk(\alpha)}{d\chi_{d,i,g}}}{\frac{dk(\alpha)}{d\alpha}}. \quad (18)$$

Monte Carlo calculations were performed with 200,000 histories per cycle, and the first 20 cycles were discarded. The total number of cycles was selected to ensure sufficient statistical accuracy for each case. The number of latent cycles for the perturbed source effect was set to ten. This number was considered sufficiently large to achieve converged solutions for the perturbed source effect.<sup>17</sup>

Reference calculations for the sensitivity coefficients were performed with the discrete ordinates transport calculation code PARTISN<sup>24</sup> using the group constants listed in Tables I–IV. The three-energy group forward and adjoint fluxes of  $k$ -eigenvalue mode were calculated with an  $S_n$  order of 8. Using these fluxes, the derivative of the denominator in Eq. (18) is

determined using a deterministic method as follows:

$$\frac{dk(\alpha)}{d\alpha} = \frac{F_\alpha}{F}. \quad (19)$$

The numerators and denominators in Eq. (19) are, respectively, defined as follows:

$$F = \int_V \sum_{g=1}^G \psi_g(r) \chi_{\alpha,g} \sum_{g=1}^G v_{\alpha} \Sigma_{f,g} \phi_g(r) dr, \quad (20)$$

$$F_\alpha = \frac{dF}{d\alpha} = \int_V \sum_{g=1}^G \psi_g(r) \frac{d\chi_{\alpha,g}}{d\alpha} \sum_{g=1}^G v_{\alpha} \Sigma_{f,g} \phi_g(r) dr + \int_V \sum_{g=1}^G \psi_g(r) \chi_{\alpha,g} \sum_{g=1}^G \frac{dv_{\alpha}}{d\alpha} \Sigma_{f,g} \phi_g(r) dr, \quad (21)$$

where

$$\frac{d\chi_{\alpha,g}}{d\alpha} = \chi_{p,g} \sum_i \frac{\beta_i \lambda_i}{(\lambda_i + \alpha)^2} - \sum_i \chi_{d,i,g} \frac{\beta_i \lambda_i}{(\lambda_i + \alpha)^2}, \quad (22)$$

$$\frac{dv_{\alpha}}{d\alpha} = -v \sum_i \frac{\beta_i \lambda_i}{(\lambda_i + \alpha)^2}. \quad (23)$$

Space integration was performed over the entire domain. Table V shows a comparison of the calculation results for  $dk(\alpha)/d\alpha$  at  $\alpha = 0.03729 \text{ s}^{-1}$  using the Monte Carlo method and deterministic method. The results of the Monte Carlo and deterministic methods agreed with a high precision, with a difference of 0.1%.

Table V  $dk(\alpha)/d\alpha$  by Monte Carlo and deterministic methods

|                                  | Monte Carlo              | PARTISN   | $ M/P^{**}-1  \times 100$ |
|----------------------------------|--------------------------|-----------|---------------------------|
| $dk(\alpha)/d\alpha \text{ (s)}$ | -0.027659<br>(0.000006)* | -0.027630 | 0.10                      |

\* Values in parentheses indicate one standard deviation.

\*\*M/P represents the ratio of the Monte Carlo method to the deterministic method.

The derivative of the numerator in Eq. (18) is determined using a deterministic method as follows:

$$\frac{dk(\alpha)}{d\chi_{d,i,g}} = \frac{F_{\chi_{d,i,g}}}{F}. \quad (24)$$

Thus,  $F$  in the denominator cancels out between the numerator and denominator in Eq. (18), which makes it unnecessary to calculate  $F$ . The numerator in Eq. (24) is defined as follows:

$$F_{\chi_{d,i,g}} = \frac{dF}{d\chi_{d,i,g}} = \int_V \psi_g(r) \frac{\beta_i \lambda_i}{\lambda_i + \alpha} \sum_{g=1}^G v_{\alpha} \Sigma_{f,g} \phi_g(r) dr. \quad (25)$$

Using Eqs. (19) and (24), the sensitivity coefficients defined in Eq. (18) can be obtained using a deterministic method. Table VI presents the sensitivity coefficients defined by Eq. (18), calculated using the Monte Carlo and deterministic methods. The results of the Monte Carlo method for all families and energy groups were within 0.4% of those obtained using the deterministic method, confirming the validity of the Monte Carlo method.

Table VI Sensitivity coefficient of  $\alpha$  with respect to delayed neutron spectrum

| Delayed neutron family | Energy group | Monte Carlo                                                   | PARTISN                 | $ M/P^{**}-1 \times 100$ |
|------------------------|--------------|---------------------------------------------------------------|-------------------------|--------------------------|
| 1                      | 1            | $-1.5560\times 10^{-2}$<br>( $5\times 10^{-6}$ ) <sup>*</sup> | $-1.5586\times 10^{-2}$ | 0.17                     |
|                        | 2            | $-4.1887\times 10^{-2}$<br>( $6.8\times 10^{-5}$ )            | $-4.1949\times 10^{-2}$ | 0.15                     |
| 2                      | 1            | $-2.4517\times 10^{-1}$<br>( $9\times 10^{-5}$ )              | $-2.4559\times 10^{-1}$ | 0.17                     |
|                        | 2            | $-4.4812\times 10^{-1}$<br>( $7.2\times 10^{-4}$ )            | $-4.4827\times 10^{-1}$ | 0.03                     |
| 3                      | 1            | $-3.6335\times 10^{-1}$<br>( $1.4\times 10^{-4}$ )            | $-3.6427\times 10^{-1}$ | 0.25                     |
|                        | 2            | $-7.1398\times 10^{-1}$<br>( $1.30\times 10^{-3}$ )           | $-7.1274\times 10^{-1}$ | 0.17                     |
| 4                      | 1            | $-1.0727$<br>( $4\times 10^{-4}$ )                            | $-1.0747$               | 0.19                     |
|                        | 2            | $-1.4849$<br>( $2.7\times 10^{-3}$ )                          | $-1.4903$               | 0.36                     |
| 5                      | 1            | $-3.3389\times 10^{-1}$<br>( $1.2\times 10^{-4}$ )            | $-3.3457\times 10^{-1}$ | 0.21                     |
|                        | 2            | $-5.9340\times 10^{-1}$<br>( $1.05\times 10^{-3}$ )           | $-5.9481\times 10^{-1}$ | 0.24                     |
| 6                      | 1            | $-1.2515\times 10^{-1}$<br>( $4\times 10^{-5}$ )              | $-1.2544\times 10^{-1}$ | 0.23                     |
|                        | 2            | $-2.1015\times 10^{-1}$<br>( $3.3\times 10^{-4}$ )            | $-2.1009\times 10^{-1}$ | 0.03                     |

<sup>\*</sup> Values in parentheses indicate standard deviations.

<sup>\*\*</sup> M/P represents the ratio of the Monte Carlo method to the deterministic method.

#### IV.C. Sensitivity Coefficients with respect to Delayed Neutron Fraction

The sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to the delayed neutron fraction is defined as follows:

$$S_{\alpha, \beta_i} = \frac{\beta_i}{\alpha} \cdot \frac{d\alpha}{d\beta_i} = \frac{\beta_i}{\alpha} \cdot \frac{\frac{dk(\alpha)}{d\beta_i}}{\frac{dk(\alpha)}{d\alpha}}. \quad (26)$$

The derivative of the numerator in Eq. (26) can be obtained using Eqs. (12)–(14). In the deterministic method, the derivative is obtained as follows:

$$\frac{dk(\alpha)}{d\beta_i} = \frac{F_{\beta_i}}{F}. \quad (27)$$

$F$  is defined in Eq. (20) and  $F_{\beta_i}$  is defined as follows:

$$F_{\beta_i} = \frac{dF}{d\beta_i} = \int_V \sum_{g=1}^G \psi_g(r) \frac{d\chi_{\alpha,g}}{d\beta_i} \sum_{g=1}^G \nu_{\alpha} \Sigma_{f,g} \phi_g(r) dr + \int_V \sum_{g=1}^G \psi_g(r) \chi_{\alpha,g} \sum_{g=1}^G \frac{d\nu_{\alpha}}{d\beta_i} \Sigma_{f,g} \phi_g(r) dr, \quad (28)$$

where

$$\frac{d\chi_{\alpha,g}}{d\beta_i} = (-\chi_{p,g} + \chi_{d,i,g}) \frac{\lambda_i}{\lambda_i + \alpha}, \quad (29)$$

$$\frac{d\nu_{\alpha}}{d\beta_i} = \nu \left( -1 + \frac{\lambda_i}{\lambda_i + \alpha} \right). \quad (30)$$

Table VII shows the results of the sensitivity coefficients with respect to  $\beta_i$  using the Monte Carlo method with Eqs. (12)–(14) and deterministic method with Eqs. (27)–(30). The sensitivity coefficients of  $\alpha$  with respect to the delayed neutron fraction obtained using the Monte Carlo method were in good agreement with those obtained by the deterministic method, with an accuracy within 0.4%.

Table VII Sensitivity coefficient of  $\alpha$  with respect to the delayed neutron fraction

| Delayed neutron family | Monte Carlo                                          | PARTISN                 | $ M/P^{**}-1 \times 100$ |
|------------------------|------------------------------------------------------|-------------------------|--------------------------|
| 1                      | $1.5186 \times 10^{-1}$<br>( $1.0 \times 10^{-4}$ )* | $1.5220 \times 10^{-1}$ | 0.22                     |
| 2                      | $7.2794 \times 10^{-1}$<br>( $1.01 \times 10^{-3}$ ) | $7.3045 \times 10^{-1}$ | 0.35                     |
| 3                      | $2.4276 \times 10^{-1}$<br>( $6.2 \times 10^{-4}$ )  | $2.4293 \times 10^{-1}$ | 0.07                     |
| 4                      | $1.2402 \times 10^{-1}$<br>( $9.0 \times 10^{-4}$ )  | $1.2419 \times 10^{-1}$ | 0.14                     |

|   |                                                       |                          |      |
|---|-------------------------------------------------------|--------------------------|------|
| 5 | $-4.1732 \times 10^{-2}$<br>( $3.40 \times 10^{-4}$ ) | $-4.1629 \times 10^{-2}$ | 0.25 |
| 6 | $-2.0436 \times 10^{-2}$<br>( $1.15 \times 10^{-4}$ ) | $-2.0446 \times 10^{-2}$ | 0.05 |

\* Values in parentheses indicate standard deviations.

\*\*M/P represents the ratio of the Monte Carlo method to the deterministic method.

#### IV.D. Sensitivity Coefficients with respect to the Precursor Decay Constant

The sensitivity coefficient of the inverse reactor period  $\alpha$  with respect to the precursor decay constant can be determined by the following equation:

$$S_{\alpha, \lambda_i} = \frac{\lambda_i}{\alpha} \cdot \frac{d\alpha}{d\lambda_i} = \frac{\lambda_i}{\alpha} \cdot \frac{\frac{dk(\alpha)}{d\lambda_i}}{\frac{dk(\alpha)}{d\alpha}}. \quad (31)$$

The derivative of the numerator in Eq. (31),  $dk(\alpha)/d\lambda_i$ , is obtained using Eqs. (15)–(17). In the deterministic method, the derivative is determined using the following equation:

$$\frac{dk(\alpha)}{d\lambda_i} = \frac{F_{\lambda_i}}{F}. \quad (32)$$

$F_{\lambda_i}$  is defined as follows:

$$\begin{aligned} F_{\lambda_i} = \frac{dF}{d\lambda_i} = & \int_V \sum_{g=1}^G \psi_g(r) \frac{d\chi_{\alpha,g}}{d\lambda_i} \sum_{g=1}^G v_{\alpha} \Sigma_{f,g} \phi_g(r) dr \\ & + \int_V \sum_{g=1}^G \psi_g(r) \chi_{\alpha,g} \sum_{g=1}^G \frac{dv_{\alpha}}{d\lambda_i} \Sigma_{f,g} \phi_g(r) dr, \end{aligned} \quad (33)$$

where

$$\frac{d\chi_{\alpha,g}}{d\lambda_i} = (-\chi_{p,g} + \chi_{d,i,g}) \frac{\beta_i \alpha}{(\lambda_i + \alpha)^2}, \quad (34)$$

$$\frac{dv_{\alpha}}{d\lambda_i} = v \frac{\beta_i \alpha}{(\lambda_i + \alpha)^2}. \quad (35)$$

Table VIII shows the calculation results for the sensitivity coefficients with respect to the precursor decay constant for each family obtained using the Monte Carlo and deterministic methods. The sensitivity coefficients of  $\alpha$  with respect to the precursor decay constant obtained using the Monte Carlo method are within 0.6% of those derived from the deterministic method for all delayed neutron families. Therefore, it can be concluded that the sensitivity coefficient calculation method using the Monte Carlo method is validated.

1 Table VIII Sensitivity coefficient of  $\alpha$  with respect to the precursor decay constant

| Family | Monte Carlo                                         | PARTISN                 | $ M/P^{**}-1 \times 100$ |
|--------|-----------------------------------------------------|-------------------------|--------------------------|
| 1      | $-4.3019\times 10^{-2}$<br>( $6.5\times 10^{-5}$ )* | $-4.3185\times 10^{-2}$ | 0.39                     |
| 2      | $-3.7933\times 10^{-1}$<br>( $4.9\times 10^{-4}$ )  | $-3.8150\times 10^{-1}$ | 0.57                     |
| 3      | $-2.6541\times 10^{-1}$<br>( $3.4\times 10^{-4}$ )  | $-2.6608\times 10^{-1}$ | 0.25                     |
| 4      | $-2.8006\times 10^{-1}$<br>( $3.4\times 10^{-4}$ )  | $-2.8025\times 10^{-1}$ | 0.07                     |
| 5      | $-2.8302\times 10^{-2}$<br>( $3.8\times 10^{-5}$ )  | $-2.8321\times 10^{-2}$ | 0.07                     |
| 6      | $-3.8899\times 10^{-3}$<br>( $4.5\times 10^{-6}$ )  | $-3.9005\times 10^{-3}$ | 0.27                     |

2 \*Values in parentheses indicate one standard deviation.

3 \*\*M/P represents the ratio of the Monte Carlo method to the deterministic method.

4

## 5 Nomenclature

|                                         |                                                                            |
|-----------------------------------------|----------------------------------------------------------------------------|
| $\alpha$                                | fundamental mode inverse reactor period                                    |
| $\mathbf{v}_e$ or $v_e$                 | neutron velocity                                                           |
| $\mathbf{r}$                            | position vector                                                            |
| $\lambda_i$                             | decay constant of the precursor family $i$                                 |
| $\boldsymbol{\Omega}$                   | angular direction vector                                                   |
| $\Sigma_t$                              | total cross section                                                        |
| $\Sigma_s$                              | scattering cross section                                                   |
| $\Sigma_f$                              | fission cross section                                                      |
| $\chi_p$                                | fission spectrum of prompt neutrons                                        |
| $\chi_{d,i}$                            | fission spectrum of delayed neutrons from the precursor family $i$         |
| $\nu$                                   | average number of neutrons per fission (prompt and delayed)                |
| $\nu_p$                                 | average number of prompt neutrons per fission                              |
| $\nu_{d,i}$                             | average number of delayed neutrons of the precursor family $i$ per fission |
| $\beta_i$                               | delayed neutron fraction of the precursor family $i$                       |
| $\phi_\alpha(\mathbf{r}, \mathbf{v}_e)$ | fundamental mode neutron flux in $\alpha$ mode                             |

|          |                                 |
|----------|---------------------------------|
| $\phi_g$ | neutron flux for energy group g |
| $\psi_g$ | adjoint flux for energy group g |
| $G$      | number of total energy groups   |

## V. CONCLUSIONS

Various nuclear data, including kinetics parameters ( $\beta_i, \chi_{d,i}, \lambda_i$ ) for delayed neutrons, are used in the calculation of the inverse reactor period  $\alpha$ . This study proposes a Monte Carlo method for calculating the sensitivity coefficients of  $\alpha$  with respect to these kinetics parameters. A two-step method is used based on the characteristic that the eigenvalue of the  $\alpha$ -mode neutron transport equation is  $k(\alpha)$ . When  $\alpha$  is exactly the inverse reactor period, the eigenvalue  $k(\alpha)$  is exactly equal to unity and does not appear explicitly in the equation. The sensitivity coefficient of  $\alpha$  with respect to the kinetics parameters is obtained by the ratio of the two derivatives of  $k(\alpha)$  with respect to  $\alpha$  and the kinetics parameters. The derivatives are determined using the DOS method.

The calculation method for the sensitivity coefficients of  $k(\alpha)$  with respect to  $\alpha$  using the DOS method was already derived in a previous paper. In this study, we derived the calculation formula for the derivatives of  $k(\alpha)$  with respect to the kinetics parameters using both the DOS method and a deterministic method. The test calculation results using the three-group constants showed good agreement between the sensitivity coefficients obtained using the Monte Carlo method and the reference solution via a deterministic approach, demonstrating the validity of this method.

## Disclosure statement

The authors report that there are no competing interests to declare.

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### List of figures

Figure 1 Flow of sensitivity coefficient of  $\alpha$  with respect to  $x$  (kinetics parameter)

Figure 2 Geometry of the test problem (not to scale)