

Preduals of Morrey-Campanato spaces

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1. INTRODUCTION

This is an announcement of my recent works.

Let $X = (X, d, \mu)$ be a space of homogeneous type, i.e. X is a topological space endowed with a quasi-distance d and a nonnegative measure μ such that

$$d(x, y) \geq 0 \quad \text{and} \quad d(x, y) = 0 \text{ if and only if } x = y,$$

$$d(x, y) = d(y, x),$$

$$(1.1) \quad d(x, y) \leq K_1 (d(x, z) + d(z, y)),$$

the balls (d -balls) $B(x, r) = B^d(x, r) = \{y \in X : d(x, y) < r\}$, $r > 0$, form a basis of neighborhoods of the point x , μ is defined on a σ -algebra of subsets of X which contains the balls, and

$$(1.2) \quad 0 < \mu(B(x, 2r)) \leq K_2 \mu(B(x, r)) < \infty,$$

where $K_i \geq 1$ ($i = 1, 2$) are constants independent of $x, y, z \in X$ and $r > 0$.

We note that every open subset of X is expressible as a countable union of balls (see [4], p.70), and so measurable.

If there are constants θ ($0 < \theta \leq 1$) and $K_3 \geq 1$ such that

$$(1.3) \quad |d(x, z) - d(y, z)| \leq K_3 (d(x, z) + d(y, z))^{1-\theta} d(x, y)^\theta, \quad x, y, z \in X,$$

then the balls are open sets. Note that (1.1) for some $K_1 \geq 1$ follows from (1.3) (Lemarié [12]). Conversely, from (1.1) it follows that there exist $\theta > 0$, $K_3 \geq 1$ and a quasi-distance which is equivalent to the original d such that (1.3) holds (Macías and Segovia [14]).

Using atoms, Coifman and Weiss [5] defined the Hardy space $H^p(X)$ as a subspace of the dual of $\text{Lip}_\alpha(X)$ and they proved that $\text{Lip}_\alpha(X)$ is the dual of $H^p(X)$. Their

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results are generalization of the case $X = \mathbb{R}^n$. In [5] $\text{Lip}_\alpha(X)$ was regarded the space of functions modulo constants. Therefore, we denote the fact above by

$$(H^p(X))^* = \text{Lip}_\alpha(X)/\mathcal{C},$$

where $\mathcal{C}$ is the space of all constant functions. Let $\mathcal{L}_{p,\phi}(X)$ be the Campanato space which is a generalization of $\text{Lip}_\alpha(X)$. In this paper we define a generalized Hardy space $H_U^{[\phi,q]}(X)$ as a subspace of the dual of $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ and prove that $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ is the dual of $H_U^{[\phi,q]}(X)$, i.e.

$$\left(H_U^{[\phi,q]}(X)\right)^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C},$$

where $1/q + 1/q' = 1$. The definition of $H^p(X)$ in [5] is a special case of ours. We note that the predual of $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ is not unique. Zorko [31] defined another predual of $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$ in the case $X = \mathbb{R}^n$. Our definition is a generalization of both definitions.

We also define a space $B_U^{\Phi,q}(X)$ generated by blocks ("block" means an atom without the cancellation property), and prove that the dual of $B_U^{\Phi,q}(X)$ is a Morrey space $L_{p,\phi}(X)$. This is

$$\left(B_U^{\Phi,q}(X)\right)^* = L_{p,\phi}(X).$$

This result is a generalization of Long [13] (1984).

It is known that $\mathcal{L}_{p,\phi}(X)/\mathcal{C} = L_{p,\phi}(X)$ under a certain condition (Nakai [24] (2006)). We show that $H_U^{\Phi,q}(X) = B_U^{\Phi,q}(X)$ under the correspondent condition.

2. NOTATIONS AND DEFINITIONS

Let (X, d, μ) be a space of homogeneous type satisfying (1.3).

Let $1 \leq p < \infty$ and $\phi : X \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$. For a ball $B = B(x, r)$, we shall write $\phi(B)$ in place of $\phi(x, r)$. For a function $f \in L_{\text{loc}}^1(X)$ and for a ball B , let $f_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$. Then the Campanato spaces $\mathcal{L}_{p,\phi}(X)$, the Morrey spaces $L_{p,\phi}(X)$ and the Hölder spaces $\Lambda_\phi(X)$ are defined to be the sets of all f such that $\|f\|_{\mathcal{L}_{p,\phi}} < \infty$, $\|f\|_{L_{p,\phi}} < \infty$ and $\|f\|_{\Lambda_\phi} < \infty$, respectively, where

$$\begin{aligned} \|f\|_{\mathcal{L}_{p,\phi}} &= \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x) - f_B|^p d\mu(x) \right)^{1/p}, \\ \|f\|_{L_{p,\phi}} &= \sup_B \frac{1}{\phi(B)} \left(\frac{1}{\mu(B)} \int_B |f(x)|^p d\mu(x) \right)^{1/p}, \\ \|f\|_{\Lambda_\phi} &= \sup_{x,y \in X, x \neq y} \frac{2|f(x) - f(y)|}{\phi(x, d(x, y)) + \phi(y, d(y, x))}. \end{aligned}$$

Then $\mathcal{L}_{p,\phi}(X)/\mathcal{C}$, $L_{p,\phi}(X)$ and $\Lambda_\phi(X)/\mathcal{C}$ are Banach spaces with the norm $\|f\|_{\mathcal{L}_{p,\phi}}$, $\|f\|_{L_{p,\phi}}$ and $\|f\|_{\Lambda_\phi}$, respectively.

If $\phi(x, r) = r^\alpha$ ($\alpha > 0$), $\Lambda_{r^\alpha}(X) = \text{Lip}_\alpha(X)$. If $p = 1$, then $\mathcal{L}_{1,\phi}(X) = \text{BMO}_\phi(X)$. If $\phi \equiv 1$, then $\mathcal{L}_{1,\phi}(X) = \text{BMO}(X)$ and $\Lambda_\phi(X) = L^\infty(X)$. If $\phi(B) = \mu(B)^{-1/p}$, then $L_{p,\phi}(X) = L^p(X)$.

If $X = \mathbb{R}^n$, $d(x, y) = |x - y|$, μ is Lebesgue measure and $\phi(x, r) = r^\alpha$, then the following are known (Campanato, Mayers, Peetre, Spanne, Janson);

$$-n/p \leq \alpha < 0 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n)/\mathcal{C} = L_{p,\phi}(\mathbb{R}^n) (= L^p(\mathbb{R}^n) \text{ if } \alpha = -n/p),$$

$$\alpha = 0 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \text{BMO}(\mathbb{R}^n) \supset L_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = L^\infty(\mathbb{R}^n),$$

$$0 < \alpha \leq 1 \Rightarrow \mathcal{L}_{p,\phi}(\mathbb{R}^n) = \Lambda_\phi(\mathbb{R}^n) = \text{Lip}_\alpha(\mathbb{R}^n).$$

The relations above were generalized to spaces of homogeneous type by Macías and Segovia [14] (1979) and Nakai [24] (2006).

For functions $\tau, \kappa : (0, +\infty) \rightarrow (0, +\infty)$, we denote $\tau(r) \sim \kappa(r)$ if there exists a constant $C > 0$ such that

$$C^{-1}\tau(r) \leq \kappa(r) \leq C\tau(r) \quad \text{for } r > 0.$$

A function $\tau : (0, +\infty) \rightarrow (0, +\infty)$ is said to be almost increasing (almost decreasing) if there exists a constant $C > 0$ such that

$$\tau(r) \leq C\tau(s) \quad (\tau(r) \geq C\tau(s)) \quad \text{for } r \leq s.$$

A function $\tau : (0, +\infty) \rightarrow (0, +\infty)$ is said to satisfy the doubling condition if there exists a constant $C > 0$ such that

$$C^{-1} \leq \frac{\tau(r)}{\tau(s)} \leq C \quad \text{for } \frac{1}{2} \leq \frac{r}{s} \leq 2.$$

Let $\mathcal{F}$ be the set of all continuous, increasing and bijective functions $\Phi : [0, +\infty) \rightarrow [0, +\infty)$. Then $\Phi(0) = 0$ and $\lim_{r \rightarrow +\infty} \Phi(r) = +\infty$ for $\Phi \in \mathcal{F}$.

Definition 2.1 ((Φ, q)-atom). Let $\Phi \in \mathcal{F}$ and $1 < q \leq \infty$. A function a on X is called a (Φ, q)-atom if there exists a ball B such that

- (i) $\text{supp } a \subset \overline{B}$,
- (ii) $\|a\|_q \leq \mu(B)^{1/q} \Phi^{-1}(1/\mu(B))$,
- (iii) $\int_X a(x) d\mu(x) = 0$,

where $\|a\|_q$ is the L^q norm of a , $\overline{B}$ is the closure of B . We denote by $A(\Phi, q)$ the set of all (Φ, q)-atoms.

Definition 2.2 ((Φ, q) -block). Let $\Phi \in \mathcal{F}$ and $1 < q \leq \infty$. A function a on X is called a (Φ, q) -block if there exists a ball B such that (i) and (ii) hold. We denote by $B(\Phi, q)$ the set of all (Φ, q) -blocks.

For $\Phi \in \mathcal{F}$ and for $B = B(x, r)$, let

$$(2.1) \quad \phi(x, r) = \phi(B) = \frac{1}{\mu(B)\Phi^{-1}(1/\mu(B))}.$$

If a is a (Φ, q) -atom, then, for a ball B satisfying (i)–(iii), we have

$$(2.2) \quad \left| \int_X a(x)g(x) d\mu(x) \right| = \left| \int_B a(x)(g(x) - g_B) d\mu(x) \right| \\ \leq \|a\|_q \left(\int_B |g(x) - g_B| d\mu(x) \right)^{1/q'} \\ \leq \mu(B)\Phi^{-1}(1/\mu(B)) \left(\frac{1}{\mu(B)} \int_B |g(x) - g_B| d\mu(x) \right)^{1/q'} \leq \|g\|_{\mathcal{L}_{q',\phi}}.$$

That is, the mapping $g \mapsto \int_X ag d\mu$ is a bounded linear functional on $\mathcal{L}_{q',\phi}(X)/\mathcal{C}$ with norm not exceeding 1.

Definition 2.3 ($H_U^{\Phi,q}(X)$). Let $\Phi, U \in \mathcal{F}$, U be concave, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and ϕ be as in (2.1). We define the space $H_U^{\Phi,q}(X) \subset (\mathcal{L}_{q',\phi}(X)/\mathcal{C})^*$ as follows:

$f \in H_U^{\Phi,q}(X)$ if and only if there exist sequences $\{a_j\} \subset A(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that

$$(2.3) \quad f = \sum_j \lambda_j a_j \text{ in } (\mathcal{L}_{q',\phi}(X)/\mathcal{C})^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty.$$

In general, the expression (2.3) is not unique. We define

$$\|f\|_{H_U^{\Phi,q}} = \inf \left\{ U^{-1} \left(\sum_j U(\lambda_j) \right) \right\},$$

where the infimum is taken over all expressions (2.3). We note that $\|f\|_{H_U^{\Phi,q}}$ is not a norm in general. Let $d(f, g) = U(\|f - g\|_{H_U^{\Phi,q}})$ for $f, g \in H_U^{\Phi,q}(X)$. Then $d(f, g)$ is a metric and $H_U^{\Phi,q}(X)$ is complete.

In the case $\Phi(r) = U(r) = r^p$, $p < 1$, then $H_U^{\Phi,q}(X) = H^p(X)$ defined by Coifman and Weiss [5] (1977). Let $I(r) = r$. Then $\|f\|_{H_I^{\Phi,q}}$ is a norm and $H_I^{\Phi,q}$ is a Banach space, which was defined by Zorko [31] (1986) in the case $X = \mathbb{R}^n$.

Definition 2.4 ($B_U^{\Phi,q}(X)$). Let $\Phi, U \in \mathcal{F}$, U be concave, $1 < q \leq \infty$, $1/q + 1/q' = 1$ and ϕ be as in (2.1). Assume that $r\Phi^{-1}(1/r)$ is almost increasing. We define the space $B_U^{\Phi,q}(X) \subset (L_{q',\phi}(X))^*$ as follows:

$f \in B_U^{\Phi,q}(X)$ if and only if there exist sequences $\{a_j\} \subset B(\Phi, q)$ and positive numbers $\{\lambda_j\}$ such that

$$(2.4) \quad f = \sum_j \lambda_j a_j \text{ in } (L_{q',\phi}(X))^* \quad \text{and} \quad \sum_j U(\lambda_j) < \infty.$$

We define

$$\|f\|_{B_U^{\Phi,q}} = \inf \left\{ U^{-1} \left(\sum_j U(\lambda_j) \right) \right\},$$

where the infimum is taken over all expressions (2.4).

Let $d(f, g) = U(\|f - g\|_{B_U^{\Phi,q}})$ for $f, g \in B_U^{\Phi,q}(X)$. Then $d(f, g)$ is a metric and $B_U^{\Phi,q}(X)$ is complete. Let $I(r) = r$. Then $\|f\|_{B_I^{\Phi,q}}$ is a norm and $B_I^{\Phi,q}$ is a Banach space.

If $X = \mathbb{R}^n$, $d(x, y) = |x - y|$, μ is Lebesgue measure, $\Phi(r) = r$ and $U(r) = r(1 + \log^+(1/r))$, then $B_U^{\Phi,q}(X)$ is the space introduced by Taibleson and Weiss [29] (1983) and Lu, Taibleson and Weiss [10] (1982).

From the definition it follows that

- If $1 < q_1 < q_2 \leq \infty$, then

$$H_U^{\Phi,q_2}(X) \subset H_U^{\Phi,q_1}(X), \quad B_U^{\Phi,q_2}(X) \subset B_U^{\Phi,q_1}(X).$$

- If $\Psi(r) \leq \Phi(Cr)$ for all $r > 0$, then

$$H_U^{\Phi,q}(X) \subset H_U^{\Psi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_U^{\Psi,q}(X).$$

- If $V(r) \leq CU(r)$ for $0 \leq r \leq 1$, then

$$H_U^{\Phi,q}(X) \subset H_V^{\Phi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_V^{\Phi,q}(X).$$

- For any concave function $U \in \mathcal{F}$,

$$H_U^{\Phi,q}(X) \subset H_I^{\Phi,q}(X), \quad B_U^{\Phi,q}(X) \subset B_I^{\Phi,q}(X).$$

In the above, the inclusion mapping are continuous.

3. MAIN RESULTS

Let $(H_U^{\Phi,q}(X))^*$ and $(B_U^{\Phi,q}(X))^*$ be the linear spaces of all continuous linear functionals on $H_U^{\Phi,q}(X)$ and on $B_U^{\Phi,q}(X)$, respectively.

Lemma 3.1. *Let Φ, q, U be as in Definition 2.3. If*

$$(3.1) \quad \sup_{0 < s < 1} \frac{U(rs)}{U(s)} \rightarrow 0 \quad (r \rightarrow 0),$$

then

$$\|\ell\|_{(H_U^{\Phi,q})^*} = \sup \left\{ |\ell(f)| : \|f\|_{H_U^{\Phi,q}} \leq 1 \right\},$$

$$\|\ell\|_{(B_U^{\Phi,q})^*} = \sup \left\{ |\ell(f)| : \|f\|_{B_U^{\Phi,q}} \leq 1 \right\}$$

are finite for all $\ell \in (H_U^{\Phi,q}(X))^*$ and for all $\ell \in (B_U^{\Phi,q}(X))^*$, respectively. $\|\ell\|_{(H_U^{\Phi,q})^*}$ and $\|\ell\|_{(B_U^{\Phi,q})^*}$ are norms.

Let $L_{\text{comp}}^q(X)$ be the set of all L^q -functions with compact support, and let

$$L_{\text{comp}}^{q,0}(X) = \left\{ f \in L_{\text{comp}}^q(X) : \int_X f d\mu = 0 \right\}.$$

Then, for $1 < q \leq \infty$, $L_{\text{comp}}^q(X)$ and $L_{\text{comp}}^{q,0}(X)$ are dense in $B_U^{\Phi,q}(X)$ and in $H_U^{\Phi,q}(X)$, respectively.

If $g \in \mathcal{L}_{q',\phi}(X)$ and $f \in L_{\text{comp}}^{q,0}(X)$, then fg is integrable.

Theorem 3.2. Let Φ, q, U, ϕ be as in Definition 2.3. If U satisfies (3.1), then

$$(H_U^{\Phi,q}(X))^* = \mathcal{L}_{q',\phi}(X)/\mathcal{C}.$$

More precisely, if $g \in \mathcal{L}_{q',\phi}(X)$, then the mapping $\ell : f \mapsto \int_X f(g + C) d\mu$ ($f \in L_{\text{comp}}^{q,0}(X)$) can be extended to a continuous linear functional on $H_U^{\Phi,q}(X)$. Conversely, if ℓ is a continuous linear functional on $H_U^{\Phi,q}(X)$, then there exists $g \in \mathcal{L}_{q',\phi}(X)$ such that $\ell(f) = \int_X f(g + C) d\mu$ for $f \in L_{\text{comp}}^{q,0}(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Corollary 3.3. Let $\Phi(r) = r$. Then, for $1 < q \leq \infty$ and for any concave function $U \in \mathcal{F}$ with (3.1),

$$(H_U^{\Phi,q}(X))^* = \text{BMO}(X)/\mathcal{C}.$$

Theorem 3.4. Let Φ, q, U, ϕ be as in Definition 2.4. If U satisfies (3.1), then

$$(B_U^{\Phi,q}(X))^* = L_{q',\phi}(X).$$

More precisely, if $g \in L_{q',\phi}(X)$, then the mapping $\ell : f \mapsto \int_X fg d\mu$ ($f \in L_{\text{comp}}^q(X)$) can be extended to a continuous linear functional on $B_U^{\Phi,q}(X)$. Conversely, if ℓ is a continuous linear functional on $B_U^{\Phi,q}(X)$, then there exists $g \in L_{q',\phi}(X)$ such that $\ell(f) = \int_X fg d\mu$ for $f \in L_{\text{comp}}^q(X)$. The norm $\|\ell\|$ is equivalent to $\|g\|_{\mathcal{L}_{q',\phi}}$.

Theorem 3.5. Assume that $\mu(X) = \infty$ and that there exists $k > 1$ s.t.

$$(3.2) \quad \mu(B) \leq \frac{1}{2}\mu(kB) \quad \text{for all balls } B.$$

Let Φ, q, U, ϕ be as in Definition 2.4 and $U(rs) \leq U(r)U(s)$ for $0 < r, s \leq 1$. If there exists $C > 0$ such that

$$(3.3) \quad \int_r^\infty \frac{1}{t\Phi^{-1}(1/t)} \frac{dt}{t} \leq C \frac{1}{r\Phi^{-1}(1/r)}, \quad 0 < r < \infty,$$

then $H_U^{\Phi, q}(X) = B_U^{\Phi, q}(X)$. More precisely, for $f \in B_U^{\Phi, q}(X)$, there exists a decomposition $f = \sum_j \lambda_j a_j$ with (Φ, q) -atoms such that

$$\langle f, g - c_g \rangle = \sum_j \lambda_j \int a_j g \quad \text{for all } g \in \mathcal{L}_{p, \phi}(X)/\mathcal{C},$$

where $c_g = \lim_{r \rightarrow \infty} g_{B(x_0, r)}$.

Remark 3.1. It is known that (3.3) is a necessary and sufficient condition for $\mathcal{L}_{q', \phi}(X)/\mathcal{C} = L_{q', \phi}(X)$ (Nakai [24] (2006)) with (2.1).

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