

On weak dividing in n -simple theories

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Weak dividing was originally defined by Shelah in [1]. After a long time, Dolich characterized that notion in simple context in [2]. Then Kim and Shi continued the investigation, in particular they proved that a theory T is stable if and only if weak dividing is symmetric in [3]. Recently, the class of simple theory was split into $\omega + 1$ subclasses by Kolesnikov in [4]. He used the notion of n -simplicity for $n \leq \omega$. I studied his paper and had some consideration about the relation with weak dividing.

At first, we recall some definitions in [4].

For $n \geq 2$, let the symbol $\text{Ind}(x; y_0, \dots, y_{n-1})$ denote the type expressing that $y_0, \dots, y_{n-1}$ are indiscernible over x .

Definition 1 Fix $1 \leq n \leq k < \omega$. Take a formula $\varphi(x, y_0, \dots, y_{n-1})$ and a partial type $p(x)$. Define $D_n[p, \varphi, k] \geq \alpha$ by induction on α .

- (1) $D_n[p, \varphi, k] \geq 0$ if p is consistent.
- (2) for α limit, $D_n[p, \varphi, k] \geq \alpha$ if $D_n[p, \varphi, k] \geq \beta$ for all $\beta < \alpha$;
- (3) $D_n[p, \varphi, k] \geq \alpha + 1$ if for every finite $r \subseteq p(x)$ there is a sequence $\{a_i | i < \omega\}$ such that for all $\bar{i} \in [\omega]^n$

$$D_n[r \cup \{\varphi(x, \bar{a}_{\bar{i}})\} \cup \text{Ind}(x; \bar{a}_{\bar{i}}), \varphi, k] \geq \alpha$$

and the set $\{\varphi(x, \bar{a}_{\bar{i}}) | \bar{i} \in [\omega]^n\}$ is $[k]^n$ -contradictory.

The expressions $D_n[p, \varphi, k] = \alpha$, $D_n[p, \varphi, k] = -1$, and $D_n[p, \varphi, k] = \infty$ are defined as usual.

Definition 2 Let $\alpha \leq \omega$. We say that a complete theory T is α -simple if for all $n < \alpha$, for all $\varphi(x, y_0, \dots, y_n)$ and $k > n + 1$ the rank $D_{n+1}[x = x, \varphi, k]$ is bounded (i.e. is less than ∞).

Definition 3 (1) A formula $\varphi(x, y_0, \dots, y_{n-1})$, a set of sequences $\{I_\eta | \eta \in ([\omega]^n)^{<\omega}\}$, and $k < \omega$ witness the n -tree property if for every $\eta \in ([\omega]^n)^\omega$, the type $\{\varphi(x, \bar{a}_{\eta|l}^{l|l}) | l < \omega\}$ is realized by $\bar{b}_\eta$ such that sequences $\bar{a}_{\eta|l}^{l|l}$ are

indiscernible over b_η for each $l < \omega$ and for every $\eta \in ([\omega]^n)^{<\omega}$ the set $\{\varphi(x, \bar{a}_\eta^\eta) | \bar{a}_\eta^\eta \in [\omega]^n\}$ is $[k]^n$ -contradictory where $\bar{a}_{\eta[l]}^{\eta[l]} := \{a_{i_0}^{\eta[l]}, \dots, a_{i_{n-1}}^{\eta[l]}\}$ for $\eta[l] = i_0 \dots i_{n-1}$.

(2) A theory T has the n -tree property if there exist a formula, a set of parameters, and a number k witnessing the n -tree property.

The next proposition is proved by the definitions.

Proposition 4 ([4]) *A theory T is α -simple if and only if it does not have an $(n+1)$ -tree property for any $n < \alpha$.*

Kolesnikov defined some notion of dividing for n -simple context.

Definition 5 For $n < \omega$, we say that a formula $\varphi(x, a_0, \dots, a_{n-1})$ n -divides over A if there is an indiscernible sequence $\{a_i | i < \omega\}$ over A and $b \models \varphi(x, a_0, \dots, a_{n-1})$ such that $\{a_0, \dots, a_{n-1}\}$ are indiscernible over b and the set $\{\varphi(x, \bar{a}_\eta) | \bar{a}_\eta \in [\omega]^n\}$ is $[k]^n$ -contradictory for some k .

Remark 6 It is clear that for $n = 1$ the definition is the same as that of dividing.

We recall the definition of weak dividing to make sure.

Definition 7 We say that $p(x) = \text{tp}(a/B)$ weakly divides over $A (\subseteq B)$ if there is a formula $\psi(x_1, \dots, x_n)$ over A such that $[p]^\psi := p(x_1) \cup \dots \cup p(x_n) \cup \{\psi(x_1, \dots, x_n)\}$ is inconsistent while $[q]^\psi$ is consistent where $q(x) = \text{tp}(a/A)$.

The next facts are easily checked.

Fact 8 Let $A \subset B$ and $\varphi(x_0, \dots, x_{n-1}, b)$ be a formula over B . Suppose that there is an indiscernible sequence $\{a_i | i < \omega\}$ over A satisfying:
 $\models \varphi(a_0, \dots, a_{n-1}, b)$ and $\{a_i | i < n\}$ are indiscernible over b . If the type $\{\varphi(x_0, \dots, x_{n-1}, b)\} \cup \text{Ind}(B; \{x_i | i < \omega\})$ is inconsistent, then there is a formula $\psi(a_0, \dots, a_{n-1}, z)$ such that $\psi(a_0, \dots, a_{n-1}, z)$ n -divides over A .

Fact 9 Let $A \subset B$ and $p(x) = \text{tp}(a/B)$. Suppose that there is a formula $\varphi(x_0, \dots, x_{n-1})$ over A and an infinite indiscernible sequence $\{a_i | i < \omega\}$ over A with $\text{tp}(a_0/A) = p \upharpoonright A$ such that

$\models \varphi(a_0, \dots, a_{n-1})$ and

the type " $\{\varphi(x_0, \dots, x_{n-1})\} \cup \text{Ind}(A; \{x_i : i < \omega\}) \cup \bigcup_{i < \omega} p(x_i)$ " is inconsistent.

Then p weakly divides over A .

Moreover if T is simple, then p divides over A .

The case is problematic when realizations of the formula can not be extended to an infinite indiscernible sequence over the original parameters. I tried to use the facts above for the argument of weak dividing in n -simple theories, but I have no result to show here.

We can define an analogy of weak dividing for n -dividing.

Definition 10 Let $A \subset B$. And let $p(x_0, \dots, x_{n-1})$ be a complete type over B such that $p(x_0, \dots, x_{n-1}) \vdash \text{Ind}(A; x_0, \dots, x_{n-1})$. We say that $p(x_0, \dots, x_{n-1})$ "weakly n -divides over A " if there are $k < \omega$ and a formula $\psi(x_0, \dots, x_{k-1})$ over A such that $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}}) \upharpoonright A$ is consistent while $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}})$ is inconsistent where $p(\bar{x}_{\bar{i}}) = p(x_{i_0}, \dots, x_{i_{n-1}})$ for $i_0 < i_1 < \dots < i_{n-1} < k$ and $k > n$.

Remark 11 When $n = 1$, "weak 1-dividing" is the same as "weak dividing".

Notation

From now, we denote $[p]^\psi$ for the type $\{\psi(x_0, \dots, x_{k-1})\} \cup \bigcup_{\bar{i} \in [k]^n} p(\bar{x}_{\bar{i}})$.

Fact 12 Let $A \subset B \subset C$.

- (1) If $\text{tp}(a/C)$ does not weakly n -divide over B , then $\text{tp}(a/B)$ does not weakly n -divide over A .
- (2) If $\text{tp}(a/C)$ does not weakly n -divide over B and $\text{tp}(a/B)$ does not weakly n -divide over A , then $\text{tp}(a/C)$ does not weakly n -divide over A .

Fact 13 Weak n -dividing has the local character.

Fact 14 If $\text{tp}(b/Aa_0 \dots a_{n-1})$ n -divides over A , then $\text{tp}(a_0 \dots a_{n-1}/Ab)$ weakly n -divides over A .

Remark 15 Naturally, we define weak n -dividing for complete types as follows:

a complete type p weakly n -divides over A if it implies a formula which weakly n -divides over A .

Lemma 16 Let $A \subset B$. And let $p(x_0, \dots, x_{n-1})$ be a complete type over B such that $p(x_0, \dots, x_{n-1}) \vdash \text{Ind}(A; x_0, \dots, x_{n-1})$.

Then the following are equivalent;

- (1) p does not weakly n -divide over A .
- (2) For any set $C := \{a_i \mid i \in I\}$ satisfying that for any n -sequence $a_{i_0}, \dots, a_{i_{n-1}} \in C$ with $i_0 < i_1 < \dots < i_{n-1}$, $\models p \upharpoonright A(a_{i_0}, \dots, a_{i_{n-1}})$, there is B' such that $\text{tp}(B/A) = \text{tp}(B'/A)$ and for any $a_{i_0}, \dots, a_{i_{n-1}} \in C$ with $i_0 < i_1 < \dots < i_{n-1}$, $\text{tp}(B'/a_{i_0} \dots a_{i_{n-1}}A) = \text{tp}(B/a_0 \dots a_{n-1}A)$.

The further characterization needs to investigate the relation between n -simple theories and n -dividing more.

References

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