

Local estimates and Maximum Principle for fully nonlinear equations in unbounded domains

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open, connected and possibly *unbounded* subset of $\mathbb{R}^N$, and let $u(x)$ be a bounded from above and upper semicontinuous function on the closure of Ω , in symbols

$$\sup_{\overline{\Omega}} u < +\infty, \quad u \in USC(\overline{\Omega}),$$

satisfying in the viscosity sense a second order fully nonlinear differential inequality of the form

$$F(x, u, Du, D^2u) \geq 0 \quad \text{in } \Omega. \quad (1)$$

In the recent paper [5], we gave an answer to the following question:

when the Maximum Principle - MP in short - holds for inequality (1), that is what assumptions on the domain Ω and/or on the operator F can ensure the validity of the implication

$$u \leq 0 \quad \text{on } \partial\Omega \implies u \leq 0 \quad \text{in } \Omega?$$

When looking at previous results about MP for unbounded domains, one can distinguish basically two kinds of results

- ♣ general comparison principles, which include MP as a special case, between viscosity subsolutions and supersolutions of fully nonlinear equations. Within this approach, the operator $F(x, u, p, X) : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \rightarrow \mathbb{R}$ is assumed to satisfy, besides the degenerate ellipticity inequality, some structural growth conditions and the *strict monotonicity* with respect to the u variable. On the other hand, no assumptions on the domain Ω are required, and even the case $\Omega = \mathbb{R}^N$ is allowed.

- ♣ for *strong solutions* of linear uniformly elliptic second order differential inequalities with bounded coefficients, that is for functions u satisfying

$$\begin{cases} \operatorname{tr}(A(x) D^2 u) + \underline{b}(x) \cdot Du + c(x) u \geq 0 & \text{a.e. in } \Omega, \\ u \in W_{\text{loc}}^{2,N}(\Omega), \quad \sup_{\Omega} u < +\infty, \end{cases}$$

MP has been obtained as a consequence of the (improved) Alexandrov-Bakelman-Pucci (**ABP** in short) estimate. In this case, a *large monotonicity* in the zero order term is allowed, namely the requirement $c(x) \leq 0$ holds, but some geometric restrictions on the domain Ω are assumed.

For the former approach, we refer to the results obtained by R. Jensen, P.L. Lions & P.E. Souganidis [9] and by H. Ishii [8], and included in the celebrated "User's guide" of M. Crandall, H. Ishii & P.L. Lions [6]. In the latter case, we refer to the results of H. Berestycki, L. Nirenberg & S.R.S. Varadhan [1] and of X. Cabré [2], as well as to the further extensions by V. Cafagna & A. Vitolo [3] and by A. Vitolo [12].

Let us observe that, in general, **MP** does *not* hold for even linear uniformly elliptic inequality not strictly monotone with respect to the u variable. As a simple example, $u(x) = 1 - 1/|x|^{N-2}$, with $N \geq 3$, is a bounded subharmonic (actually, harmonic) function in the exterior domain $\Omega = \mathbb{R}^N \setminus \overline{B}_1(0)$ and constantly equals zero on the boundary, while being strictly positive inside Ω . Thus, widely speaking, some extra assumptions are needed in order to obtain **MP**.

In this notes, after recalling the method pursued for linear operators, we present the results obtained in [5], which extend it to viscosity solutions of fully nonlinear inequalities.

2 ABP estimate in the linear case.

Let u be a bounded from above strong solution of the following linear differential inequality

$$\begin{cases} \operatorname{tr}(A(x) D^2 u) + \underline{b}(x) \cdot Du + c(x) u \geq f(x) & \text{a.e. in } \Omega, \\ u \in W_{\text{loc}}^{2,N}(\Omega), \quad \sup_{\Omega} u < +\infty, \end{cases}$$

with *bounded* coefficients satisfying

$$\lambda I_N \leq A(x) \leq \Lambda I_N, \quad c(x) \leq 0 \quad \text{for a.e. } x \in \Omega,$$

and source term such that

$$f \in L^N(\Omega).$$

The **ABP** estimate assumes different forms according to the boundedness properties of the domain Ω .

2.1 ABP for bounded domains.

In the standard case of a bounded domain, the **ABP** estimate states that (see e. g. [7])

$$\sup_{\Omega} u \leq \limsup_{x \rightarrow \partial\Omega} u + C \operatorname{diam}(\Omega) \|f^-\|_{L^N(\Omega)},$$

where f^- is the negative part of the function f and $C > 0$ is a constant depending on N , on the ellipticity constants λ and Λ , and on the product $\operatorname{diam}(\Omega) \|\underline{b}\|_{L^\infty(\Omega)}$.

2.2 ABP for domains having finite measure.

In this case, by assuming further that $f \in L^\infty(\Omega)$, H. Berestycki, L. Nirenberg & S.R.S. Varadhan [1] proved that

$$\sup_{\Omega} u \leq \limsup_{x \rightarrow \partial\Omega} u + C \operatorname{meas}(\Omega)^{\frac{2}{N}} \|f^-\|_{L^\infty(\Omega)},$$

with $C > 0$ depending on N , λ , Λ , and on the product $\operatorname{meas}(\Omega)^{\frac{1}{N}} \|\underline{b}\|_{L^\infty(\Omega)}$.

2.3 ABP for certain unbounded domains.

The general case of an unbounded domain has been considered by X. Cabré [2], under the following geometric condition that will be referred to as condition **(G)** :

there exist constants $\sigma, \tau \in (0, 1)$ and $R(\Omega) > 0$ such that, for all $y \in \Omega$, there is a ball B_{R_y} , containing y and having radius $R_y \leq R(\Omega)$, which satisfies

$$\operatorname{meas}(B_{R_y} \setminus \Omega_{y,\tau}) \geq \sigma \operatorname{meas}(B_{R_y}),$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y .

Roughly speaking, the requirement $R_y \leq R(\Omega)$ for all $y \in \Omega$ imposes in a measure theoretic sense that there is "enough boundary" uniformly near to every point of Ω . The positive constant $R(\Omega)$ plays the role of the diameter for an unbounded domain. Examples of domains satisfying condition **(G)** include all the domain having finite measure, in which case we have $R(\Omega) = (2 \operatorname{meas}(\Omega) / \operatorname{meas}(B_1))^{1/N}$, and all the cylinders, for which $R(\Omega)$ equals the diameter of their bounded projections.

If Ω satisfies **(G)**, the improved **ABP** estimate obtained in [2] states that

$$\sup_{\Omega} u \leq \limsup_{x \rightarrow \partial\Omega} u + C R(\Omega) \|f^-\|_{L^N(\Omega)},$$

with $C > 0$ depending on N , λ , Λ , and on the product $R(\Omega) \|\underline{b}\|_{L^\infty(\Omega)}$.

3 ABP and MP in the fully nonlinear case.

Let u be a bounded from above viscosity solution of the following fully nonlinear differential inequality

$$\begin{cases} F(x, u, Du, D^2u) \geq f(x) & \text{in } \Omega, \\ u \in USC(\overline{\Omega}), \quad \sup_{\overline{\Omega}} u < +\infty. \end{cases} \quad (\mathbf{P})$$

Here we assume that $f \in C(\Omega) \cap L^\infty(\Omega)$. Furthermore, the continuous real valued function $F : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathcal{S}^N \rightarrow \mathbb{R}$ (with $\mathcal{S}^N$ being the set of $N \times N$ real symmetric matrices) is assumed to satisfy, besides the degenerate ellipticity inequality

$$F(x, t, p, X) \geq F(x, t, p, Y) \quad (\mathbf{F}_1)$$

for all $x \in \Omega$, $t \in \mathbb{R}$, $p \in \mathbb{R}^N$ and $X, Y \in \mathcal{S}^N$ with $X - Y \geq O$, the following bound from above

$$F(x, t, p, X) \leq \mathcal{P}_{\lambda, \Lambda}^+(X) + b(x) |p| \quad (\mathbf{F}_2)$$

for all $x \in \Omega$, $p \in \mathbb{R}^N$, $X \in \mathcal{S}^N$ and $t \geq 0$. We assume that $b \in C(\Omega) \cap L^\infty(\Omega)$ is a nonnegative function and we denote by $\mathcal{P}_{\lambda, \Lambda}^+$ the Pucci maximal operator, defined as (see [4, 6])

$$\mathcal{P}_{\lambda, \Lambda}^+(X) := \sup_{A \in \mathcal{A}} \text{Tr}(AX) = \Lambda \text{Tr}(X^+) - \lambda \text{Tr}(X^-),$$

where $\mathcal{A} = \mathcal{A}(\lambda, \Lambda) = \{A \in \mathcal{S}^N : \lambda I \leq A \leq \Lambda I\}$, and X^+ and X^- are nonnegative definite matrices such that $X = X^+ - X^-$ and $X^+ X^- = O$.

Let us point out that assumptions $(\mathbf{F}_1)$, $(\mathbf{F}_2)$ are satisfied by any uniformly elliptic proper operator F having linear growth with respect to first order terms. Furthermore, if F satisfies $(\mathbf{F}_1)$ and its principal part $F(x, 0, 0, X)$ is linear with respect to X , then condition $(\mathbf{F}_2)$ implies the uniform ellipticity of $F(x, 0, 0, X)$. Indeed, by using $(\mathbf{F}_2)$ with $X = \pm Q$ and $Q \geq O$, it follows that

$$F(x, 0, 0, Q) \leq \mathcal{P}_{\lambda, \Lambda}^+(Q) = \Lambda \text{Tr}(Q), \quad F(x, 0, 0, -Q) \leq \mathcal{P}_{\lambda, \Lambda}^+(-Q) = -\lambda \text{Tr}(Q),$$

and then, by linearity,

$$\lambda \text{Tr}(Q) \leq F(x, 0, 0, Q) \leq \Lambda \text{Tr}(Q), \quad \forall Q \geq O.$$

On the other hand, assumptions $(\mathbf{F}_1)$, $(\mathbf{F}_2)$ include also nonlinear, possibly degenerate, elliptic operators, such as the following one

$$F(x, t, p, X) = \Lambda \left(\sum_{i=1}^N \varphi(\mu_i^+) \right) - \lambda \left(\sum_{i=1}^N \psi(\mu_i^-) \right) + H(x, t, p),$$

where μ_i , $i = 1, \dots, N$, are the eigenvalues of the matrix $X \in \mathcal{S}^N$, $\varphi, \psi : [0, +\infty) \rightarrow [0, +\infty)$ are continuous and nondecreasing functions such that $\varphi(s) \leq s$ and $\psi(s) \geq s$ for all $s \geq 0$, and $H(x, t, p)$ is a continuous function such that $H(x, t, p) \leq b(x)|p|$ for all $x \in \Omega$, $t \geq 0$ and $p \in \mathbb{R}^N$.

3.1 ABP for bounded domains.

When the domain Ω is bounded, the **ABP** estimate has also in the fully nonlinear case the form

$$\sup_{\Omega} u \leq \sup_{x \in \partial\Omega} u^+ + C \text{diam}(\Omega) \|f^-\|_{L^N(\Omega)},$$

where $C > 0$ is a constant depending on N , on the ellipticity constants λ and Λ , and on the product $\text{diam}(\Omega) \|b\|_{L^\infty(\Omega)}$. This has been proved by L. Caffarelli & X. Cabré [4] if the operator F does not depend on the gradient variable, and then extended by A. Persello [11] to complete fully nonlinear operators satisfying $(\mathbf{F}_1)$ – $(\mathbf{F}_2)$, and by S. Koike & T. Takahashi [10] to operators having superlinear growth with respect to the gradient.

3.2 ABP and MP for certain unbounded domains

Here and henceforth we assume that the domain Ω satisfies the following condition, which will be referred to as **(wG)**

there exist constants $\sigma, \tau \in (0, 1)$ such that, for all $y \in \Omega$, there is a ball B_{R_y} containing y which satisfies

$$\text{meas}(B_{R_y} \setminus \Omega_{y, \tau}) \geq \sigma \text{meas}(B_{R_y}),$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y .

Note that condition **(wG)** is exactly the same as **(G)** but not requiring any uniform bound on the radii R_y 's. Typical examples of domains satisfying **(wG)** (and failing **(G)**) are cones, for which $R_y = O(|y|)$ as $|y| \rightarrow \infty$.

Under assumption **(wG)** we have the following localized version of **ABP** estimate.

Theorem 1 (see [5]) *Let u be a solution of **(P)**, with F and Ω satisfying respectively assumptions **(F₁)**–**(F₂)** and **(wG)**. Then, for every $y \in \Omega$, there exists a constant $\theta_y \in (0, 1)$, depending on $N, \lambda, \Lambda, \sigma, \tau$ and on y through the quantity $R_y \|b\|_{L^\infty(\Omega_{y,\tau})}$, such that*

$$w^+(y) \leq (1 - \theta_y) \sup_{\Omega} w^+ + \theta_y \sup_{\partial\Omega} w^+ + R_y \|f^-\|_{L^N(\Omega_{y,\tau})}. \quad (2)$$

If either the operator F does not depend on the gradient or the domain Ω satisfies condition **(G)**, then the constant θ_y appearing in (2) is independent of y . In this case, from (2) with $f \equiv 0$ we immediately obtain the following

Corollary 2 *Assume that F satisfies **(F₁)**–**(F₂)** and that **(wG)** holds true for Ω . If either **(F₂)** is satisfied with $b \equiv 0$ or Ω satisfies **(G)**, then **MP** holds for the operator F in the domain Ω .*

In order to obtain a global **ABP** estimate for fully nonlinear inequalities in unbounded domains we need to assume, besides condition **(wG)** on Ω and assumptions **(F₁)**–**(F₂)** on F , a further requirement coupling the geometry of the domain with the growth of the first order coefficients. Precisely, we have the following result.

Theorem 3 (see [5]) *Let Ω, F and u be as in Theorem 1. If further*

$$\sup_{y \in \Omega} R_y \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty, \quad (*)$$

where R_y and $\Omega_{y,\tau}$ are as in **(wG)** and b is as in **(F₂)**, then

$$\sup_{\Omega} w \leq \sup_{\partial\Omega} w^+ + C \sup_{y \in \Omega} R_y \|f^-\|_{L^N(\Omega_{y,\tau})}$$

for some positive constant C depending on $N, \lambda, \Lambda, \sigma, \tau$ and $\sup_{y \in \Omega} R_y \|b\|_{L^\infty(\Omega_{y,\tau})}$.

For $f \geq 0$, Theorem 3 immediately yields the following

Corollary 4 *Under the same assumptions of Theorem 3, the Maximum Principle holds for the operator F in the domain Ω .*

4 Examples and further extensions.

4.1 On the necessity of condition (*) for **MP**.

For a complete second order operator condition **(wG)** alone is in general not enough for **MP** to hold. A counterexample (see [12]) is given by the function

$$u(x) = u(x_1, x_2) = (1 - e^{1-x_1^2}) (1 - e^{1-x_2^2}),$$

with $0 < \alpha < 1$. Indeed, u is bounded and strictly positive in the plane cone

$$\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 > 1, x_2 > 1\},$$

and it satisfies

$$u \equiv 0 \quad \text{on } \partial\Omega, \quad \Delta u + \underline{b}(x) \cdot Du = 0 \quad \text{in } \Omega,$$

where the vector-field $\underline{b}$ is given by

$$\underline{b}(x) = \underline{b}(x_1, x_2) = \left(\frac{\alpha}{x_1^{1-\alpha}} + \frac{1-\alpha}{x_1}, \frac{\alpha}{x_2^{1-\alpha}} + \frac{1-\alpha}{x_2} \right).$$

Notice that Ω satisfies (wG) with $R_y = O(|y|)$ as $|y| \rightarrow \infty$ and, on the other hand, condition (F₂) holds with $b(x) = |\underline{b}(x)|$. Since for every $y \in \Omega$ and for any choice of B_{R_y} we have $\|b\|_{L^\infty(\Omega_{y,r})} \geq 1$ and $\sup_{y \in \Omega} R_y = +\infty$, condition (*) clearly fails in this example.

4.2 An application.

Let us look at some special non trivial cases in which condition (*) is fulfilled.

- (a) Consider the half cylinder $\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1, x_N > 0\}$. Since Ω satisfies condition (G), then (*) is satisfied if b in assumption (F₂) is any nonnegative bounded and continuous function.
- (b) Ω is a convex set with "parabolic" boundary, i.e.

$$\Omega = \{(x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N > |x'|^q\}$$

with $q > 1$. Then, Ω satisfies assumption (wG) with radii $R_y = O(|y|^{1/q})$ as $|y| \rightarrow \infty$. In this case, requirement (*) imposes to the function b a rate of decay $b(y) = O(1/|y|^{1/q})$ as $|y| \rightarrow \infty$. If so, the balls B_{R_y} in (wG) can be chosen in such a way that $\|b\|_{L^\infty(\Omega_{y,r})} = O(1/|y|^{1/q})$ as $|y| \rightarrow \infty$ and (*) is fulfilled.

- (c) Ω is the strictly convex cone $\{x \in \mathbb{R}^N \setminus \{0\} : x/|x| \in \Gamma\}$ where Γ is a proper subset of the unit half-sphere $S_+^{N-1} = \{x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x| = 1, x_N > 0\}$. In this case, condition (wG) is satisfied with $R_y = O(|y|)$ for $|y| \rightarrow \infty$ and condition (*) requires on the coefficient b the rate of decay $b(y) = O(1/|y|)$ as $|y| \rightarrow \infty$.

Note that cases (a) and (c) can be seen as limiting cases of situation (b) when, respectively, $q \rightarrow +\infty$ and $q = 1$.

4.3 MP for domains not satisfying (wG).

The validity of MP can be extended to even more general domains, not satisfying (wG), by repeatedly applying the argument of Corollary 4.

More precisely, let F be a second order operator satisfying (F₁)-(F₂) and assume that there exists a closed set $H \subset \Omega$ with the following properties

- (i) MP holds for F in each connected component of $\Omega \setminus H$;

- (ii) (wG) holds for all points of H , i.e. there exist constants $\sigma, \tau \in (0, 1)$ such that for all $y \in H$ there is a ball B_{R_y} of radius R_y containing y such that

$$|B_{R_y} \setminus \Omega_{y,\tau}| \geq \sigma |B_{R_y}|,$$

where $\Omega_{y,\tau}$ is the connected component of $\Omega \cap B_{R_y/\tau}$ containing y ;

- (iii)

$$\sup_{y \in H} R_y \|b\|_{L^\infty(\Omega_{y,\tau})} < \infty,$$

where R_y and $\Omega_{y,\tau}$ are as in (ii) and b is as in (F_2) .

In this situation we have the following

Theorem 5 (see [3, 5, 12]) Assume that F satisfies conditions (F_1) – (F_2) and that assumptions (i), (ii) and (iii) above hold for Ω . Then, MP holds for operator F in Ω .

As a consequence of the above result, MP can be obtained in non-convex, perhaps degenerate cones. For instance, if F satisfies (F_2) with a coefficient $b(x)$ such that $b(x) = O(1/|x|)$ as $|x| \rightarrow \infty$, then MP holds for F in the cat plane $\Omega = \mathbb{R}^2 \setminus \{(x_1, 0) \in \mathbb{R}^2 : x_1 \leq 0\}$, as it follows from Theorem 5 with e.g. $H = \{(x_1, -x_1) \in \mathbb{R}^2 : x_1 < 0\}$.

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