

Branched Coverings of Complex Manifolds

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Introduction. The theory of branched coverings is one of good examples of amalgamation of different branches of mathematics : topology, complex analysis and algebraic geometry. See, for example, Zariski [31], Fox [6], Kato [17], Hirzebruch [11], Höfer [13], Ishida [15], Fukui [7], Gaffney-Lazarsfeld [8], etc..

It need not to be mentioned that the theory of (Galois) branched coverings is a geometric counterpart of the Galois theory of function fields.

In this article, we present a theory of Galois and abelian branched coverings of complex manifolds, emphasizing existence theorems and examples mainly along the line of Namba [22]. In the last section, we discuss the equivalence problem of Kummer coverings after Kato [18].

Chapter 1. Galois Coverings.

1. Definition of Branched Coverings. First of all, we give a definition of branched coverings of complex manifolds. Since we treat infinite coverings as well as finite coverings, we define branched coverings as follows:

Definition 1.1. Let M be an n -dimensional connected complex manifold. A branched covering of M is an irreducible normal complex space X together with a surjective holomorphic mapping $\pi : X \longrightarrow M$ satisfying the following 4 conditions:

- i) Every fiber of π is discrete.
- ii) $R_\pi = \{p \in X \mid \pi^* : \mathcal{O}_{M, \pi(p)} \longrightarrow \mathcal{O}_{X, p} \text{ is not isomorphic}\}$ and $B_\pi = \pi(R_\pi)$ are hypersurfaces (i.e., pure codimension 1) of X and M , respectively, called the ramification locus and the branch locus of π , respectively. (Here, $\mathcal{O}_{X, p}$ is the local ring of germs of holomorphic functions around p .)
- iii) $\pi : X - \pi^{-1}(B_\pi) \longrightarrow M - B_\pi$ is a topological (i.e., unbranched) covering.
- iv) For every point $q \in B_\pi$, there is an open neighborhood W of q in M such that, for every connected component U of

$\pi^{-1}(W)$, $\pi^{-1}(q) \cap U$ consists of one point and $\pi|_U : U \longrightarrow W$ is a surjective proper mapping (hence a finite mapping).

If R_π is empty, then $\pi : X \longrightarrow M$ should be called an unbranched covering. But we call such a covering also a branched covering by abuse of language. A branched covering is said to be finite if every fiber is a finite set. The mapping degree of $\pi : X - \pi^{-1}(B_\pi) \longrightarrow M - B_\pi$ is called the degree of π . Using the purity of branch loci (see Fischer [4]), we have easily

Proposition 1. 2. An irreducible normal complex space X together with a surjective finite proper holomorphic mapping $\pi : X \longrightarrow M$ is a finite branched covering, and vice versa.

Let $\pi : X \longrightarrow M$ and $\pi' : X' \longrightarrow M$ be branched coverings of M . A morphism of π to π' is, by definition, a surjective holomorphic mapping $\phi : X \longrightarrow X'$ such that $\pi' \circ \phi = \pi$. Thus we have the category of branched coverings of M . ϕ is an isomorphism if $\phi : X \longrightarrow X'$ is biholomorphic. In this case, we say that π and π' are isomorphic. In particular, if $X = X'$ and $\pi = \pi'$, then an isomorphism is called a covering transformation of π . The set G_π of all covering transformations of π forms a group under compositions, called the covering transformation group. G_π acts on every fiber of π . A branched covering $\pi : X \longrightarrow M$ is called a Galois covering if G_π acts transitively on every fiber. $\pi : X \longrightarrow M$ is called an abelian (resp. a cyclic) covering if π is a Galois covering and G_π is an abelian (resp. a cyclic) group.

We denote by $\text{Sing } B_\pi$ the singular locus of the branch

locus B_π . It can be shown that, for every point $q \in B_\pi - \text{Sing} B_\pi$, every point $p \in \pi^{-1}(q)$ is a non-singular point of both X and $\pi^{-1}(B_\pi)$. Moreover, for any sufficiently small open neighborhood W of q with a coordinate system $(w_1, \dots, w_n)$ such that $q = (0, \dots, 0)$ and $B_\pi \cap W = \{w_n = 0\}$, there is an open neighborhood U of p with a coordinate system $(z_1, \dots, z_n)$ such that U is a connected component of $\pi^{-1}(W)$, $p = (0, \dots, 0)$ and π is locally given by

$$\pi|_U : (z_1, \dots, z_n) \longrightarrow (w_1, \dots, w_n) = (z_1, \dots, z_{n-1}, z_n^e),$$

where e is a positive integer, (see Roan [25] and Namba [22]). For an irreducible component C of $\pi^{-1}(B_\pi)$, the integer e is constant for points of $C - \pi^{-1}(\text{Sing} B)$, and is called the ramification index of π along C . (For convenience, the ramification index of π along an irreducible hypersurface of X which is not contained in $\pi^{-1}(B_\pi)$ is defined to be 1.) If π is a Galois covering, then, for any irreducible component D_1 of B_π , the ramification index e of π along irreducible components of $\pi^{-1}(D_1)$ is constant. In this case, e is called the ramification index of π along D_1 .

Let a hypersurface B of M be given. Suppose for simplicity that B has a finite number of irreducible components $D_1, \dots, D_s$:

$$B = D_1 \cup \dots \cup D_s.$$

Let $e_1, \dots, e_s$ be positive integers greater than one, and

$$D = e_1 D_1 + \dots + e_s D_s$$

be a positive divisor on M .

Definition 1.3. A branched covering $\pi : X \longrightarrow M$ is

said to branch along D (resp. at most along D) if (i) $B_\pi = B$ (resp. $B_\pi \subset B$) and (ii) for every j ($1 \leq j \leq s$) and for every irreducible component C of $\pi^{-1}(D_j)$, the ramification index of π along C is e_j (resp. divides e_j).

For branched coverings $\pi : X \longrightarrow M$ and $\pi' : X' \longrightarrow M$ of M , we denote $\pi \geq \pi'$ or $\pi' \leq \pi$ if there is a morphism of π to π' . If $\pi \geq \pi'$ and π branches at most along D , then π' branches at most along D . If π is a Galois covering, $\pi \geq \pi'$ and $\pi \leq \pi'$, then π and π' are isomorphic.

Definition 1. 4. A Galois covering $\pi : X \longrightarrow M$ is called a D-universal covering if (i) π branches along D and (ii) for any covering $\pi' : X' \longrightarrow M$ which branches at most along D , the relation $\pi \geq \pi'$ holds.

By the above remark, a D-universal covering is unique up to isomorphisms, if it exists. We denote it by

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M.$$

We now propose the following two problems:

Problem 1. When does a D-universal covering exist?

Problem 2. When does a finite Galois covering which branches along D exist?

As for a compact Riemann surface M , the problems were answered completely by Bundgaard-Nielsen [1] and Fox [5]:

Theorem 1. 5. Let M be a compact Riemann surface of genus g , $p_1, \dots, p_s$ be points of M , $e_1, \dots, e_s$ be positive integers greater than 1, and $D = e_1 p_1 + \dots + e_s p_s$ be a positive divisor on M . Then the following three conditions are equivalent:

- (i) There does not exist a D -universal covering of M .
- (ii) There does not exist a finite Galois covering $\pi : X \rightarrow M$ which branches along D .
- (iii) Either (iii-1) $g = 0$ and $s = 1$ or (iii-2) $g = 0$, $s = 2$ and $e_1 \neq e_2$.

Example 1. 6. If M is a compact Riemann surface and $\tilde{\pi} : \tilde{M}(D) \rightarrow M$ exists, then $\tilde{\pi}$ is an infinite covering, unless $M = \tilde{M}(D) = \mathbb{P}^1$, the complex projective line, and $\tilde{\pi}$ is isomorphic to one of the following rational functions, (see Klein [20], Hochstadt [12]):

$$(1) \quad w = z^m \quad (m = 1, 2, \dots),$$

$$D = m(\infty) + m(0), \quad \tilde{G} \simeq C_m \text{ (m-th cyclic group)}.$$

$$(2) \quad w = -(z^m - 1)^2 / 4z^m,$$

$$D = m(\infty) + 2(0) + 2(1), \quad \tilde{G} \simeq D_m \text{ (m-th dihedral group)}.$$

$$(3) \quad w = (z^4 + 2\sqrt{3}z^2 - 1)^3 / (z^4 - 2\sqrt{3}z^2 - 1)^3,$$

$$D = 3(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq A_4.$$

$$(4) \quad w = (z^8 + 14z^4 + 1)^3 / 108z^4(z^4 - 1)^4,$$

$$D = 4(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq S_4.$$

$$(5) \quad w = \frac{(z^{20} - 228z^{15} + 494z^{10} + 228z^5 + 1)^3}{-1728z^5(z^{10} + 11z^5 - 1)^5}$$

$$D = 5(\infty) + 3(0) + 2(1), \quad \tilde{G} \simeq A_5.$$

(Here (α) is the point divisor of $\alpha \in \mathbb{P}^1$, $\tilde{G} = G_{\tilde{\pi}}$ and A_n (resp. S_n) is the alternating (resp. symmetric) group of n letters.)

2. D-universal coverings. In this section, we give answers to the problems at the end of §1, using language of fundamental groups.

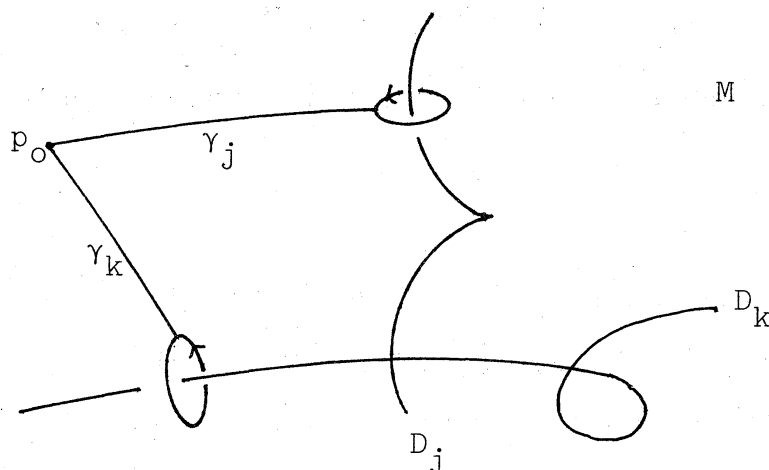


Figure 1

Take a point $p_0 \in M - B$ and fix it once for all. Let γ_j be a loop in $M - B$ starting and terminating at p_0 , encircling a point $p \in D_j - \text{Sing}B$ in the positive sense as in Figure 1.

γ_j is called a normal loop of D_j . We identify γ_j with its homotopy class in $\pi_1(M - B, p_0)$. Let

$$J = \langle \gamma_1^{e_1}, \dots, \gamma_s^{e_s} \rangle^{\pi_1}$$

be the smallest normal subgroup of $\pi_1(M - B, p_0)$ which contains $\gamma_1^{e_1}, \dots, \gamma_s^{e_s}$.

Definition 2. 1. A subgroup K of $\pi_1(M - B, p_0)$ with $J \subset K$ is said to be D-faithful if the following condition is satisfied: If γ_j^d belongs to K , then $d \equiv 0 \pmod{e_j}$ for every j ($1 \leq j \leq s$).

For every point $p \in \text{Sing} B$, take a sufficiently small ball W (with respect to a metric on M) with the center p such that

$$\pi_1(W - B) \simeq \pi_{1, \text{loc}, p}(M - B), \text{ (the local fundamental group at } p\text{).}$$

Let

$$i_* : \pi_1(W - B) \longrightarrow \pi_1(M - B, p_0)$$

be the homomorphism induced by the inclusion mapping $i : W - B \hookrightarrow M - B$.

Definition 2. 2. A subgroup K of $\pi_1(M - B, p_0)$ with $J \subset K$ is said to be locally cofinite if $i_*^{-1}(K)$ is a subgroup of $\pi_1(W - B)$ of finite index for every point $p \in \text{Sing } B$.

Theorem 2. 3. For any covering $\pi : X \longrightarrow M$ which branches at most along D , $K = \pi_*(\pi_1(X - \pi^{-1}(B)))$ contains J and is locally cofinite. Conversely, for any locally cofinite subgroup $K (\supset J)$ of $\pi_1(M - B, p_0)$, there exists a unique (up to isomorphisms) covering $\pi : X \longrightarrow M$ which branches at most along D such that $\pi_*(\pi_1(X - \pi^{-1}(B))) = K$. In this case, π branches along D if and only if K is D -faithful.

For the proof of the converse, we construct a topological covering $\pi' : X' \longrightarrow M - B$ such that $K = \pi_*^1(\pi_1(X'))$, and then we extend π' to

$$\pi : X \longrightarrow M$$

using a theorem in Grauert-Remmert [9], (see also Grothendieck-Raynaud [10], p.340). Topologically, this is so called a Fox completion, (see Fox[6]). See Namba [22] for detail. By

Theorem 2.3,

Theorem 2.4. There exists a finite Galois covering $\pi : X \rightarrow M$ which branches along D if and only if there exists a normal subgroup K of $\pi_1(M - B, p_0)$ of finite index which contains J and is D -faithful. The correspondence $\pi \rightarrow K = \pi_*(\pi_1(X - \pi^{-1}(B)))$ between (isomorphism classes of) such π 's and such K 's is one-to-one. In this case, G_π is isomorphic to $\pi_1(M - B, p_0)/K$.

In fact, for such a normal subgroup K , we have

$$\frac{\pi_1(W-B)}{i_*^{-1}(K)} \sim \frac{i_*(\pi_1(W-B))}{K \cap i_*(\pi_1(W-B))} \sim \frac{K \cdot i_*(\pi_1(W-B))}{K} \subset \frac{\pi_1(M-B, p_0)}{K}$$

under the above notation. Hence K is necessarily locally cofinite.

Now, put

$$\tilde{K} = \bigcap K,$$

where the intersection $\bigcap$ runs over all subgroups K of $\pi_1(M - B, p_0)$ which contain J and are locally cofinite. $\tilde{K}$ is then a normal subgroup of $\pi_1(M - B, p_0)$ which contains J .

Theorem 2.5. A D -universal covering $\tilde{\pi} : \tilde{M}(D) \rightarrow M$ exists if and only if $\tilde{K}$ is locally cofinite and D -faithful. In this case, $\tilde{K} = \tilde{\pi}_*(\pi_1(\tilde{M}(D) - \tilde{\pi}^{-1}(B)))$ and $G_{\tilde{\pi}} \sim \pi_1(M - B, p_0)/\tilde{K}$. Moreover, $\tilde{M}(D)$ is simply connected.

It is easy to see that $\tilde{M}(D)$ is simply connected. In fact, if $\mu : \tilde{X} \rightarrow \tilde{M}(D)$ is a (topological) universal covering

of $\tilde{M}(D)$, then $\tilde{\pi} \cdot \mu : \tilde{X} \longrightarrow M$ is a covering which branches along D such that $\tilde{\pi} \cdot \mu \geq \tilde{\pi}$. By the D -universality of $\tilde{\pi}$, we have $\tilde{\pi} \cdot \mu \leq \tilde{\pi}$. Hence μ is an isomorphism.

Theorem 2. 6. Let $\pi : X \longrightarrow M$ be a Galois covering which branches along D . Suppose that X is non-singular and simply connected. Then π is D -universal. In this case, $\tilde{K} = J$ and $G_\pi = \pi_1(M - B, p_0)/J$.

In this theorem, the condition of the non-singularity of X can not be dropped, as the following example shows:

Example 2. 7. Put $M = \mathbb{C}^2$ and let (u, v) be the coordinate system on $\mathbb{C}^2$. Put $D_1 = \{u = 0\}$, $D_2 = \{v = 0\}$ and $D = 2D_1 + 2D_2$. Put $X = \{(u, v, w) \in \mathbb{C}^3 \mid w^2 = uv\}$ and

$$\pi : (u, v, w) \in X \longmapsto (u, v) \in \mathbb{C}^2.$$

Then π is a cyclic covering of degree 2 which branches along D . X is simply connected, for X is a cone. But π is not D -universal. In fact, putting $Y = \mathbb{C}^2$ and

$$\mu : (x, y) \in Y \longmapsto (u, v, w) = (x^2, y^2, xy) \in X,$$

the composition $\pi \cdot \mu : Y \longrightarrow \mathbb{C}^2$ is a covering of degree 4 which branches along D and $\pi \cdot \mu \geq \pi$. (By Theorem 2. 6, $\pi \cdot \mu$ is D -universal.)

For the rest of this section, we suppose that B is simple normally crossing.

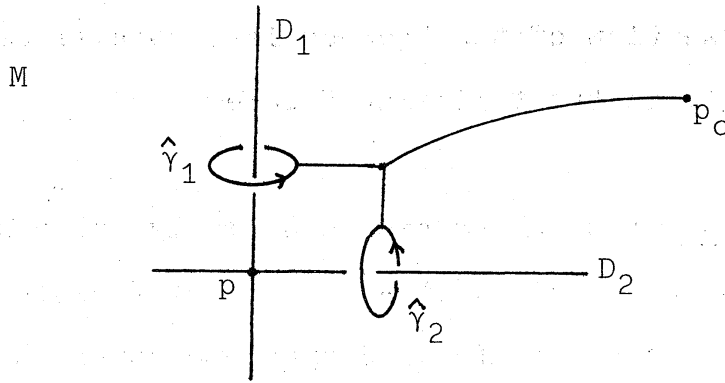


Figure 2

For any point $p \in \text{Sing} B$, let $(w_1, \dots, w_n)$ be a local coordinate system around p such that $p = (0, \dots, 0)$ and

$$B = \{(w_1, \dots, w_n) \mid w_k = \dots = w_n = 0\}.$$

locally. Let

$$\{w_j = 0\} = D_j \quad (k \leq j \leq n),$$

locally, say. Let $\hat{\gamma}_j$ be a loop in $M - B$ starting and terminating at p_0 , encircling a point of $D_j - \text{Sing} B$ near p in the positive sense as in Figure 2. Then $\hat{\gamma}_j$ is conjugate to γ_j in $\pi_1(M - B, p_0)$. $\hat{\gamma}_k, \dots, \hat{\gamma}_n$ are mutually commutative. For a sufficiently small ball W with the center p , we have

$$\pi_1(W - B) = (\hat{\gamma}_k)^{\mathbb{Z}} \dots (\hat{\gamma}_n)^{\mathbb{Z}}$$

and

$$(\hat{\gamma}_k^{e_k})^{\mathbb{Z}} \dots (\hat{\gamma}_n^{e_n})^{\mathbb{Z}} \subset i_*^{-1}(J) \subset \pi_1(W - B).$$

Hence J is locally cofinite, so that $\tilde{K} = J$. Thus

Theorem 2.8. If B is simple normally crossing, then a D -universal covering $\tilde{\pi} : \tilde{M}(D) \rightarrow M$ exists if and only if J is D -faithful. In this case, $J = \tilde{K}$ and $G_{\tilde{\pi}} \cong \pi_1(M - B, p_0)/J$. Moreover, if (under the above notation), $(\hat{\gamma}_k^{e_k})^{\mathbb{Z}} \dots (\hat{\gamma}_n^{e_n})^{\mathbb{Z}} = i_*^{-1}(J)$ for every point $p \in \text{Sing} B$, then $\tilde{M}(D)$ is non-singular.

The last assertion of the theorem is a special case of Kato [17], as well as the following theorem.

Theorem 2. 9. Let B be simple normally crossing. Let K be a normal subgroup of $\pi_1(M - B, p_0)$ of finite index which contains J and is D -faithful. Suppose moreover that, for any point $p \in \text{Sing} B$, K satisfies the following condition :
(under the above notation)

if $\hat{\gamma}_k^{d_k} \cdots \hat{\gamma}_n^{d_n} \in K$, then $d_k \equiv 0 \pmod{e_k}, \dots, d_n \equiv 0 \pmod{e_n}$.

Then the irreducible normal complex space X is non-singular, where $\pi : X \rightarrow M$ is the finite Galois covering which branches along D and corresponds to K under Theorem 2. 4.

3. Examples. It is not easy in general to apply the results of §2 to concrete examples. (Even the calculation of $\pi_1(M - B, p_0)$ is not easy.) In this section, we discuss two examples.

Case 1. Put $M = \mathbb{C}^2$, $B = D_1 = \{(x, y) \in \mathbb{C}^2 \mid x^3 = y^2\}$, ℓ : a positive integer greater than 1, and $D = \ell D_1$.

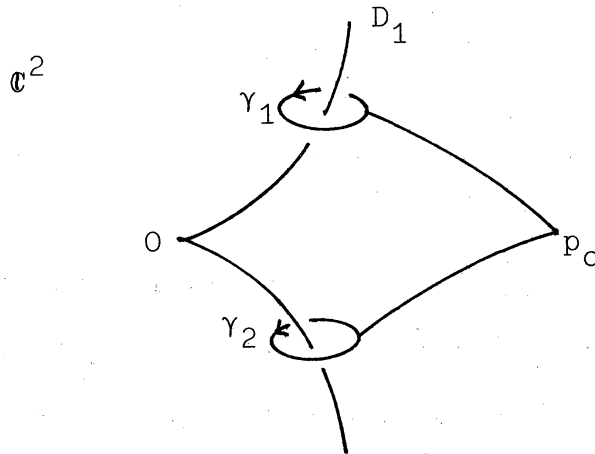


Figure 3

As is well known, $\pi_1(\mathbb{C}^2 - B, p_0)$ is isomorphic to 3rd braid group B_3 ; taking the loops γ_1 and γ_2 as in Figure 3, we have

$$\pi_1(\mathbb{C}^2 - B, p_0) = \langle \gamma_1, \gamma_2 \mid \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 \rangle.$$

Here the right hand side means the group generated by γ_1 and γ_2 with the generating relation $\gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2$. Since $\gamma_2 = (\gamma_2 \gamma_1)^{-1} \gamma_1 (\gamma_2 \gamma_1)$, γ_2 is conjugate to γ_1 . Let J be the smallest normal subgroup of $\pi_1(\mathbb{C}^2 - B, p_0)$ containing γ_1^ℓ (and so γ_2^ℓ). Then

$$\pi_1(\mathbb{C}^2 - B, p_0)/J \cong G_\ell = \langle a, b \mid a^\ell = b^\ell = 1, aba = bab \rangle$$

by the correspondence: $\gamma_1 \mapsto a, \gamma_2 \mapsto b$. We identify these groups through the isomorphism.

The cyclic covering

$$\pi_\ell: X_\ell \longrightarrow M = \mathbb{C}^2,$$

corresponding to the kernel of the homomorphism

$$f_\ell: G_\ell \longrightarrow \mathbb{Z}/\ell\mathbb{Z}, (f_\ell(a) = f_\ell(b) = 1)$$

is given by

$$\begin{aligned} \pi_\ell: X_\ell = \{(x, y, z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} &\longrightarrow M = \mathbb{C}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

and branches along $D = \ell D_1$. But π_ℓ is not D -universal.

The following argument on the structure of G_ℓ was informed by Mr. Mizutani. See also Coxeter [2]. First of all,

Lemma 3.1. $c = (aba)^2$ is an element of the center $Z(G_\ell)$ of G_ℓ .

Next, consider the Schwarz' triangular group

$$G(2, 3, \ell) = \langle S, T \mid S^2 = T^3 = (ST)^\ell = 1 \rangle,$$

and the homomorphism,

$$g : G_\ell \longrightarrow G(2, 3, \ell)$$

defined by $g(a) = ST$ and $g(b) = TS$.

Proposition 3. 2. The following sequence is exact:

$$1 \longrightarrow \langle c \rangle \longrightarrow G_\ell \xrightarrow{g} G(2, 3, \ell) \longrightarrow 1$$

From this proposition, we have the following table:

ℓ	$\text{ord}(c)$	$G(2, 3, \ell)$	G_ℓ	$\text{ord } G_\ell$
2	1	S_3	S_3	6
3	2	A_4	$SL(2, \mathbb{Z}/3\mathbb{Z})$	24
4	4	S_4	$G_4/Z(G_4) \simeq S_4,$ $Z(G_4) \simeq \mathbb{Z}/4\mathbb{Z}$	96
5	10	A_5	$SL(2, \mathbb{Z}/5\mathbb{Z}) \times (\mathbb{Z}/5\mathbb{Z})$	600
6	∞	infinite group	infinite solvable group	∞
≥ 7	∞	infinite group	infinite unsolvable group	∞

If ℓ satisfies $2 \leq \ell \leq 5$, then (under the notations of

§2) $\tilde{K} = J$ and there exists a D-universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{C}^2.$$

In this case, $\tilde{\pi}$ is a finite Galois covering such that $G_{\tilde{\pi}} \simeq G_\ell$.

Moreover, we have $\tilde{M}(D) = \mathbb{C}^2$ and $\tilde{\pi}$ is the composition

$$\tilde{M}(D) = \mathbb{C}^2 \xrightarrow{\mu} X_\ell \xrightarrow{\pi_\ell} M = \mathbb{C}^2,$$

where μ is the projection

$$\mu : \tilde{M}(D) = \mathbb{C}^2 \longrightarrow X_\ell = \mathbb{C}^2/H,$$

where H is a finite subgroup of $GL(2, \mathbb{C})$. The origin of X_ℓ in this case is called the Klein singularity, (see Pinkham [24]).

If $\ell = 6$, then we have

Proposition 3. 3. The kernel of $f_6 : G_6 \longrightarrow \mathbb{Z}/6\mathbb{Z}$ ($f_6(a) = f_6(b) = 1$) is given by $\langle a^{-1}b, ab^{-1} \rangle$ and is isomorphic to $N = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \mid i, j, k \in \mathbb{Z} \right\}$.

The isomorphism is given by

$$a^{-1}b \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad ab^{-1} \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

We identify $\ker(f_6)$ with N through the isomorphism. For any positive odd integer r ,

$$N(r) = \left\{ \begin{pmatrix} 1 & i & j \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \in N \mid i \equiv j \equiv k \equiv 0 \pmod{r} \right\}$$

is a normal subgroup of G_6 of index $6r^3$ and is D -faithful.

Hence

Proposition 3. 4. For any positive odd integer r , there is a Galois covering $v_r : Y_r \longrightarrow M = \mathbb{C}^2$ of degree $6r^3$ branching along $6D_1$. $v_r \leq v_{r'}$ if and only if $r|r'$.

Since

$$\bigcap_{r:\text{odd}} N(r) = \{1\},$$

We have

Proposition 3. 5. If $D = 6D_1$, then there does not exist a D -universal covering of $M = \mathbb{C}^2$.

Case 2. Put $M = \mathbb{P}^2$ (the complex projective plane), $B = D_1 \cup D_2$, D_1 = the closure in $\mathbb{P}^2$ of the affine curve $\{(x, y) \in \mathbb{C}^2 \mid x^3 - y^2 = 0\}$, $D_2 = L_\infty$ (the line at infinity), ℓ, m : positive integers greater than 1, and $D = \ell D_1 + m D_2$.

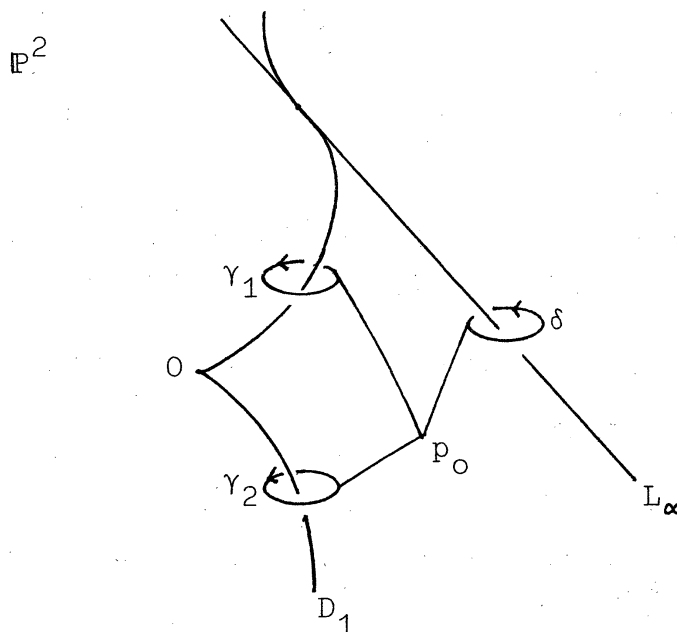


Figure 4

Taking the loops γ_1 , γ_2 and δ as in Figure 4, we have

$$\pi_1(M - B, p_0) = \langle \gamma_1, \gamma_2, \delta \mid \gamma_1 \gamma_2 \gamma_1 = \gamma_2 \gamma_1 \gamma_2 = \delta^{-1} \rangle.$$

Let J be the smallest normal subgroup of $\pi_1(M - B, p_0)$ which contains γ_1^ℓ , γ_2^ℓ and δ^m . Then

$$\pi_1(\mathbb{P}^2 - B, p_0)/J \simeq G_{\ell, m},$$

where

$$G_{\ell,m} = \langle \alpha, \beta, \delta \mid \alpha^\ell = \beta^\ell = \delta^m = 1, \alpha\beta\alpha = \beta\alpha\beta = \delta^{-1} \rangle,$$

$$(\gamma_1 \mapsto \alpha, \gamma_2 \mapsto \beta, \delta \mapsto \delta).$$

Let G_ℓ be the group in Case 1. There is a surjective homomorphism

$$h : G_\ell \longrightarrow G_{\ell,m}$$

defined by $h(a) = \alpha$ and $h(b) = \beta$.

For simplicity, we assume that m is a positive even integer.

Then the following sequence is exact:

$$1 \longrightarrow \langle c^{m/2} \rangle \longrightarrow G_\ell \xrightarrow{h} G_{\ell,m} \longrightarrow 1.$$

In particular, if the pair (ℓ, m) is one of the following table:

ℓ	2	3	4	5
m	2	4	8	20

then $G_{\ell,m} \cong G_\ell$ and there exists a D-universal covering

$$\tilde{\pi} : \tilde{M}(D) \longrightarrow M = \mathbb{P}^2 \quad (D = \ell D_1 + m D_2).$$

In this case, $\tilde{\pi}$ is a finite Galois covering such that $G_{\tilde{\pi}} \cong G_{\ell,m} \cong G_\ell$.

If $\ell = 6$, then we have by Proposition 3.3,

Proposition 3.6. For any positive integer m such that $m \equiv 2 \pmod{4}$, there is a Galois covering $\phi_m : Z_m \longrightarrow \mathbb{P}^2$ of degree $6(m/2)^3$ branching along $D = 6D_1 + mL_\infty$. $\phi_m \leq \phi_{m'}$ if and only if $m|m'$.

On the other hand, since the sequence

$$1 \longrightarrow \langle c \rangle / \langle c^{m/2} \rangle \longrightarrow G_{\ell, m} \longrightarrow G(2, 3, \ell) \longrightarrow 1$$

is exact, we have in particular (putting $m = 2$),

$$G_{\ell, 2} \simeq G(2, 3, \ell).$$

Putting $\ell = 2$, we identify $G_{6, 2}$ with $G(2, 3, 6)$ through the isomorphism. It is well known that $G(2, 3, 6)$ has the normal subgroup L such that

$$G(2, 3, 6)/L \simeq \mathbb{Z}/6\mathbb{Z},$$

$$L \simeq \mathbb{Z} \oplus \mathbb{Z} \text{ (the direct sum).}$$

Identifying L with $\mathbb{Z} + \mathbb{Z}$ through the isomorphism, consider, for any positive integer q , the normal subgroup

$$L_q = \{(j, k) \in \mathbb{Z} \oplus \mathbb{Z} \mid j \equiv k \equiv 0 \pmod{q}\}$$

of index $6q^2$ of $G(2, 3, 6)$. Since

$$\bigcap_q L_q = \{1\},$$

we have

Proposition 3. 7. If $D = 6D_1 + 2L_\infty$, then there does not exist a D -universal covering of $\mathbb{P}^2$.

By another method (see Namba [23]), we can show

Proposition 3. 8. For any positive integer k , there exists a finite Galois covering $\pi : X \longrightarrow \mathbb{P}^2$ branching along $D = 6kD_1 + 2kL_\infty$.

4. Existence of Finite Galois Coverings. As for Problem 2 in §1, it is desirable to give (sufficient) conditions for the existence without using language of fundamental groups. Theorem 1.5 is such a theorem. In this section, we give such theorems.

Let $L_1, \dots, L_s$ be distinct lines on $\mathbb{P}^2$ and put $B = L_1 \cup \dots \cup L_s$. Put

$$\Delta = \{p \in B \mid m_p(B) \geq 3\},$$

where $m_p(B)$ is the multiplicity at p of the curve B . Δ is a finite point set.

Theorem 4. 1. Suppose that $L_j \cap \Delta$ is non-empty for every j ($1 \leq j \leq s$). Then, for any positive integers $e_1, \dots, e_s$ greater than 1, there exists a finite Galois covering $\pi : X \rightarrow \mathbb{P}^2$ branching along $D = e_1 L_1 + \dots + e_s L_s$.

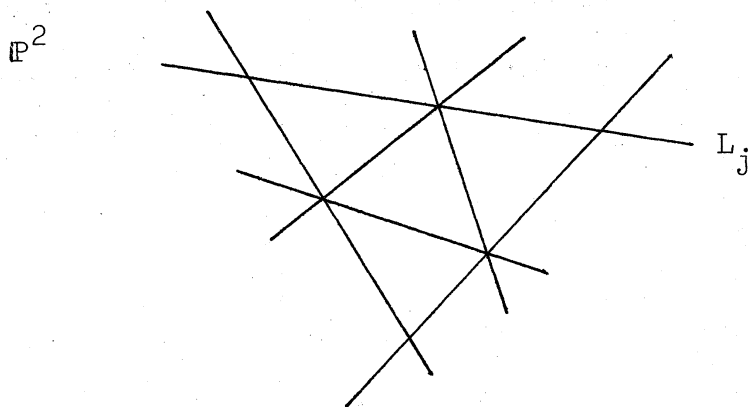


Figure 5

See Kato [16] for the proof of the theorem. We generalize the theorem as follows:

Theorem 4. 2. Let M be an n (≥ 2) dimensional projective

manifold and $D_1, \dots, D_s$ be distinct irreducible hypersurfaces of M . Suppose that there are fixed component free linear pencils $\Lambda_1, \dots, \Lambda_t$ on M such that (i) every D_j is a member of some Λ_k and (ii) every Λ_k contains at least three D_j 's as its members. Then, for any positive integers $e_1, \dots, e_s$ greater than 1, there exists a finite Galois covering $\pi: X \rightarrow M$ branching along $D = e_1 D_1 + \dots + e_s D_s$.

Note that Theorem 4. 1 follows from Theorem 4. 2, putting $M = \mathbb{P}^2$, $D_j = L_j$ ($1 \leq j \leq s$) and Λ_k = the linear pencil given by the projection with the center point $p_k \in \Delta$. See Namba [22] for the proofs of Theorem 4. 2 and the following theorem:

Theorem 4. 3. Let $C_1, \dots, C_s$ be distinct irreducible conics on $\mathbb{P}^2$ such that, for any C_j , there is a C_k which is tangent to C_j at two distinct points. Then, for any positive integers $e_1, \dots, e_s$ greater than 1, there exists a finite Galois covering $\pi: X \rightarrow \mathbb{P}^2$ branching along $D = e_1 D_1 + \dots + e_s D_s$.

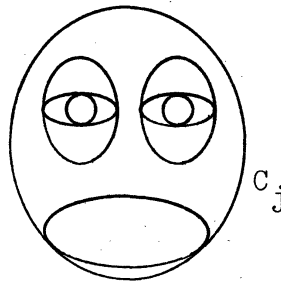
 $\mathbb{P}^2$


Figure 6

Chapter 2. Abelian Coverings.

5. Abelian D-universal coverings. Let M be an n -dimensional connected complex manifold. Let $B = D_1 \cup \dots \cup D_s$, $D = e_1 D_1 + \dots + e_s D_s$, $p_0 \in M - B$, γ_j ($1 \leq j \leq s$) and $J = \langle \gamma_1^{e_1}, \dots, \gamma_s^{e_s} \rangle^{\pi_1}$ be as in §1 and §2. Put

$$\hat{J} = J \cdot [\pi_1(M-B, p_0), \pi_1(M-B, p_0)],$$

where $[G, G]$ is the commutator subgroup of G . Then we can easily prove the following lemma.

Lemma 5. 1. $\pi_1(M - B, p_0)/\hat{J} \cong H_1(M - B; \mathbb{Z})/(\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s)).$

Here $H_1(M - B; \mathbb{Z})$ is the first homology group of $M-B$ and $\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s)$ is the subgroup of $H_1(M - B; \mathbb{Z})$ generated by $e_1 \gamma_1, \dots, e_s \gamma_s$, which are regarded as elements of $H_1(M - B; \mathbb{Z})$.

Moreover, we can prove:

Proposition 5. 2. $\hat{J}$ is a normal subgroup of $\pi_1(M - B, p_0)$ which contains J and is locally cofinite.

The covering $\pi_0 : X_0 \rightarrow M$ which branches at most along D , corresponding to $\hat{J}$ by Theorem 2. 1 is an abelian covering. Moreover, for any abelian covering $\pi : X \rightarrow M$ which branches at most along D , the relation $\pi_0 \geq \pi$ holds.

Definition 5. 3. An abelian covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \rightarrow M$ is called an abelian D-universal covering if (i) $\tilde{\pi}_{ab}$ branches

along D and (ii) for any abelian covering $\pi : X \longrightarrow M$ which branches at most along D , the relation $\tilde{\pi}_{ab} \geq \pi$ holds.

By the above consideration, if an abelian D -universal covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ exists, then it must be isomorphic to $\pi_0 : X_0 \longrightarrow M$. Conversely, if $\pi_0 : X_0 \longrightarrow M$ branches along D , then it is an abelian D -universal covering. Thus

Theorem 5. 4. There exists an abelian D -universal covering $\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ if and only if the following condition is satisfied: if $d\gamma_j \in \mathbb{Z}(e_1\gamma_1) + \dots + \mathbb{Z}(e_s\gamma_s)$, then $d \equiv 0 \pmod{e_j}$ for every $1 \leq j \leq s$. In this case, the covering transformation group of $\tilde{\pi}_{ab}$ is isomorphic to $\tilde{G}_{ab} = H_1(M - B; \mathbb{Z})/(\mathbb{Z}(e_1\gamma_1) + \dots + \mathbb{Z}(e_s\gamma_s))$.

For example, let $M = \mathbb{C}^2$, $B = D_1$ and $D = \ell D_1$ be as in Case 1 of §3. Then we have $H_1(\mathbb{C}^2 - B; \mathbb{Z}) = \mathbb{Z}\gamma_1$ and the condition in Theorem 5. 4 is clearly satisfied. In this case, $\tilde{\pi}_{ab} : \tilde{\mathbb{C}}_{ab}^2(D) \longrightarrow \mathbb{C}^2$ is nothing but the cyclic covering

$$\begin{aligned} \pi_\ell : X_\ell = \{(x, y, z) \in \mathbb{C}^3 \mid z^\ell = x^3 - y^2\} &\longrightarrow \mathbb{C}^2 \\ (x, y, z) &\longmapsto (x, y) \end{aligned}$$

considered in Case 1 of §3.

Let $M = \mathbb{P}^2$, $B = D_1 \cup L_\infty$ and $D = \ell D_1 + mL_\infty$ be as in Case 2 of §3. Then we have

$$H_1(\mathbb{P}^2 - B; \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \mathbb{Z}\gamma_2)/\mathbb{Z}(3\gamma_1 + \gamma_2).$$

Hence the condition in Theorem 5. 4 is equivalent in this case to the condition: $\ell/(3, \ell) = m$, where $(3, \ell)$ is the GCD of 3 and ℓ . If this is the case, $\tilde{\pi}_{ab} : \tilde{\mathbb{P}}_{ab}^2(D) \rightarrow \mathbb{P}^2$ is a finite

covering.

In general, if $M = \mathbb{P}^n$, D_j is an irreducible hypersurface of degree d_j ($1 \leq j \leq s$) and $B = D_1 \cup \dots \cup D_s$, then we have

$H_1(\mathbb{P}^n - B; \mathbb{Z}) = (\mathbb{Z}\gamma_1 + \dots + \mathbb{Z}\gamma_s) / (\mathbb{Z}(d_1\gamma_1 + \dots + d_s\gamma_s))$.
Thus

Theorem 5. 5. Let D_j be distinct irreducible hypersurfaces of degree d_j ($1 \leq j \leq s$) of the complex projective space $\mathbb{P}^n$.

Put $D = e_1 D_1 + \dots + e_s D_s$. Then there exists an abelian D -

universal covering $\tilde{\pi}_{ab} : \widetilde{\mathbb{P}^n_{ab}}(D) \longrightarrow \mathbb{P}^n$ if and only if

$e_j / (d_j, e_j)$ divides

$$\langle e_1 / (d_1, e_1), \dots, e_{j-1} / (d_{j-1}, e_{j-1}), e_{j+1} / (d_{j+1}, e_{j+1}), \dots, e_s / (d_s, e_s) \rangle$$

for every j ($1 \leq j \leq s$), where $(\dots)$ and $\langle \dots \rangle$ denote the

GCD and LCM of the components, respectively. In this case,

$\tilde{\pi}_{ab}$ is a finite covering.

As for a compact Riemann surface M , Theorem 5. 4 can be rewritten as

Theorem 5. 6. Let p_j ($1 \leq j \leq s$) be distinct points on a compact Riemann surface M of genus g . Put $D = e_1 p_1 + \dots + e_s p_s$. Then there exists an abelian D -universal covering

$\tilde{\pi}_{ab} : \tilde{M}_{ab}(D) \longrightarrow M$ if and only if e_j divides

$$\langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_s \rangle$$

for every j ($1 \leq j \leq s$). In this case, $\tilde{\pi}_{ab}$ is an infinite covering if $g \geq 1$.

Finally, as for finite abelian coverings of a complex manifold M , we have

Theorem 5. 7. Let M be a connected complex manifold, $B = D_1 \cup \dots \cup D_s$, $D = e_1 D_1 + \dots + e_s D_s$ and γ_j ($1 \leq j \leq s$) be as before. Then there exists a one-to-one correspondence $\pi \longrightarrow K = K(\pi)$ between isomorphism classes of finite abelian coverings $\pi : X \longrightarrow M$ which branches at most along D , and subgroups K of finite index of

$$\tilde{G}_{ab} = H_1(M - B; \mathbb{Z}) / (\mathbb{Z}(e_1 \gamma_1) + \dots + \mathbb{Z}(e_s \gamma_s)).$$

The correspondence satisfies (1) $G_\pi \simeq \tilde{G}_{ab}/K(\pi)$, (2) $\pi_1 \leq \pi_2$ if and only if $K(\pi_1) \supset K(\pi_2)$ and (3) π branches along D if and only if $K(\pi)$ satisfies the following condition: if $d\gamma_j \in K(\pi)$, then $d \equiv 0 \pmod{e_j}$ for $1 \leq j \leq s$.

6. Finite Abelian Coverings of Projective Manifolds. In this section, we suppose that M is a projective manifold. We discuss finite abelian coverings of M . Here are two typical examples of abelian coverings.

Example 6. 1. Let $\hat{\pi} : L \longrightarrow M$ be a holomorphic line bundle on M and $\xi = \{\xi_\alpha\}$ be a holomorphic section of $L^{\otimes e}$ (the e -times tensor product of L for a positive integer e greater than 1), where ξ_α is a holomorphic function on an open set U_α on which L is trivial. Suppose that the zero divisor (ξ) of ξ has no multiple component:

$$(\xi) = D_1 + \dots + D_s,$$

where D_j are distinct prime divisors. Put

$$D = e(\xi) = eD_1 + \dots + eD_s.$$

Put

$$X = \bigcup_{\alpha} \{(p, z_{\alpha}) \in U_{\alpha} \times \mathbb{C} \mid z_{\alpha}^e = \xi_{\alpha}(p)\}.$$

Then X can be considered as an irreducible normal hypersurface of the bundle space L . Put

$$\pi = \hat{\pi}|_X : X \longrightarrow M.$$

Then π is a cyclic covering which branches along D .

Example 6. 2. Let L be a holomorphic line bundle on M and $\xi_1, \dots, \xi_s$ be holomorphic sections of L . Suppose that $D_1 = (\xi_1), \dots, D_s = (\xi_s)$ are distinct prime divisors such that $D_1 \cap \dots \cap D_s = \emptyset$. For a positive integer e greater than 1, put

$$B = D_1 \cup \dots \cup D_s,$$

$$D = eD_1 + \dots + eD_s.$$

Consider the Kummer extension

$$F = \mathbb{C}(M)((\xi_1/\xi_s)^{1/e}, \dots, (\xi_{s-1}/\xi_s)^{1/e})$$

of the field $\mathbb{C}(M)$ of meromorphic functions on M . Let

$$\pi : X \longrightarrow M$$

be the F-normalization of M , (see Iitaka [14]). Then π is a finite abelian covering of M which branches along D such that $G_{\pi} \simeq (\mathbb{Z}/e\mathbb{Z})^{s-1}$. The covering $\pi : X \longrightarrow M$ is called a Kummer covering. In this case, we can prove that, if B is simple normally crossing, then X is non-singular.

Now, let $B = D_1 \cup \dots \cup D_s$ and $D = e_1 D_1 + \dots + e_s D_s$ be as in §1. We rewrite Theorem 5. 7 using language of rational divisors. A rational D-divisor is a rational divisor $\hat{E}$ on M

of the following type:

$$\hat{E} = (a_1/e_1)D_1 + \dots + (a_s/e_s)D_s + E,$$

where a_j ($1 \leq j \leq s$) are integers and E is an integral divisor. Rational D-divisors form an additive group $\text{Div}^{\mathbb{Q}}(M, D)$. Let $\text{Div}_0^{\mathbb{Q}}(M, D)$ be the subgroup of $\text{Div}^{\mathbb{Q}}(M, D)$ consisting of all $\hat{E}$ such that

$$\begin{aligned} c_{\mathbb{Q}}(\hat{E}) &= (a_1/e_1)c_{\mathbb{Q}}([D_1]) + \dots + (a_s/e_s)c_{\mathbb{Q}}([D_s]) + c_{\mathbb{Q}}(E) \\ &= 0 \in H^2(M; \mathbb{Q}), \end{aligned}$$

where $[D_j]$ is the line bundle determined by D_j and $c_{\mathbb{Q}}: \text{Pic}(M) \rightarrow H^2(M; \mathbb{Q})$ is the homomorphism of rational Chern class.

Two rational D-divisors $\hat{E}$ and $\hat{E}'$ are said to be linearly equivalent, $\hat{E} \sim \hat{E}'$, if $\hat{E} - \hat{E}'$ is an integral and principal divisor. Consider the additive group

$$\text{Pic}_0^{\mathbb{Q}}(M, D) = \text{Div}_0^{\mathbb{Q}}(M, D)/\sim.$$

Theorem 6. 3. There exists a one-to-one correspondence $\pi \rightarrow S = S(\pi)$ between isomorphism classes of finite abelian coverings $\pi: X \rightarrow M$ which branches at most along D , and subgroups S of finite index of $\text{Pic}_0^{\mathbb{Q}}(M, D)$. The correspondence satisfies (1) $G_{\pi} \simeq S(\pi)$ and (2) $\pi_1 \leq \pi_2$ if and only if $S(\pi_1) \subset S(\pi_2)$.

Theorem 6. 4. There exists a finite abelian covering $\pi: X \rightarrow M$ which branches along D if and only if there is a subgroup S of finite index of $\text{Pic}_0^{\mathbb{Q}}(M, D)$ with the following condition: for every j ($1 \leq j \leq s$), there is an element $\hat{E}(j)/\sim \in S$ such that $(a_j, e_j) = 1$, where

$\hat{E}(j) = (a_1/e_1)D_1 + \cdots + (a_j/e_j)D_j + \cdots + (a_s/e_s)D_s + E$,
 (E : an integral divisor).

For the proofs of the above theorems, we make use of the theory of harmonic integrals by de Rham-Kodaira [3].

For example, the cyclic covering $\pi : X \longrightarrow M$ in Example 6.1 corresponds to

$$S = \{(a/e)(D_1 + \cdots + D_s) - aE \mid 0 \leq a \leq e-1\} / \sim,$$

where E is an integral divisor on M such that $[E] = L$.

The Kummer covering $\pi : X \longrightarrow M$ in Example 6.2 corresponds to

$$S = S_{12} + S_{23} + \cdots + S_{n-1,n} + S_{n,1},$$

where

$$S_{12} = \{(a/e)D_1 - (a/e)D_2 \mid 0 \leq a \leq e-1\} / \sim, \text{ etc..}$$

As applications of Theorem 6.4,

Theorem 6.5. Let $D_1, \cdots, D_s$ ($s \geq 2$) be linearly equivalent distinct prime divisors on a projective manifold M . Suppose that, for every j ($1 \leq j \leq s$), e_j divides

$$\langle e_1, \cdots, e_{j-1}, e_{j+1}, \cdots, e_s \rangle.$$

Then there exists a finite abelian covering $\pi : X \longrightarrow M$ which branches along $D = e_1D_1 + \cdots + e_sD_s$.

Theorem 6.6. Let $p_1, \cdots, p_s$ be distinct points of a compact Riemann surface M . Put $D = e_1p_1 + \cdots + e_sp_s$, ($e_j \geq 2$). Then there exists a finite abelian covering $\pi : X \longrightarrow M$ which branches along D if and only if, for every j ($1 \leq j \leq s$), e_j divides

$$\langle e_1, \dots, e_{j-1}, e_{j+1}, \dots, e_s \rangle.$$

Finally, we move D and consider various $\text{Pic}_0^{\mathbb{Q}}(M, D)$'s. Let $\text{Div}^{\mathbb{Q}}(M)$ be the additive group of all rational divisors on M , and $\text{Div}_0^{\mathbb{Q}}(M)$ be the subgroup of $\text{Div}^{\mathbb{Q}}(M)$ consisting of all rational divisors whose rational Chern classes vanish. Two rational divisors $\hat{E}$ and $\hat{E}'$ are said to be linearly equivalent, $\hat{E} \sim \hat{E}'$, if $\hat{E} - \hat{E}'$ is integral and principal. Consider the additive group

$$\text{Pic}_0^{\mathbb{Q}}(M) = \text{Div}_0^{\mathbb{Q}}(M)/\sim.$$

Let $\mathbb{C}(M)$ be the field of meromorphic functions on M . Note that isomorphism classes of finite Galois (resp. abelian) branched coverings $\pi : X \rightarrow M$ and (isomorphism classes of) finite Galois (resp. abelian) extensions $F/\mathbb{C}(M)$ of $\mathbb{C}(M)$ are in one-to-one correspondence under

$$\pi \longrightarrow F = \mathbb{C}(X),$$

$$F \longrightarrow F\text{-normalization of } M.$$

Then, by Theorem 6. 3, we have

Theorem 6. 7. For a projective manifold M , there exists a one-to-one correspondence $F \rightarrow S = S(F)$ between the set of all (isomorphism classes of) finite abelian extensions $F/\mathbb{C}(M)$ and the set of all finite subgroups S of $\text{Pic}_0^{\mathbb{Q}}(M)$. The correspondence satisfies (1) $S(F) \simeq \text{Gal}(F/\mathbb{C}(M))$ and (2) $F_1 \subset F_2$ if and only if $S(F_1) \subset S(F_2)$.

Note that the class field theory for fields of algebraic functions (of one variable) asserts the dual version of this

theorem, using the generalized Jacobian variety, (see Serre [27]).

The content of this section can be generalized to finite Galois coverings of a projective manifold, using language of unitary flat generalized vector bundles, along the line of Weil [30]. See Namba [22] for detail.

7. Equivalence Problem and Automorphism Groups of Kummer Coverings. Let $\pi : X \longrightarrow M$ be a Galois covering of M branching along $D = e_1 D_1 + \dots + e_s D_s$ with the covering transformation group $G = G_\pi$. In this case, we also write in this section

$$\pi : (G, X) \longrightarrow (M : D).$$

For a second Galois covering $\pi' : (G', X') \longrightarrow (M' : D')$, a biholomorphism $h : X \longrightarrow X'$ is referred to as an equivalence, written $h : \pi \approx \pi'$, if there is a biholomorphism $\bar{h} : M \longrightarrow M'$ making a commutative diagram:

$$\begin{array}{ccc} X & \xrightarrow{h} & X' \\ \pi \downarrow & & \downarrow \pi' \\ M & \xrightarrow{\bar{h}} & M' \end{array}$$

For a biholomorphism $h : X \longrightarrow X'$, if we put

$$G'^h = \{h^{-1}g'h \mid g' \in G'\},$$

Then we have

$$h : \pi \approx \pi' \iff G = G'^h.$$

Let $E(\pi)$ be the subgroup of $\text{Aut}(X)$ consisting of equivalences of π onto itself. We have an obvious short exact sequence:

$$\{1\} \longrightarrow G \longrightarrow E(\pi) \longrightarrow \text{Aut}(M, D),$$

where $\text{Aut}(M, D) = \{f \in \text{Aut}(M) \mid f^*D = D\}$.

Definition 7.1. A Galois covering $\pi : X \longrightarrow M$ is said to be rigid, if $E(\pi) = \text{Aut}(X)$.

Equivalence Problem. Are which kinds of Galois coverings rigid?

This problem in the case of cyclic branched coverings of $\mathbb{P}^1$ was proposed by H. Shiga in Wakabayashi's problem session, Wakabayashi [29].

The second named author, Namba [21] showed that cyclic branched coverings of $\mathbb{P}^1$ are rigid under some conditions. Moreover, by making use of a theorem of Matsumura-Monsky, he proved that an m -fold cyclic covering $\pi : X \longrightarrow \mathbb{P}^n$ branching along a non-singular hypersurface of degree m in $\mathbb{P}^n$ is rigid, provided that

- (i) $m \geq 4$, if $n = 1$,
- (ii) $m \geq 3$, if $n \geq 2$, and
- (iii) $(m, n) \neq (4, 2)$.

T. Kato [19] improved the results of Namba in the case of cyclic branched coverings of $\mathbb{P}^1$.

Let $L = L_1 + \cdots + L_s$ be a reduced divisor of $\mathbb{P}^n$ consisting of s distinct hyperplanes $L_1, \dots, L_s$, which will be referred to as a hyperplane configuration of $\mathbb{P}^n$.

A Kummer covering

$$\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$$

of $\mathbb{P}^n$ branching along mL is nothing but a branched covering

obtained as the Fox completion of a covering spread

$X_0 \longrightarrow \mathbb{P}^n - L \subset \mathbb{P}^n$ associated with a $\mathbb{Z}/m\mathbb{Z}$ -Hurewicz homomorphism

$$\pi_1(\mathbb{P}^n - L, *) \longrightarrow H_1(\mathbb{P}^n - L; \mathbb{Z}) \longrightarrow H_1(\mathbb{P}^n - L; \mathbb{Z}/m\mathbb{Z}).$$

Thus

$$G \simeq H_1(\mathbb{P}^n - L; \mathbb{Z}/m\mathbb{Z}) = (\mathbb{Z}/m\mathbb{Z})^{s-1}$$

and G is generated by covering transformations $g_1, \dots, g_s$ corresponding to the normal loops $\gamma_1, \dots, \gamma_s$ of $L_1, \dots, L_s$, respectively.

We are interested in the case where $n = 2$.

Let q be an r -ple point of L ; $q = L_{i_1} \cap \dots \cap L_{i_r}$.

Let

$$\phi : B_q(\mathbb{P}^2) \longrightarrow \mathbb{P}^2$$

be the blowing up of $\mathbb{P}^2$ at q . Then $\phi^{-1}(q) = E$ is a non-singular rational curve and we have a reduced divisor

$$p_1 + \dots + p_r$$

on E , where

$$p_k = \overline{(\phi^* L_k - E)} \cap E$$

for $k = 1, \dots, r$.

Definition 7. 2. If a Kummer covering of E branching along $m(p_1 + \dots + p_r)$ is rigid, then $(\mathbb{P}^2 : mL)$ is said to be rigid at q . We shall say that $(\mathbb{P}^2 : mL)$ is locally rigid, if for each r -ple ($r \geq 4$) point q of L , $(\mathbb{P}^2 : mL)$ is rigid at q .

In M. Kato [18], the first named author proved essentially

Theorem 7. 3. Let $\pi : (G, X) \longrightarrow (\mathbb{P}^2 : mL)$ and $\pi' : (G', X') \longrightarrow (\mathbb{P}^2 : mL')$ be kummer coverings of $\mathbb{P}^2$ such that L and L' are line configurations of $\mathbb{P}^2$. Suppose that

- (1) $m \geq 6$,
- (2) each L_j contains at least three singular points of L and
- (3) $(\mathbb{P}^2 : mL)$ is locally rigid.

If a biholomorphism $h : X \longrightarrow X'$ exists, then $h : \pi \approx \pi'$. In particular, $\pi : X \longrightarrow \mathbb{P}^2$ is rigid.

Since the Kummer covering $\pi : (G, X) \longrightarrow (\mathbb{P}^n : mL)$ is an abelian mL -universal covering, it follows that a natural homomorphism

$$E(\pi) \longrightarrow \text{Aut}(\mathbb{P}^n, L) (= \text{Aut}(\mathbb{P}^n, mL), (m > 0))$$

is surjective. Thus we have

Corollary 7. 4. Under the assumption of Theorem 7. 3, we have a short exact sequence:

$$\{1\} \longrightarrow G(\simeq (\mathbb{Z}/m\mathbb{Z})^{s-1}) \longrightarrow \text{Aut}(X) \longrightarrow \text{Aut}(\mathbb{P}^2, L) \longrightarrow \{1\}.$$

The following results about rigidity of a Kummer covering $\pi : X \longrightarrow \mathbb{P}^1$ branching along $m(p_1 + \cdots + p_s)$ are known:

- Theorem 7. 5. (1) if $\chi(X) \geq 0$, i.e., either $s = 2$ or $s = 3$ and $m \leq 3$, then π is not rigid.
- (2) if $s = 3$ and $m \geq 4$, then π is rigid (see Namba [21]).
- (3) if $s \geq 4$ and $m \geq 5(s - 1)$, then π is rigid (see M. Kato [18]).

Theorem 7. 6. (Sakurai-Suzuki [26]). Suppose that $\chi(X) < 0$, $s \geq 4$ and that for any subset P' of $\{p_1, \dots, p_s\}$ with $\#P' \geq 4$, $\text{Aut}(P^1, p') = \{1\}$. Then π is rigid.

Remark 7. 7. Recently, Sakurai is improving the result above extensively. He has announced in February, 1987, that π is rigid, if $\chi(X) < 0$ and $m \geq 11$. It is plausible that π is rigid, if $\chi(X) < 0$, i. e., $\text{Aut}(X)$ is finite.

The proof of Theorem 7. 3 is based on the following facts:

- (I) If X is a surface of general type, then $\text{Aut}(X)$ is finite.
- (II) The covering transformation group G is generated by 'complex reflections' $g_1, \dots, g_s$ of the surface X .
- (III) If a finite unitary reflection group of $\mathbb{C}^2$ contains a unitary reflection of order ≥ 6 , then it is abelian, refer to Shephard-Todd [28].

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