

On the Spin-Boson Model

Asao Arai (新井朝雄)*

Department of Mathematics, Hokkaido University, Sapporo 060, Japan

Masao Hirokawa (廣川真男)

Advanced Research Laboratory, Hitachi Ltd., Hatoyama, Saitama 350-03, Japan

The existence and uniqueness of ground states of the spin-boson Hamiltonian H_{SB} are considered. The main results in the case of massive bosons include: (i)(existence) there exists a ground state *without restriction for the strength of the coupling constant*; (ii)(uniqueness) under a mild (nonperturbative) condition for the parameters contained in H_{SB} , H_{SB} has only one ground state; (iii) (degeneracy) under a certain condition for the parameters of H_{SB} which is weaker than that of (ii), the number of the ground states is at most two. In the case of massless bosons, the existence of a ground state of H_{SB} is shown as a limit of ground states of the massive case. The methods used are *nonperturbative*. A generalization of the model is proposed.

Contents

- 1 . Introduction and the main results
- 2 . Some basic facts
- 3 . A finite volume approximation
- 4 . Convergence of the finite volume approximation
- 5 . Proof of the main results
 - 5.1. Proof of Theorem 1.1
 - 5.2. Proof of Theorem 1.2
 - 5.3. Proof of Theorem 1.3
 - 5.4. Proof of Theorem 1.4
- 6 . A generalization of the model

1. Introduction and the main results

The spin-boson model, which describes a two-level quantum system coupled to a quantized Bose field, has been investigated as a simplified model for atomic systems interacting

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with a quantized radiation or phonon field ([1, 2, 5, 6, 7, 9, 14] and references therein). The ground states of the model are of particular interest. Spohn [14] discussed properties of ground states defined as zero-temperature limits of positive temperature equilibrium states. Analysis related to the work of Spohn was made by Amann [1] in terms of the notion of algebraic ground states, although it treats only a discrete version of the model. Recently attention has been paid to the ground states as the eigenvectors of the Hamiltonian H_{SB} of the model with eigenvalue equal to the infimum of its spectrum to analyze spectral properties of H_{SB} and the process of radiative decay in the model [8, 9]. In [8] Hübner and Spohn showed that, under certain conditions for the dispersion ω for bosons, the coupling function, the coupling constant α and the spectral gap μ of the unperturbed two-level system, there exists a unique ground state of H_{SB} and identify the spectrum of H_{SB} .

In this paper we focus our attention on the existence and uniqueness of ground states of the spin-boson Hamiltonian H_{SB} . We first consider the case where the bosons are massive (i.e., $m := \inf_k \omega(k) > 0$) and show that, *as far as the existence of the ground states is concerned, no restriction is needed for the coupling constant α* , which greatly improves the result on the existence of ground states in [8] (in the massive case). The basic idea to do it is as follows: we first do a unitary transformation for H_{SB} to convert it to an operator more tractable in a sense and then apply the method of constructive quantum field theory [7] to the latter operator. Moreover, by employing the min-max principle, under an additional condition for the parameters m, μ and α , which is nonperturbative, we show that H_{SB} has a unique ground state. We also suggest the possibility for H_{SB} to have degenerate ground states by showing that, under a weaker condition for m, μ and α , there exist at most two ground states of H_{SB} . In the case of massless bosons (i.e., $m = 0$), we construct a ground state as a weak limit of ground states in the massive case.

We now describe our main results. For mathematical generality, we consider the situation where bosons move in the ν -dimensional Euclidean space $\mathbb{R}^\nu$ with $\nu \geq 1$. We take the Hilbert space of bosons to be

$$\mathcal{F} = \mathcal{F}(L^2(\mathbb{R}^\nu)) = \bigoplus_{n=0}^{\infty} [\otimes_s^n L^2(\mathbb{R}^\nu)], \quad (1.1)$$

the symmetric Fock space over $L^2(\mathbb{R}^\nu)$ ($\otimes_s^n \mathcal{K}$ denotes the n -fold symmetric tensor product of a Hilbert space $\mathcal{K}$, $\otimes_s^0 \mathcal{K} := \mathbb{C}$). Let ω and λ be functions on $\mathbb{R}^\nu$ satisfying the following conditions

- (A.1) For all $k \in \mathbb{R}^\nu$, $\omega(k) \geq 0$ and there exist constants $\gamma > 0$ and $C > 0$ such that

$$|\omega(k) - \omega(k')| \leq C|k - k'|^\gamma, \quad k, k' \in \mathbb{R}^\nu. \quad (1.2)$$

- (A.2) The function λ is real-valued and continuous with $\lambda, \lambda/\sqrt{\omega}, \lambda/\omega \in L^2(\mathbb{R}^\nu)$ and there exist constants $q > \nu/2$ and $K_0 > 0$ such that, for all $|k| \geq K_0$,

$$\left| \frac{\lambda(k)}{\omega(k)} \right| \leq \frac{D}{1 + |k|^q}$$

with D a constant (which may depend on q and K_0).

Throughout this paper, we assume (A.1) and (A.2).

A typical example of ω satisfying (A.1) is $\omega(k) = \sqrt{|k|^2 + m_0^2}$ with $m_0 \geq 0$ a constant.

We denote by $d\Gamma(\omega)$ the second quantization of the multiplication operator ω on $L^2(\mathbb{R}^\nu)$ and set

$$H_b = d\Gamma(\omega) = \int d^\nu k \omega(k) a(k)^* a(k), \quad (1.3)$$

where $a(k)$ is the operator-valued distribution kernel of the smeared annihilation operator $a(f) = \int a(k) f(k)^* d^\nu k$ ($f \in L^2(\mathbb{R}^\nu)$) on $\mathcal{F}$ (f^* denotes the complex conjugate of f). The Hamiltonian of the spin-boson model is defined by

$$H_{\text{SB}} = \frac{1}{2} \mu \sigma_z \otimes I + I \otimes H_b + \alpha \sigma_x \otimes (a(\lambda)^* + a(\lambda)) \quad (1.4)$$

acting in the Hilbert space

$$\mathcal{H} = \mathbb{C}^2 \otimes \mathcal{F} = \mathcal{F} \oplus \mathcal{F}, \quad (1.5)$$

where σ_x, σ_z are the standard Pauli matrices, $\mu > 0$ and $\alpha \in \mathbb{R}$ are constants denoting the spectral gap of the unperturbed two-level system and the coupling constant, respectively, and I denotes identity.

For a linear operator T on a Hilbert space, we denote its domain by $D(T)$. It is well known that H_{SB} is self-adjoint with $D(H_{\text{SB}}) = D(I \otimes H_b)$ and

$$H_{\text{SB}} \geq -\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2, \quad (1.6)$$

where $\|\cdot\|_{L^2}$ denotes the norm of $L^2(\mathbb{R}^\nu)$.

For a self-adjoint operator T bounded from below, we denote by $E(T)$ the infimum of the spectrum $\sigma(T)$ of T :

$$E(T) = \inf \sigma(T). \quad (1.7)$$

In this paper, an eigenvector of T with eigenvalue $E(T)$ is called a *ground state* of T (if it exists). We say that T has a (resp. unique) ground state if $\dim \ker(T - E(T)) \geq 1$ (resp. $\dim \ker(T - E(T)) = 1$).

The following estimate for $E(H_{\text{SB}})$ is well known (see (2.10) below) :

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H_{\text{SB}}) \leq -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega\|_{L^2}^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2. \quad (1.8)$$

Let

$$m := \inf_{k \in \mathbb{R}^\nu} \omega(k) \quad (1.9)$$

We have the following result on the existence of ground states of H_{SB} :

THEOREM 1.1. Assume (A.1), (A.2) and $m > 0$. Then H_{SB} has purely discrete spectrum in the interval $[E(H_{\text{SB}}), E(H_{\text{SB}}) + m)$. In particular, H_{SB} has a ground state.

Remark: Theorem 1.1 implies that, under the same assumption, $\inf \sigma_{\text{ess}}(H_{\text{SB}}) \geq E(H_{\text{SB}}) + m$, where $\sigma_{\text{ess}}(\cdot)$ denotes essential spectrum, i.e., H_{SB} has a spectral gap. In a forthcoming paper, we shall show that, in fact, $\sigma_{\text{ess}}(H_{\text{SB}}) = [E(H_{\text{SB}}) + m, \infty)$.

To state our result on the uniqueness of ground states, we introduce

$$K_\varepsilon(\alpha, \mu) = \min \left\{ m(1 - \varepsilon), \frac{\mu}{2} \right\} - \frac{4\alpha^2 \mu^2}{\varepsilon} \left\| \frac{\lambda}{\omega \sqrt{\omega}} \right\|_{L^2}^2 - 2|\alpha| \mu \left\| \frac{\lambda}{\omega} \right\|_{L^2}, \quad (1.10)$$

with λ such that $\lambda/\omega\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$.

Remark: If $m > 0$, then $\lambda \in L^2(\mathbb{R}^\nu)$ implies that, for all $s > 0$, $\lambda/\omega^s \in L^2(\mathbb{R}^\nu)$.

THEOREM 1.2. Assume (A.1), (A.2) and $m > 0$. Suppose that

$$\sup_{0 < \varepsilon < 1} K_\varepsilon(\alpha, \mu) > \frac{\mu}{2} \left(1 - e^{-2\alpha^2 \|\lambda/\omega\|_{L^2}^2} \right) \quad (1.11)$$

Then H_{SB} has a unique ground state.

Remark: By applying regular perturbation theory (e.g., [12, Chapt.XII]), one can easily show that there exists a constant $\alpha_0 > 0$ such that, for all $\alpha \in (-\alpha_0, \alpha_0)$, H_{SB} has a unique ground state. For arbitrarily fixed $m > 0$ and $\mu > 0$, (1.11) is satisfied if $|\alpha|$ is sufficiently small. Thus Theorem 1.2 may be regarded as a result which improves the one obtained by regular perturbation theory. Note that (1.11) is a nonperturbative estimate in α , since the right hand side (RHS) of (1.11) is non-polynomial in α . We believe that (1.11) is a relatively good estimate to ensure the uniqueness of ground states of H_{SB} (see the proof of Theorem 1.2 in §5.2).

As is easily seen, in the case $\mu = 0$, H_{SB} has two-fold degenerate ground states. This fact suggests that H_{SB} with $\mu > 0$ also may have degenerate ground states. In this respect, we have the following result:

THEOREM 1.3. Assume (A.1), (A.2) and $m > 0$. Suppose that

$$m > \frac{\mu}{2} \left(1 - e^{-2\alpha^2 \|\lambda/\omega\|_{L^2}^2} \right). \quad (1.12)$$

Then the following (a) and (b) hold:

- (a) There are at most two eigenvalues (counting multiplicity) of H_{SB} in the interval $[E(H_{\text{SB}}), -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega\|_{L^2}^2} - \alpha^2 \|\lambda/\sqrt{\omega}\|_{L^2}^2]$.
- (b) The Hamiltonian H_{SB} has at most two ground states, i.e., $\dim \ker(H_{\text{SB}} - E(H_{\text{SB}})) \leq 2$.

Remark: Condition (1.11) implies (1.12), i.e., the latter condition is weaker than the former.

In the case of *massless bosons*, we have the following result on the existence of ground states of H_{SB} :

THEOREM 1.4. *Assume (A.1), (A.2) and $m = 0$. Suppose, in addition, that $\omega\lambda \in L^2(\mathbb{R}^\nu)$ and*

$$|\alpha| < \frac{1}{\|\lambda/\omega\|_{L^2}}. \quad (1.13)$$

Then H_{SB} has a ground state.

Remark: To our best knowledge, Theorem 1.4 is the first which establishes the existence of ground states of the spin-boson Hamiltonian H_{SB} in the case of *massless bosons*.

The present paper is organized as follows. In Section 2 we review some basic facts on the spin-boson Hamiltonian H_{SB} . We recall a well known unitary transformation which converts H_{SB} to an operator H simpler in a sense. We analyze the operator H . To prove the existence of ground states of H , we introduce in Section 3 a finite volume approximation H_V ($V > 0$) for H . In Section 4 we prove that H_V converges to H in the norm resolvent sense as $V \rightarrow \infty$. In Section 5 we prove Theorems 1.1 – 1.4. In the last section we propose a generalization of the model.

2. Some basic facts

It is well known that, for all $f \in L^2(\mathbb{R}^\nu)$, the operator

$$P(f) := i\{a(f)^* - a(f)\} \quad (2.1)$$

is essentially self-adjoint on the finite particle subspace

$$\mathcal{F}_0 = \{\Psi = \{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{F} \mid \text{only finitely many } \Psi_n \text{'s are not zero}\}. \quad (2.2)$$

We denote the closure of $P(f)$ by the same symbol. Let

$$U_\pm = e^{\pm i\alpha P(\lambda/\omega)}. \quad (2.3)$$

Then

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} U_+ & U_- \\ U_+ & -U_- \end{pmatrix} \quad (2.4)$$

is unitary on $\mathcal{H}$. Moreover, we have

$$U^{-1}H_{\text{SB}}U = H - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \quad (2.5)$$

with

$$H = I \otimes H_b + \frac{\mu}{2}(A \otimes U_+^2 + A^* \otimes U_-^2), \quad (2.6)$$

where

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \quad (2.7)$$

Based on (2.5), we shall consider, instead of H_{SB} , the operator H defined by (2.6). An advantage of this approach is in that the perturbation term

$$H_I := \frac{\mu}{2}(A \otimes U_+^2 + A^* \otimes U_-^2) \quad (2.8)$$

of H is a bounded self-adjoint operator. The operator norm $\|H_I\|$ of H_I can be exactly computed:

LEMMA 2.1. *We have*

$$\|H_I\| = \frac{\mu}{2}. \quad (2.9)$$

PROOF: We need only to use the relation $H_I = \frac{\mu}{2}U^{-1}(\sigma_z \otimes I)U$ and the fact $\|\sigma_z \otimes I\| = 1$. ■

It follows from (2.9) and the variational principle (cf. [2, 4]) that

$$-\frac{\mu}{2} \leq E(H) \leq -\frac{\mu}{2}e^{-2\alpha^2\|\lambda/\omega\|_{L^2}^2} < 0. \quad (2.10)$$

LEMMA 2.2. *Assume, in addition to (A.1) and (A.2), that $\omega\lambda \in L^2(\mathbb{R}^\nu)$. Let Ψ be any eigenvector of H_{SB} . Then $\Psi \in D((I \otimes H_b)^{3/2})$.*

PROOF: By the assumption, we have $H_{\text{SB}}\Psi = E\Psi$, $\Psi \in D(H_{\text{SB}}) = D(I \otimes H_b)$ with E an eigenvalue of H_{SB} . Hence

$$(I \otimes H_b)\Psi = E\Psi - \frac{\mu}{2}(\sigma_z \otimes I)\Psi - \alpha\sigma_x \otimes [a(\lambda)^* + a(\lambda)]\Psi.$$

The vectors on the RHS except for the last one is in $D(I \otimes H_b)$. We denote by $a(\cdot)^\#$ either $a(\cdot)^*$ or $a(\cdot)$. It is known that, if $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$, then $a^\#(f)$ maps $D(H_b)$ into $D(H_b^{1/2})$ [3, Lemma 2.4]. Hence $\sigma_x \otimes [a(\lambda)^* + a(\lambda)]\Psi \in D((I \otimes H_b)^{1/2})$. Thus we conclude that $(I \otimes H_b)\Psi \in D((I \otimes H_b)^{1/2})$, which implies the desired result. ■

Let

$$N = d\Gamma(I) = \int d^\nu k a(k)^* a(k), \quad (2.11)$$

the number operator on $\mathcal{F}$.

In general we denote by $(\cdot, \cdot)_\mathcal{K}$ and $\|\cdot\|_\mathcal{K}$ the inner product (complex linear in the second variable) and the norm of a Hilbert space $\mathcal{K}$, respectively, but, we sometimes omit the subscript $\mathcal{K}$ if there is no danger of confusion.

LEMMA 2.3. Assume, in addition to (A.1) and (A.2), that $\omega\lambda \in L^2(\mathbb{R}^\nu)$. Then, for every normalized ground state Ω of H_{SB} ,

$$(\Omega, I \otimes N\Omega)_{\mathcal{H}} \leq \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2. \quad (2.12)$$

PROOF: Let f be a function such that $\omega f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ (then $f \in L^2(\mathbb{R}^\nu)$). It follows from Lemma 2.2 and a mapping property of $a(f)^\#$ [3, Lemma 2.3] that $a(f)\Omega \in D(I \otimes H_b) = D(H_{\text{SB}})$. Since $H_{\text{SB}} - E(H_{\text{SB}}) \geq 0$, we have

$$\begin{aligned} 0 &\leq (I \otimes a(f)\Omega, [H_{\text{SB}} - E(H_{\text{SB}})] I \otimes a(f)\Omega) \\ &= (I \otimes a(f)\Omega, [H_{\text{SB}}, I \otimes a(f)]\Omega) \\ &= (I \otimes a(f)\Omega, (-I \otimes a(\omega f) - \alpha(\sigma_x \otimes I)(f, \lambda)_{L^2})\Omega). \end{aligned}$$

Hence

$$(\Omega, I \otimes a(f)^* a(\omega f)\Omega) + \alpha(f, \lambda)_{L^2}(\sigma_x \otimes a(f)\Omega, \Omega) \leq 0. \quad (2.13)$$

There exists a sequence $\{f_n\}_{n=1}^\infty$ of functions such that $\omega f_n, f_n/\sqrt{\omega} \in L^2(\mathbb{R}^\nu)$ for all $n \geq 1$ and $\{\sqrt{\omega} f_n\}_{n=1}^\infty$ is a complete orthonormal system of $L^2(\mathbb{R}^\nu)$. By (2.13), we have for all $N = 1, 2, 3, \dots$

$$\sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n)\Omega) + \alpha(\sigma_x \otimes a(F_N)\Omega, \Omega) \leq 0,$$

where $F_N = \sum_{n=1}^N (f_n, \lambda)_{L^2} f_n$. It is not so difficult to show that

$$\begin{aligned} \lim_{N \rightarrow \infty} \sum_{n=1}^N (\Omega, I \otimes a(f_n)^* a(\omega f_n)\Omega) &= (\Omega, I \otimes N\Omega), \\ \lim_{N \rightarrow \infty} (\sigma_x \otimes a(F_N)\Omega, \Omega) &= (\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega). \end{aligned}$$

Hence $(\Omega, I \otimes N\Omega) + \alpha(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega) \leq 0$. Since $(\Omega, I \otimes N\Omega) \geq 0$, it follows that $(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega)$ is real and

$$(\Omega, I \otimes N\Omega) \leq -\alpha \left(\sigma_x \otimes a\left(\frac{\lambda}{\omega}\right)\Omega, \Omega \right). \quad (2.14)$$

Applying the well known estimate

$$\|a(f)\Psi\|_{\mathcal{F}} \leq \|f\|_{L^2} \|N^{1/2}\Psi\|_{\mathcal{F}}, \quad f \in L^2(\mathbb{R}^\nu), \Psi \in D(N^{1/2}), \quad (2.15)$$

to the RHS of (2.14), we obtain

$$(\Omega, I \otimes N\Omega) \leq |\alpha| \left\| \frac{\lambda}{\omega} \right\|_{L^2} \|(I \otimes N)^{1/2}\Omega\|,$$

which implies (2.12). ■

Inequality (2.12) gives an upper bound for the mean of boson numbers in any normalized ground state of H_{SB} . Note that inequality (2.12) is independent of whether bosons are massive or massless.

3. A finite volume approximation

Let $V > 0$ be a parameter and

$$\Gamma_V = \frac{2\pi\mathbb{Z}^\nu}{V} = \left\{ k = (k_1, \dots, k_\nu) \mid k_j = \frac{2\pi n_j}{V}, n_j \in \mathbb{Z}, j = 1, \dots, \nu \right\}. \quad (3.1)$$

Let

$$\mathcal{F}_V = \mathcal{F}(\ell^2(\Gamma_V)) = \bigoplus_{n=0}^{\infty} [\otimes_s^n \ell^2(\Gamma_V)] \quad (3.2)$$

the symmetric Fock space over $\ell^2(\Gamma_V)$, which describes state vectors of bosons in the finite box $[-V/2, V/2]^\nu$. Each element Ψ in $\otimes_s^n \ell^2(\Gamma_V)$ can be identified with a piecewise constant function in $\otimes_s^n L^2(\mathbb{R}^\nu)$ which is a constant on each cube of volume $(2\pi/V)^{n\nu}$ centered about a lattice point

$$(k_1, \dots, k_n) \in \Gamma_V \times \dots \times \Gamma_V = \Gamma_V^n.$$

With this identification, $\mathcal{F}_V$ is regarded as a closed subspace of $\mathcal{F}$.

For each $k = (k_1, \dots, k_\nu) \in \Gamma_V$, we define a function $\chi_{k,V}$ on $\mathbb{R}^\nu$ by

$$\chi_{k,V}(\ell) = \chi_{[k_1 - \frac{\pi}{V}, k_1 + \frac{\pi}{V}]}(\ell_1) \cdots \chi_{[k_\nu - \frac{\pi}{V}, k_\nu + \frac{\pi}{V}]}(\ell_\nu), \quad \ell = (\ell_1, \dots, \ell_\nu) \in \mathbb{R}^\nu, \quad (3.3)$$

where $\chi_{[a,b]}$ denotes the characteristic function of the interval $[a,b]$. We introduce

$$a_V(k) := \left(\frac{V}{2\pi} \right)^{\nu/2} a(\chi_{k,V}) = \left(\frac{V}{2\pi} \right)^{\nu/2} \int_{[-\pi/V, \pi/V]^\nu} a(k + \ell) d\ell. \quad (3.4)$$

It is easy to see that, for all $k, \ell \in \Gamma_V$,

$$[a_V(k), a_V(\ell)^*] = \delta_{k\ell}, \quad [a_V(k), a_V(\ell)] = 0, \quad (3.5)$$

on $\mathcal{F}_0$.

We define

$$\omega_V(k) = \omega(k_V), \quad k \in \mathbb{R}^\nu, \quad (3.6)$$

with k_V a lattice point closed to k :

$$k_V \in \Gamma_V, \quad |k_j - (k_V)_j| \leq \frac{\pi}{V}, \quad j = 1, \dots, \nu. \quad (3.7)$$

Let

$$H_{b,V} := d\Gamma(\omega_V) = \int d^\nu k \omega_V(k) a(k)^* a(k). \quad (3.8)$$

LEMMA 3.1. We have

$$D(H_{b,V}) = D(H_b) \quad (3.9)$$

and there exists a constant $c > 0$ independent of V such that, for all $\Psi \in D(N)$,

$$\|(H_b - H_{b,V})\Psi\| \leq \frac{c}{V^\gamma} \|N\Psi\|. \quad (3.10)$$

PROOF: By (1.2) and (3.7), we have for all $k \in \mathbb{R}^\nu$, $|\omega(k) - \omega(k_V)| \leq c/V^\gamma$ with $c = C\pi^\gamma \nu^{\gamma/2}$, from which (3.9) and (3.10) follow. ■

The following fact is well known:

LEMMA 3.2. The operator $H_{b,V}$ is reduced by $\mathcal{F}_V$ and

$$H_{b,V} \upharpoonright \mathcal{F}_V = \sum_{k \in \Gamma_V} \omega(k) a_V(k)^* a_V(k).$$

For notational simplicity, we set

$$g(k) = \frac{\alpha \lambda(k)}{\omega(k)}. \quad (3.11)$$

For $K > 0$, we define a function $g_{K,V}$ on $\mathbb{R}^\nu$ by

$$g_{K,V} = \sum_{k \in \Gamma_V, |k_j| \leq K, j=1, \dots, \nu} g(k) \chi_{k,V}.$$

LEMMA 3.3. The function $g_{K,V}$ converges in $L^2(\mathbb{R}^\nu)$ as $K \rightarrow \infty$.

PROOF: For a constant $K > 0$, we put

$$S_{K,V} = \sum_{k \in \Gamma_V, |k_j| \leq K, j=1, \dots, \nu} \left(\frac{2\pi}{V} \right)^\nu |g(k)|^2$$

Then, by the growth condition for λ/ω in (A.2), we have

$$\begin{aligned} S_{K,V} &\leq \sum_{k \in \Gamma_V, |k| \leq K_0} \left(\frac{2\pi}{V} \right)^\nu |g(k)|^2 + \alpha^2 D^2 \sum_{k \in \Gamma_V, |k| \geq K_0} \left(\frac{2\pi}{V} \right)^\nu \frac{1}{(1 + |k|^q)^2} \\ &\leq \sum_{k \in \Gamma_V, |k| \leq K_0} \left(\frac{2\pi}{V} \right)^\nu |g(k)|^2 + \alpha^2 D^2 \int_{\mathbb{R}^\nu} \frac{1}{(1 + |k|^q)^2} dk < \infty. \end{aligned}$$

Hence $S_{K,V}$ is uniformly bounded in K . Since $S_{K,V}$ is monotone non-decreasing in K , it follows that the infinite series $S_V := \sum_{k \in \Gamma_V} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2$ converges. Let $K' \geq K$. Then we have $(g_{K,V}, g_{K',V})_{L^2} = S_{K,V} \rightarrow S_V$ ($K \rightarrow \infty$), which implies that $\{g_{K,V}\}_K$ is a Cauchy net. ■

We write

$$g_V = L^2 - \lim_{K \rightarrow \infty} g_{K,V} = \sum_{k \in \Gamma_V} g(k) \chi_{k,V}. \quad (3.12)$$

Then we have

$$P(g_V) = i \left(\frac{2\pi}{V}\right)^{\nu/2} \sum_{k \in \Gamma_V} g(k) (a_V(k)^* - a_V(k)) \quad (3.13)$$

on $\mathcal{F}_0$.

Let

$$U_\pm(V) = e^{\pm i P(g_V)}. \quad (3.14)$$

and

$$H_V = I \otimes H_{b,V} + \frac{\mu}{2} \{A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2\}. \quad (3.15)$$

LEMMA 3.4. *The operator H_V is self-adjoint with $D(H_V) = D(I \otimes H_b)$ and bounded from below with*

$$H_V \geq -\frac{\mu}{2}. \quad (3.16)$$

PROOF: Since the operator

$$H_I(V) := \frac{\mu}{2} \{A \otimes U_+(V)^2 + A^* \otimes U_-(V)^2\} \quad (3.17)$$

is bounded, the Kato-Rellich theorem gives the self-adjointness of H_V with $D(H_V) = D(I \otimes H_{b,V}) = D(I \otimes H_b)$ (Lemma 3.1). Inequality (3.16) follows from the fact $\|H_I(V)\| = \frac{\mu}{2}$, which can be proven in the same way as in Lemma 2.1. ■

In the next section, we show that H_V is a finite volume approximation for H in a suitable sense.

4. Convergence of the finite volume approximation

In this section we prove the following theorem:

THEOREM 4.1. *For all $z \in \mathbb{C}$ with $\text{Im } z \neq 0$ or $z < -\mu/2$,*

$$\lim_{V \rightarrow \infty} \|(H_V - z)^{-1} - (H - z)^{-1}\| = 0. \quad (4.1)$$

To prove this theorem, we prepare some lemmas.

LEMMA 4.2.

$$\lim_{V \rightarrow \infty} \|g_V - g\|_{L^2} = 0. \quad (4.2)$$

PROOF: By the growth condition for λ/ω in (A.2), one can easily show that

$$\|g_V\|_{L^2}^2 = \sum_{k \in \Gamma_V} \left(\frac{2\pi}{V}\right)^\nu |g(k)|^2 \rightarrow \int_{\mathbb{R}^\nu} d^\nu k |g(k)|^2 = \|g\|_{L^2}^2 \quad (V \rightarrow \infty). \quad (4.3)$$

Let $f \in C_0^\infty(\mathbb{R}^\nu)$ and $\text{supp } f \subset \{k \in \mathbb{R}^\nu \mid |k_j| \leq K_f, j = 1, \dots, \nu\}$ with a constant K_f . Then we have

$$(f, g_V)_{L^2} = \sum_{\ell \in \Gamma_V} \left(\frac{2\pi}{V}\right)^\nu f(\ell)^* g(\ell) + I_V,$$

where

$$I_V = \sum_{\ell \in \Gamma_V, |\ell_j| \leq K_f, j=1, \dots, \nu} g(\ell) \int_{[\ell_1 - \frac{\pi}{V}, \ell_1 + \frac{\pi}{V}] \times \dots \times [\ell_\nu - \frac{\pi}{V}, \ell_\nu + \frac{\pi}{V}]} [f(k)^* - f(\ell)^*] d^\nu k.$$

Since f is uniformly continuous, for any $\varepsilon > 0$, there exists a constant $V_0 > 0$ such that, if $|k_j - \ell_j| \leq \pi/V_0$, then $|f(k) - f(\ell)| \leq \varepsilon$. Hence, for all $V \geq V_0$, we have $|I_V| \leq D_V \varepsilon$, where $D_V = \sum_{\ell \in \Gamma_V, |\ell_j| \leq K_f, j=1, \dots, \nu} \left(\frac{2\pi}{V}\right)^\nu g(\ell)$. Note that

$$\lim_{V \rightarrow \infty} D_V = D := \int_{[-K_f, K_f]^\nu} |g(k)| d^\nu k \leq \left(\int_{[-K_f, K_f]^\nu} |g(k)|^2 d^\nu k \right)^{1/2} (2K_f)^\nu < \infty.$$

Hence $\overline{\lim}_{V \rightarrow \infty} |I_V| \leq D\varepsilon$. Since $\varepsilon > 0$ is arbitrary, we conclude that $\lim_{V \rightarrow \infty} I_V = 0$. Thus we obtain

$$(f, g_V)_{L^2} \rightarrow (f, g)_{L^2} \quad (V \rightarrow \infty). \quad (4.4)$$

By (4.3), (4.4) and a limiting argument using the denseness of $C_0^\infty(\mathbb{R}^\nu)$ in $L^2(\mathbb{R}^\nu)$, we obtain (4.2). ■

We say that two self-adjoint operators T_1 and T_2 on a Hilbert space *strongly commute* if their spectral measures commute.

LEMMA 4.3. Let T_1 and T_2 be strongly commuting self-adjoint operators on a Hilbert space. Then, for all $\psi \in D(T_1) \cap D(T_2)$,

$$\|(e^{iT_1} - e^{iT_2})\psi\| \leq \|(T_1 - T_2)\psi\|.$$

PROOF: Let E_j be the spectral measure of T_j . Then there exists a unique two-dimensional spectral measure E such that, for all Borel sets B_1, B_2 in $\mathbb{R}$, $E(B_1 \times B_2) = E_1(B_1)E_2(B_2)$. In terms of E , we have

$$T_j = \int \lambda_j dE(\lambda_1, \lambda_2), \quad e^{iT_j} = \int e^{i\lambda_j} dE(\lambda_1, \lambda_2), \quad j = 1, 2.$$

By the functional calculus and the inequality $|e^{ix} - e^{iy}| \leq |x - y|$, $x, y \in \mathbb{R}$, we have for all $\psi \in D(T_1) \cap D(T_2)$

$$\begin{aligned} \|(e^{iT_1} - e^{iT_2})\psi\|^2 &= \int_{\mathbb{R}^2} |e^{i\lambda_1} - e^{i\lambda_2}|^2 d\|E(\lambda_1, \lambda_2)\psi\|^2 \\ &\leq \int_{\mathbb{R}^2} |\lambda_1 - \lambda_2|^2 d\|E(\lambda_1, \lambda_2)\psi\|^2 \\ &= \|(T_1 - T_2)\psi\|^2. \end{aligned}$$

Thus the desired result follows. ■

LEMMA 4.4.

$$\|(U_{\pm}(V)^2 - U_{\pm}^2)(N + I)^{-1/2}\| \leq 4\|g_V - g\|. \quad (4.5)$$

PROOF: For all real-valued functions $f_1, f_2 \in L^2(\mathbb{R}^\nu)$ and all $s, t \in \mathbb{R}$, $e^{itP(f_1)}$ commutes with $e^{isP(f_2)}$ (e.g., [11, Theorem X.43]). Hence, by a general theorem (e.g., [10, Theorem VIII.13], $P(f_1)$ and $P(f_2)$ strongly commute. Applying this fact, we conclude that $P(g)$ and $P(g_V)$ strongly commute. Hence, by Lemma 4.3, we have for all $\Psi \in \mathcal{F}_0$,

$$\begin{aligned} \|(U_{\pm}(V)^2 - U_{\pm}^2)\Psi\| &\leq 2\|(P(g_V) - P(g))\Psi\| \\ &\leq 2(\|a(g_V - g)\Psi\| + \|a(g_V - g)^*\Psi\|). \end{aligned}$$

By (2.15) and the complementary estimate to it

$$\|a(f)^*\Phi\| \leq \|f\|_{L^2} \|(N + I)^{1/2}\Phi\|, \quad \Phi \in D(N^{1/2}), f \in L^2(\mathbb{R}^\nu),$$

we obtain

$$\|(U_{\pm}(V)^2 - U_{\pm}^2)\Psi\| \leq 4\|g_V - g\| \cdot \|(N + I)^{1/2}\Psi\|.$$

Since $\mathcal{F}_0$ is a core of $N^{1/2}$, we can extend this inequality, via a simple limiting argument, to all $\Psi \in D(N^{1/2})$. Thus (4.5) follows. ■

Proof of Theorem 4.1

We prove (4.1) in the case $\text{Im } z \neq 0$ (the other case can be similarly treated). Writing

$$I \otimes H_b = H - H_I$$

and using Lemma 2.1, we have

$$\|I \otimes H_b \Psi\| \leq \|H \Psi\| + \frac{\mu}{2} \|\Psi\|, \quad \Psi \in D(I \otimes H_b).$$

Let $L = I \otimes N + I$. By the fact that $\|N \Phi\| \leq \|H_b \Phi\|/m, \Phi \in D(H_b)$, we obtain

$$\|(L - I) \Psi\| \leq \frac{1}{m} \left(\|H \Psi\| + \frac{\mu}{2} \|\Psi\| \right), \quad \Psi \in D(I \otimes H_b),$$

which implies that, for all $z \in \mathbb{C} \setminus \mathbb{R}$, $L(H - z)^{-1}$ is bounded. By Lemma 3.1, $(I \otimes H_b - I \otimes H_{b,V})L^{-1}$ is bounded with

$$\|(I \otimes H_b - I \otimes H_{b,V})L^{-1}\| \leq \frac{c}{V^\gamma}. \quad (4.6)$$

We write

$$\begin{aligned} (H_V - z)^{-1} - (H - z)^{-1} &= (H_V - z)^{-1} (I \otimes H_b - I \otimes H_{b,V}) L^{-1} L (H - z)^{-1} \\ &\quad + (H_V - z)^{-1} (H_I - H_I(V)) L^{-1/2} L^{1/2} (H - z)^{-1}. \end{aligned}$$

Hence

$$\begin{aligned} \|(H_V - z)^{-1} - (H - z)^{-1}\| &\leq \frac{1}{|\operatorname{Im} z|} \left(\|(H_b - H_{b,V})L^{-1}\| \cdot \|L(H - z)^{-1}\| \right. \\ &\quad \left. + \|(H_I - H_I(V))L^{-1/2}\| \cdot \|L^{1/2}(H - z)^{-1}\| \right). \end{aligned}$$

We have

$$H_I - H_I(V) = \frac{\mu}{2} \{A \otimes (U_+^2 - U_+(V)^2) + A^* \otimes (U_-^2 - U_-(V)^2)\}.$$

Hence, by Lemma 4.4, $\|(H_I - H_I(V))L^{-1/2}\| \leq 4\mu \cdot \|g_V - g\|$, which, combined with Lemma 4.2, implies that $\lim_{V \rightarrow \infty} \|(H_I - H_I(V))L^{-1/2}\| = 0$. By (4.6), we have $\lim_{V \rightarrow \infty} \|(H_b - H_{b,V})L^{-1}\| = 0$. Thus we obtain (4.1). ■

5. Proof of the main results

5.1. Proof of Theorem 1.1

Let

$$\mathcal{H}_V = \mathbb{C}^2 \otimes \mathcal{F}_V.$$

LEMMA 5.1. *The operator $H_V \upharpoonright \mathcal{H}_V$ has purely discrete spectrum.*

PROOF: It is well known or easy to see that $I \otimes H_{b,V} \upharpoonright \mathcal{H}_V$ has compact resolvent. Since $H_I(V)$ is bounded, it follows that $H_I(V)(I \otimes H_{b,V} + i)^{-1} \upharpoonright \mathcal{H}_V$ is compact. Hence, by a general theorem [12, §XIII.4, Corollary 2], $\sigma_{\text{ess}}(H_V \upharpoonright \mathcal{H}_V) = \sigma_{\text{ess}}(I \otimes H_{b,V} \upharpoonright \mathcal{H}_V) = \emptyset$. Thus the desired result follows. ■

LEMMA 5.2.

$$H_V \upharpoonright \mathcal{H}_V^\perp \geq E(H_V) + m.$$

PROOF: We decompose $L^2(\mathbb{R}^\nu)$ as $L^2(\mathbb{R}^\nu) = F_{1V} \oplus F_{1V}^\perp$ with $F_{1V} = L^2(\mathbb{R}^\nu) \cap \mathcal{F}_V$. Then

$$\mathcal{F} = \mathcal{F}_V \otimes \mathcal{F}(F_{1V}^\perp) = \bigoplus_{j=0}^{\infty} \mathcal{F}^{(j)},$$

where $\mathcal{F}^{(j)} = \mathcal{F}_V \otimes [\otimes_s^j F_{1V}^\perp]$. Hence $\mathcal{F}_V^\perp = \bigoplus_{j=1}^{\infty} \mathcal{F}^{(j)}$ and $\mathcal{H}_V^\perp = \mathbb{C}^2 \otimes \mathcal{F}_V^\perp = \bigoplus_{j=1}^{\infty} \mathbb{C}^2 \otimes \mathcal{F}^{(j)}$. On each $\mathbb{C}^2 \otimes \mathcal{F}^{(j)}$, H_V has the form $S \otimes I + I \otimes T$ with $S = H_V \upharpoonright \mathcal{H}_V$ and T is a sum of j copies of $H_{b,V}$, each acting on a single factor $F_{1V}^\perp$. Since $T \geq jm$ on $\otimes_s^j F_{1V}^\perp$, the assertion of the lemma follows. ■

LEMMA 5.3 [13, LEMMA 4.6]. *Let T_n and T be self-adjoint operators on a Hilbert space, which are bounded from below. Suppose that $T_n \rightarrow T$ in norm resolvent sense as $n \rightarrow \infty$ and T_n has purely discrete spectrum in $[E(T_n), E(T_n) + c)$ with some constant $c > 0$. Then, $\lim_{n \rightarrow \infty} E(T_n) = E(T)$ and T has purely discrete spectrum in $[E(T), E(T) + c)$.*

We are now ready to prove Theorem 1.1 : By Lemmas 5.1 and 5.2, H_V has purely discrete spectrum in $[E(H_V), E(H_V) + m)$. By this fact and Theorem 4.1, we can apply Lemma 5.3 to conclude that H has purely discrete spectrum in $[E(H), E(H) + m)$, which, combined with (2.5), implies Theorem 1.1.

5.2. Proof of Theorem 1.2

The basic idea of proof is to use the min-max principle for H [12, Theorem XIII.1]. Let

$$\mu_2(H) = \sup_{\Phi \in \mathcal{H}} U_H(\Phi)$$

with $U_H(\Phi) = \inf_{\Psi \in D(H), \|\Psi\|=1, \Psi \in [\Phi]^\perp} (\Psi, H\Psi)$, where $[\Phi]^\perp = \{\Psi \in \mathcal{H} | (\Psi, \Phi) = 0\}$. We estimate $\mu_2(H)$ from below. For this purpose, we write

$$H = I \otimes H_b + \frac{\mu}{2} \sigma_x \otimes I + W,$$

where

$$W = \frac{\mu}{2} \{A \otimes (U_+^2 - I) + A^* \otimes (U_-^2 - I)\}.$$

For $\varepsilon > 0$, we set

$$D_\varepsilon(\alpha, \mu) = \frac{4\alpha^2\mu^2}{\varepsilon} \left\| \frac{\lambda}{\omega\sqrt{\omega}} \right\|_{L^2}^2 + 2|\alpha|\mu \left\| \frac{\lambda}{\omega} \right\|_{L^2}.$$

LEMMA 5.4. For all $\varepsilon > 0$ and $\Psi \in D(I \otimes H_b)$,

$$|(\Psi, W\Psi)| \leq \varepsilon(\Psi, I \otimes H_b\Psi) + D_\varepsilon(\alpha, \mu)\|\Psi\|^2. \quad (5.1)$$

PROOF: By the fact $\|A\| = \|A^*\| = 1$ and Lemma 4.3, we have for all $\Psi \in D(I \otimes H_b)$

$$\begin{aligned} \|W\Psi\| &\leq \frac{\mu}{2} (\|I \otimes (U_+^2 - I)\Psi\| + \|I \otimes (U_-^2 - I)\Psi\|) \\ &\leq 2|\alpha|\mu \|I \otimes P(\lambda/\omega)\Psi\| \\ &\leq 2|\alpha|\mu (\|I \otimes a(\lambda/\omega)\Psi\| + \|I \otimes a(\lambda/\omega)^*\Psi\|). \end{aligned}$$

On the other hand, the following estimates are well known:

$$\begin{aligned} \|a(f)\psi\| &\leq \|f/\sqrt{\omega}\|_{L^2} \|H_b^{1/2}\psi\|, \\ \|a(f)^*\psi\| &\leq \|f/\sqrt{\omega}\|_{L^2} \|H_b^{1/2}\psi\| + \|f\|_{L^2} \|\psi\|, \quad f, f/\sqrt{\omega} \in L^2(\mathbb{R}^\nu), \psi \in D(H_b^{1/2}). \end{aligned}$$

Hence

$$\|W\Psi\| \leq 4|\alpha|\mu \left\| \frac{\lambda}{\omega\sqrt{\omega}} \right\|_{L^2} \|(I \otimes H_b)^{1/2}\Psi\| + 2|\alpha|\mu \|\Psi\| \left\| \frac{\lambda}{\omega} \right\|_{L^2}.$$

Using this estimate and the elementary inequality $xy \leq \varepsilon x^2 + \frac{y^2}{4\varepsilon}$ holding for all $x, y, \varepsilon > 0$, we obtain (5.1). ■

We now proceed to proof of Theorem 1.2. Let Ω_0 be the Fock vacuum in $\mathcal{F}$: $\Omega_0 = \{1, 0, 0, \dots\}$ and

$$\Phi_0 = \begin{pmatrix} \Omega_0 \\ -\Omega_0 \end{pmatrix}.$$

Then it is easy to see that

$$[\Phi_0]^\perp = \left\{ \Psi = \begin{pmatrix} \Psi_1 \\ \Psi_2 \end{pmatrix} \in \mathcal{H} \mid \Psi_1^{(0)} = \Psi_2^{(0)} \right\},$$

where we write $\Psi_j = \{\Psi_j^{(n)}\}_{n=0}^\infty \in \mathcal{F}$, $\Psi_j^{(n)} \in \otimes_s^n L^2(\mathbb{R}^\nu)$. Let $\Psi \in [\Phi_0]^\perp$. Then, by the fact $H_b\Omega_0 = 0$ and $H_b \upharpoonright \otimes_s^n L^2(\mathbb{R}^\nu) \geq nm$, we have

$$(\Psi, I \otimes H_b\Psi) \geq \sum_{j=1}^2 \sum_{n=1}^\infty (\Psi_j^{(n)}, H_b\Psi_j^{(n)}) \geq m \sum_{j=1}^2 \sum_{n=1}^\infty \|\Psi_j^{(n)}\|^2.$$

Noting the fact $\Psi_1^{(0)} = \Psi_2^{(0)}$, we have

$$\begin{aligned}
\frac{\mu}{2}(\Psi, \sigma_x \otimes I \Psi) &= \frac{\mu}{2}\{(\Psi_1, \Psi_2) + (\Psi_2, \Psi_1)\} \\
&= \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} + \frac{\mu}{2} \sum_{n=1}^{\infty} \{(\Psi_1^{(n)}, \Psi_2^{(n)}) + (\Psi_2^{(n)}, \Psi_1^{(n)})\} \\
&\geq \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \mu \sum_{n=1}^{\infty} \|\Psi_1^{(n)}\| \|\Psi_2^{(n)}\| \\
&\geq \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2} \|\Psi\|^2.
\end{aligned}$$

These estimates and Lemma 5.4 give

$$\begin{aligned}
(\Psi, H \Psi) &\geq m(1 - \varepsilon) \sum_{j=1}^2 \sum_{n=1}^{\infty} \|\Psi_j^{(n)}\|^2 + \frac{\mu}{2}\{|\Psi_1^{(0)}|^2 + |\Psi_2^{(0)}|^2\} - \frac{\mu}{2} \|\Psi\|^2 - D_\varepsilon(\alpha, \mu) \|\Psi\|^2 \\
&\geq \left\{ M_\varepsilon - \frac{\mu}{2} - D_\varepsilon(\alpha, \mu) \right\} \|\Psi\|^2,
\end{aligned}$$

where ε is an arbitrary constant satisfying $0 < \varepsilon < 1$ and $M_\varepsilon = \min \{m(1 - \varepsilon), \frac{\mu}{2}\}$. Since this inequality holds for all $\Psi \in [\Phi_0]^\perp$, we obtain $\mu_2(H) \geq C_0$ with

$$C_0 = \sup_{0 < \varepsilon < 1} \left\{ M_\varepsilon - \frac{\mu}{2} - D_\varepsilon(\alpha, \mu) \right\}.$$

This estimate and the min-max principle imply that $E(H)$ is a simple eigenvalue of H if $E(H) < C_0$. By (2.10), if $C_0 > -\mu e^{-2\alpha^2 \|\lambda/\omega\|^2}/2$ (this condition is equivalent to condition (1.11)), then $E(H) < C_0$ and hence H has a unique ground state. Thus the desired result follows.

5.3. Proof of Theorem 1.3

Let

$$\mu_3(H) = \sup_{\Phi_1, \Phi_2 \in \mathcal{H}} U_H(\Phi_1, \Phi_2)$$

with $U_H(\Phi_1, \Phi_2) = \inf_{\Psi \in D(H); \|\Psi\|=1, \Psi \in [\Phi_1, \Phi_2]^\perp} (\Psi, H \Psi)$, where $[\Phi_1, \Phi_2]^\perp$ denotes the orthogonal complement of $\{\alpha \Phi_1 + \beta \Phi_2 | \alpha, \beta \in \mathbb{C}\}$. Let

$$\Phi_1 = \begin{pmatrix} \Omega_0 \\ \Omega_0 \end{pmatrix}, \quad \Phi_2 = \begin{pmatrix} \Omega_0 \\ -\Omega_0 \end{pmatrix}.$$

Then we have

$$[\Phi_1, \Phi_2]^\perp = \mathbb{C}^2 \otimes \mathcal{G} = \mathcal{G} \oplus \mathcal{G}$$

with $\mathcal{G} = \bigoplus_{n=1}^{\infty} \otimes_s^n L^2(\mathbb{R}^\nu)$. For all $\Psi = (\Psi_+, \Psi_-) \in [\Phi_1, \Phi_2]^\perp$ ($\Psi_\pm \in \mathcal{G}$), we have

$$(\Psi, H\Psi) \geq (\Psi_+, H_b\Psi_+) + (\Psi_-, H_b\Psi_-) - \frac{\mu}{2}\|\Psi\|^2.$$

It is easy to see that $(\Psi_\pm, H_b\Psi_\pm) \geq m\|\Psi_\pm\|^2$. Hence we obtain $(\Psi, H\Psi) \geq (m - \frac{\mu}{2})\|\Psi\|^2$, which implies that

$$\mu_3(H) \geq m - \frac{\mu}{2}. \quad (5.2)$$

Assume (1.12). Then, by (5.2) and (2.10), we have

$$\mu_3(H) > -\frac{\mu}{2}e^{-2\|\lambda/\omega\|_{L^2}^2} \geq E(H).$$

Hence, by the min-max principle, there are at most two eigenvalues (counting multiplicity) of H in the interval $[E(H), -\frac{\mu}{2}e^{-\|\lambda/\omega\|_{L^2}^2}]$. In particular, H has at most two ground states. These facts and (2.5) imply Theorem 1.3. ■

5.4. Proof of Theorem 1.4

We apply the following fact (which may be more or less known):

LEMMA 5.5. *Let $A_n, n = 1, 2, \dots$, and A be self-adjoint operators on a Hilbert space $\mathcal{K}$ having a common core D such that, for all $\psi \in D$, $A_n\psi \rightarrow A\psi$ as $n \rightarrow \infty$. Let ψ_n be a normalized eigenvector of A_n with eigenvalue E_n : $A_n\psi_n = E_n\psi_n$ such that $E := \lim_{n \rightarrow \infty} E_n$ and $w\text{-}\lim_{n \rightarrow \infty} \psi_n = \psi \neq 0$ exist, where $w\text{-}\lim$ denotes weak limit. Then ψ is an eigenvector of A with eigenvalue E . In particular, if ψ_n is a ground state of A_n , then ψ is a ground state of A .*

PROOF: By the present assumption and a general theorem [10, Theorem VIII.25(a)], A_n converges to A in the strong resolvent sense as $n \rightarrow \infty$. Hence, for all $\phi \in \mathcal{K}$ and $z \in \mathbb{C} \setminus \mathbb{R}$, we have

$$\begin{aligned} & |(\phi, (A_n - z)^{-1}\psi_n) - (\phi, (A - z)^{-1}\psi)| \\ &= |((A_n - z^*)^{-1}\phi - (A - z^*)^{-1}\phi, \psi_n)| + |((A - z^*)^{-1}\phi, \psi_n - \psi)| \\ &\leq \|((A_n - z^*)^{-1}\phi - (A - z^*)^{-1}\phi)\| + |((A - z^*)^{-1}\phi, \psi_n - \psi)| \\ &\rightarrow 0 \quad (n \rightarrow \infty), \end{aligned}$$

i.e., $\lim_{n \rightarrow \infty} (\phi, (A_n - z)^{-1}\psi_n) = (\phi, (A - z)^{-1}\psi)$. By the spectral theorem, we have $(\phi, (A_n - z)^{-1}\psi_n) = (E_n - z)^{-1}(\phi, \psi_n)$. Hence we obtain $(\phi, (A - z)^{-1}\psi) = (\phi, (E - z)^{-1}\psi)$ for all $\phi \in \mathcal{K}$, which implies that $(A - z)^{-1}\psi = (E - z)^{-1}\psi$. Thus $\psi \in D(A)$ and $A\psi = E\psi$. If ψ_n is a ground state of A_n , then $(\phi, A_n\phi) \geq E_n\|\phi\|^2$ for all $\phi \in D$. Taking the limit $n \rightarrow \infty$ in this inequality, we obtain $(\phi, A\phi) \geq E\|\phi\|^2$. Since D is a core for A , the last inequality extends to all $\phi \in D(A)$, which, combined with the preceding result, implies that $E = \inf \sigma(A)$. Thus ψ is a ground state of A . ■

We now turn to the spin-boson Hamiltonian in the case $\inf_{k \in \mathbb{R}^{\nu}} \omega(k) = 0$. To employ the results in the case of massive bosons, we define for $m > 0$

$$\omega_m(k) = \omega(k) + m.$$

Then (1.2) with ω replaced by ω_m holds for all $m > 0$. We introduce

$$H_{\text{SB}}(m) = \frac{1}{2} \mu \sigma_z \otimes I + I \otimes H_b(m) + \alpha \sigma_x \otimes (a(\lambda)^* + a(\lambda))$$

with $H_b(m) = d\Gamma(\omega_m)$.

LEMMA 5.6. *Let $\mathcal{D} = \mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b)]$, where $\hat{\otimes}$ denotes algebraic tensor product. Then $\mathcal{D}$ is a common core for all $H_{\text{SB}}(m)$ and H_{SB} . Moreover, for all $\Psi \in \mathcal{D}$, $H_{\text{SB}}(m)\Psi \rightarrow H_{\text{SB}}\Psi$ as $m \rightarrow 0$.*

PROOF: The first half of the lemma is well known (note that $\mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b)] = \mathbb{C}^2 \hat{\otimes} [\mathcal{F}_0 \cap D(H_b(m))]$). The second half follows from a direct computation. ■

We are now ready to prove Theorem 1.4. By Theorem 1.1, there exists a ground state $\Omega(m)$ of $H_{\text{SB}}(m)$: $H_{\text{SB}}(m)\Omega(m) = E(H_{\text{SB}}(m))\Omega(m)$. Without loss of generality, we can assume that $\|\Omega(m)\| = 1$. By (1.8), we have

$$-\frac{\mu}{2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_m}} \right\|_{L^2}^2 \leq E(H_{\text{SB}}(m)) \leq -\frac{\mu}{2} e^{-2\alpha^2 \|\lambda/\omega_m\|_{L^2}^2} - \alpha^2 \left\| \frac{\lambda}{\sqrt{\omega_m}} \right\|_{L^2}^2.$$

By using the Lebesgue dominated convergence theorem, one can easily show that

$$\lim_{m \rightarrow 0} \left\| \frac{\lambda}{\sqrt{\omega_m}} \right\|_{L^2}^2 = \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2, \quad \lim_{m \rightarrow 0} \left\| \frac{\lambda}{\omega_m} \right\|_{L^2}^2 = \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2. \quad (5.3)$$

Hence $\{E(H_{\text{SB}}(m))\}_m$ is uniformly bounded in m . Thus there exists a sequence $\{m_j\}_{j=1}^{\infty}$ with $m_1 > m_2 > \dots > m_j \rightarrow 0$ ($j \rightarrow \infty$) such that

$$E := \lim_{j \rightarrow \infty} E(H_{\text{SB}}(m_j))$$

and

$$\Omega := \text{w-} \lim_{j \rightarrow \infty} \Omega(m_j)$$

exist. We need only to show that $\Omega \neq 0$ (then, by Lemmas 5.6 and 5.5, Ω is a ground state of H_{SB}).

Let P_0 be the orthogonal projection from $\mathcal{F}$ onto the Fock vacuum state $\{c\Omega_0 | c \in \mathbb{C}\}$. It is easy to see that

$$I \otimes P_0 \geq I - I \otimes N.$$

If $\omega\lambda$ and λ are in $L^2(\mathbb{R}^\nu)$, then $\omega_m\lambda \in L^2(\mathbb{R}^\nu)$. By these facts and Lemma 2.3, we have

$$(\Omega(m), I \otimes P_0 \Omega(m)) \geq 1 - (\Omega(m), I \otimes N \Omega(m)) \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega_m} \right\|_{L^2}^2. \quad (5.4)$$

Since the range of $I \otimes P_0$ is finite dimensional (in fact, two dimensional), we have

$$\lim_{j \rightarrow \infty} (\Omega(m_j), I \otimes P_0 \Omega(m_j)) = (\Omega, I \otimes P_0 \Omega).$$

From this fact, (5.4) and the second formula in (5.3), we obtain

$$(\Omega, I \otimes P_0 \Omega) \geq 1 - \alpha^2 \left\| \frac{\lambda}{\omega} \right\|_{L^2}^2.$$

Under condition (1.13), the RHS is strictly positive. Hence $\Omega \neq 0$. ■

6. A generalization of the model

In this section we propose a generalization of the spin-boson model discussed in the preceding sections. We expect that the generalization clarify the general properties of the spin-boson model. We also have in mind applications to quantum spin systems on an infinite lattice in which spins interact with bosons too.

Let $\mathcal{H}$ be a Hilbert space and A (resp. B) be a self-adjoint (resp. symmetric) operator on $\mathcal{H}$. The Hamiltonian of the genelaized spin-boson model we propose is given by

$$H = A \otimes I + I \otimes d\Gamma(\omega) + B \otimes (a(\lambda)^* + a(\lambda))$$

acting in the Hilbert space $\mathcal{H} \otimes \mathcal{F}$.

Supposse that A, B are bounded and $\lambda, \lambda/\sqrt{\omega}, \lambda/\omega$ are in $L^2(\mathbf{R}^d)$. Then

$$L_{A,B} := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\|\lambda/\omega\|_{L^2} B t} A e^{i\|\lambda/\omega\|_{L^2} B t} e^{-t^2/2} dt - \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 B^2$$

is a bounded self-adjoint operator. We can show [4] that

$$- \|A\| - \|B\|^2 \left\| \frac{\lambda}{\sqrt{\omega}} \right\|_{L^2}^2 \leq E(H) \leq E(L_{A,B}). \quad (6.1)$$

In the case of the original spin-boson model (i.e., the case $H = H_{\text{SB}}$), (6.1) is just (1.8). Thus estimate (6.1) clarifies a general structure of (1.8). The results on ground states of H_{SB} also can be generalized to the case of H . We can also develop scattering theory concerning the pair $\langle A \otimes I + I \otimes d\Gamma(\omega), H \rangle$. For the details, see [4].

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