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Multiple zeta-star values and multiple integrals

By

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Abstract

We prove a kind of integral expressions for finite multiple harmonic sums and multiple zeta-star values. Moreover, we introduce a class of multiple integrals, associated with some combinatorial data (called 2-labeled posets). This class includes both multiple zeta and zeta-star values of Euler-Zagier type, and also several other types of multiple zeta values. We show that these integrals can be used to obtain some relations among such zeta values quite transparently.

§ 1. Integral expression of finite multiple harmonic sums

We begin with the finite multiple harmonic sums

$$s_{\mathbf{k}}(N)=\sum_{N=m_1\geq\cdots\geq m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}},$$

where $\mathbf{k}=(k_1,\dots,k_n)$ is an n-tuple of positive integers and N is a positive integer. When $k_1\geq 2$, by definition, their sum gives the multiple zeta-star values (MZSVs for short):

$$\zeta^\star(\mathbf{k})=\sum_{m_1\geq\cdots\geq m_n\geq 1}\frac{1}{m_1^{k_1}\cdots m_n^{k_n}}=\sum_{N=1}^{\infty}s_{\mathbf{k}}(N). \tag{1.1}$$

One of the basic properties of these finite multiple sums is the following relation, called the duality:

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Theorem 1.1 ([H, K]). *For any index* $\mathbf{k} \in (\mathbb{Z}_{\geq 1})^n$ *and* $N \geq 0$*, we have*

$$\sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} s_{\mathbf{k}}(i+1) = s_{\mathbf{k}^*}(N). \tag{1.2}$$

Here $\mathbf{k}^*$ *denotes the 'transpose' of* $\mathbf{k}$ *(see below).*

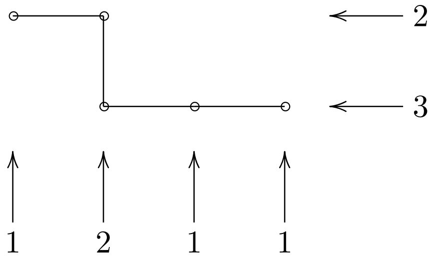
For an index $\mathbf{k} = (k_1, \ldots, k_n)$, we set

$$|\mathbf{k}| := k_1 + \cdots + k_n, \quad A(\mathbf{k}) := \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{n-1}\}.$$

Then the transpose $\mathbf{k}^*$ is the index determined by the property

$$|\mathbf{k}| = |\mathbf{k}^*|, \quad \{1, \ldots, |\mathbf{k}| - 1\} = A(\mathbf{k}) \amalg A(\mathbf{k}^*).$$

For example, the transpose of $(2,3)$ is $(1,2,1,1)$. It can be illustrated by the following picture:



The identity (1.2) is somewhat analogous to the well-known duality

$$\zeta(a_1+1, \underbrace{1, \ldots, 1}_{b_1-1}, \ldots, a_s+1, \underbrace{1, \ldots, 1}_{b_s-1}) = \zeta(b_s+1, \underbrace{1, \ldots, 1}_{a_s-1}, \ldots, b_1+1, \underbrace{1, \ldots, 1}_{a_1-1}) \tag{1.3}$$

of the multiple zeta values (MZVs)

$$\zeta(\mathbf{k}) = \sum_{m_1 > \cdots > m_n > 0} \frac{1}{m_1^{k_1} \cdots m_n^{k_n}}.$$

Since the latter duality follows immediately from the iterated integral expression

$$\begin{aligned} &\zeta(a_1+1, \underbrace{1, \ldots, 1}_{b_1-1}, \ldots, a_s+1, \underbrace{1, \ldots, 1}_{b_s-1}) \\ &= \int_0^1 \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{a_1} \circ \underbrace{\frac{dt}{1-t} \circ \cdots \circ \frac{dt}{1-t}}_{b_1} \circ \cdots \circ \underbrace{\frac{dt}{t} \circ \cdots \circ \frac{dt}{t}}_{a_s} \circ \underbrace{\frac{dt}{1-t} \circ \cdots \circ \frac{dt}{1-t}}_{b_s}, \end{aligned} \tag{1.4}$$

it is natural to ask for a similar integral expression of finite multiple sum $s_{\mathbf{k}}(N)$ from which (1.2) follows. Here is an answer:

Theorem 1.2. *Let* $\mathbf{k} = (k_1, \ldots, k_n)$ *be an index, and put* $k = |\mathbf{k}| = k_1 + \cdots + k_n$. *Moreover, define*

$$J(\mathbf{k}) = \{0, k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_{n-1}\} = A(\mathbf{k}) \cup \{0\},$$
$$\Delta(\mathbf{k}) = \left\{ (t_1, \ldots, t_k) \in [0,1]^k \;\middle|\; \begin{array}{l} t_j < t_{j+1} \text{ if } j \notin J(\mathbf{k}), \\ t_j > t_{j+1} \text{ if } j \in J(\mathbf{k}) \end{array} \right\}.$$

Then, for $N \geq 1$, *we have*

$$s_{\mathbf{k}}(N) = \int_{\Delta(\mathbf{k})} t_1^{N-1} dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k), \tag{1.5}$$

where $\omega_0(t) = \frac{dt}{t}$, $\omega_1(t) = \frac{dt}{1-t}$ *and*

$$\delta(j) = \begin{cases} 0 & (j - 1 \notin J(\mathbf{k})), \\ 1 & (j - 1 \in J(\mathbf{k})). \end{cases} \tag{1.6}$$

Remark. We include 0 in the set $J(\mathbf{k})$ to set $\delta(1) = 1$ in (1.6), though this value is not used in the above theorem. We need it in Corollary 1.3.

Proof. We consider the case $\mathbf{k} = (2, 1, 2)$ as an example, and then the general pattern will be understood. In this case, we should prove

$$s_{(2,1,2)}(N) = \int_{t_1 < t_2 > t_3 > t_4 < t_5} t_1^{N-1} dt_1 \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} \frac{dt_4}{1 - t_4} \frac{dt_5}{t_5}.$$

(Here, implicitly, the inequalities $0 \leq t_i \leq 1$ are assumed.) The right-hand side is computed by repeating single integrals, i.e.,

$$\begin{aligned} \int_0^{t_2} t_1^{N-1} dt_1 &= \frac{t_2^N}{N}, \\ \frac{1}{N} \int_{t_3}^1 t_2^N \frac{dt_2}{t_2} &= \frac{1 - t_3^N}{N^2}, \\ \frac{1}{N^2} \int_{t_4}^1 (1 - t_3^N) \frac{dt_3}{1 - t_3} &= \sum_{N \geq m \geq 1} \frac{1 - t_4^m}{N^2 m}, \\ \sum_{N \geq m \geq 1} \frac{1}{N^2 m} \int_0^{t_5} (1 - t_4^m) \frac{dt_4}{1 - t_4} &= \sum_{N \geq m \geq l \geq 1} \frac{t_5^l}{N^2 m l}, \\ \sum_{N \geq m \geq l \geq 1} \frac{1}{N^2 m l} \int_0^1 t_5^l \frac{dt_5}{t_5} &= \sum_{N \geq m \geq l \geq 1} \frac{1}{N^2 m l^2}. \end{aligned}$$

The last sum is exactly $s_{(2,1,2)}(N)$. $\square$

To deduce Theorem 1.1 from Theorem 1.2, note that there is a bijection

$$\Delta(\mathbf{k}) \ni (t_1, \dots, t_k) \longmapsto (1 - t_1, \dots, 1 - t_k) \in \Delta(\mathbf{k}^*) \tag{1.7}$$

and that the map δ^* associated with $\mathbf{k}^*$ as in (1.6) satisfies $\delta^*(j) = 1 - \delta(j)$ for $j = 2, \dots, k$. Hence, by changing of the integral variables $s_j = 1 - t_j$, we obtain

$$\begin{aligned} s_{\mathbf{k}^*}(N) &= \int_{\Delta(\mathbf{k}^*)} s_1^{N-1} ds_1 \, \omega_{\delta^*(2)}(s_2) \cdots \omega_{\delta^*(k)}(s_k) \\ &= \int_{\Delta(\mathbf{k})} (1 - t_1)^{N-1} dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k) \\ &= \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} \int_{\Delta(\mathbf{k})} t_1^i dt_1 \omega_{\delta(2)}(t_2) \cdots \omega_{\delta(k)}(t_k) \\ &= \sum_{i=0}^{N-1} (-1)^i \binom{N-1}{i} s_{\mathbf{k}}(i+1). \end{aligned}$$

By (1.1) and (1.5), we also obtain an integral expression of MZSVs.

Corollary 1.3. *In the same notation as in Theorem 1.2, assume $k_1 \geq 2$. Then*

$$\zeta^\star(\mathbf{k}) = \int_{\Delta(\mathbf{k})} \omega_{\delta(1)}(t_1) \cdots \omega_{\delta(k)}(t_k). \tag{1.8}$$

Example 1.4. For $\mathbf{k} = (2, 1)$, we have

$$\zeta^\star(2,1) = \int_{t_1 < t_2 > t_3} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3}. \tag{1.9}$$

This integral can be divided into two parts:

$$\int_{t_1 < t_2 > t_3} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3} = \left(\int_{t_1 < t_3 < t_2} + \int_{t_3 < t_1 < t_2} \right) \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2} \frac{dt_3}{1 - t_3}.$$

By the iterated integral expression (1.4) of MZVs, the right-hand side is equal to $\zeta(2,1) + \zeta(2,1)$. Therefore, from the integral expression (1.9), one obtains

$$\zeta^\star(2,1) = 2\zeta(2,1).$$

Note that this is different from the relation

$$\zeta^\star(2,1) = \zeta(2,1) + \zeta(3)$$

obtained from the series expressions. By comparing these two relations, one proves Euler's famous relation $\zeta(2,1) = \zeta(3)$.

More generally, from the integral and series expressions of $\zeta^\star(k-1,1)$, one can show the sum formula for double zeta values

$$\zeta(k-1,1)+\zeta(k-2,2)+\cdots+\zeta(2,k-2)=\zeta(k)\quad(k\geq 3)$$

in a similar manner.

§ 2. Multiple integrals associated with 2-labeled finite posets

Now we define a class of integrals which includes both MZVs (1.4) and MZSVs (1.8). Recall that a finite poset is a finite set endowed with a partial order. In the following, we omit the word 'finite' since we only consider finite posets.

Definition 2.1.

(1) A *2-labeled poset* is a pair $X=(X,\delta_X)$ consisting of a poset X and a map $\delta_X\colon X\to\{0,1\}$, called the *labeling map*. The *weight*, denoted by $|X|$, is the number of elements of the underlying set X, and the *depth*, denoted by $\mathrm{dep}(X)$, is the number of $x\in X$ such that $\delta(x)=1$.

(2) A 2-labeled poset X is said *admissible* if $\delta_X(x)=1$ for all minimal $x\in X$ and $\delta_X(x)=0$ for all maximal $x\in X$.

(3) For any poset X, we put

$$\Delta(X):=\big\{(t_x)_{x\in X}\in[0,1]^X\;\big|\;t_x<t_y\text{ if }x<y\big\}.$$

(4) For an admissible 2-labeled poset X, we define the associated integral by

$$I(X):=\int_{\Delta(X)}\prod_{x\in X}\omega_{\delta_X(x)}(t_x).\tag{2.1}$$

Here $\omega_0(t)=\frac{dt}{t}$ and $\omega_1(t)=\frac{dt}{1-t}$ are the same notation as in Theorem 1.2.

Remark. For the empty 2-labeled poset, denoted $\varnothing$, we put $I(\varnothing)=1$. This is compatible with the usual definition $\zeta(\varnothing)=\zeta^\star(\varnothing)=1$, where $\varnothing$ denotes the index of length 0.

We use Hasse diagrams to indicate 2-labeled posets, with vertices $\circ$ and $\bullet$ corresponding to $\delta(x)=0$ and 1, respectively. For example,

$$X=\{1<2<3<4<5\}\text{ and }\big(\delta(1),\ldots,\delta(5)\big)=(1,0,1,0,0)$$

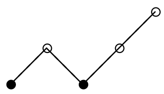
is represented as the diagram

This 2-labeled poset is admissible, and we have

$$I(X) = \int_{t_1<t_2<t_3<t_4<t_5} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}\frac{dt_3}{1-t_3}\frac{dt_4}{t_4}\frac{dt_5}{t_5} = \zeta(3,2).$$

In general, the iterated integral expression of a MZV is equal to the integral $I(X)$ associated with an admissible 2-labeled *totally ordered* set X, and the weight and the depth of the MZV coincide with those of X.

Another example: Corollary 1.3 for $\mathbf{k} = (2,3)$ gives

$\zeta^\star(2,3) = I\Bigg($  $\Bigg).$

Here we collect some basic constructions on 2-labeled posets, and relations between the associated integrals.

Definition 2.2.

(1) For 2-labeled posets X and Y, one can naturally define their direct sum $X \amalg Y$: Its underlying poset is the direct sum of finite sets X and Y endowed with the partial order

$$\begin{aligned} x \le y \text{ in } X \amalg Y \iff & x, y \in X \text{ and } x \le y \text{ in } X \text{ or} \\ & x, y \in Y \text{ and } x \le y \text{ in } Y. \end{aligned}$$

The map $\delta_{X \amalg Y} : X \amalg Y \to \{0,1\}$ is the direct sum of the maps $\delta_X : X \to \{0,1\}$ and $\delta_Y : Y \to \{0,1\}$.

(2) Let $X = (X, \le)$ be a poset, and $a, b \in X$ not comparable, i.e., neither $a \le b$ nor $b \le a$ hold. Then we denote by X_a^b the poset with the same underlying set X and endowed with the order $\le_a^b$ defined by

$$x \le_a^b y \iff x \le y, \text{ or } x \le a \text{ and } b \le y.$$

We call X_a^b the refinement of X obtained by imposing $a < b$. If X is a 2-labeled poset, then X_a^b also becomes a 2-labeled poset with the same labeling map.

(3) For a 2-labeled poset X, we define its *transpose* X^* as the 2-labeled poset consisting of the same underlying set as X endowed with the reversed order (i.e. $x \le y$ in X^* if and only if $y \le x$ in X), and the labeling map $\delta_{X^*}(x) = 1 - \delta_X(x)$.

Proposition 2.3.

(1) If X and Y are admissible 2-labeled posets, then $X \amalg Y$ is admissible and

$$I(X \amalg Y) = I(X) \cdot I(Y). \tag{2.2}$$

(2) If X is an admissible 2-labeled poset, and a and $b \in X$ are not comparable, then both X_a^b and X_b^a are admissible and

$$I(X) = I(X_a^b) + I(X_b^a). \tag{2.3}$$

(3) If X is an admissible 2-labeled poset, then the transpose X^* is admissible and

$$I(X^*) = I(X). \tag{2.4}$$

Proof. All assertions are easily verified. Note that (2.4) is shown by making the change of variables

$$\Delta(X) \ni (t_x) \longmapsto (1 - t_x) \in \Delta(X^*),$$

which is a generalization of (1.7). □

Remark. The shuffle relation for the MZVs can be derived from the identities (2.2) and (2.3) (see also the remark after Corollary 2.4). On the other hand, the identity (2.4) is a natural generalization of the duality (1.3) for the MZVs.

Corollary 2.4. *For any 2-labeled poset X, the integral $I(X)$ can be expressed as the sum of a finite number of MZVs of weight $|X|$ and depth* $\operatorname{dep}(X)$*.*

Proof. By using (2.3) several times, we express $I(X)$ as a sum of the integrals associated with 2-labeled totally ordered sets, i.e., the integral expressions of MZVs. Each of these 2-labeled totally ordered sets consists of the same underlying set and labeling map as X and a total order extending the partial order of X. In particular, these have the same weight and depth as X. □

Remark. There is an algebraic formalism for the integrals $I(X)$, similar to the well-known pair of the shuffle algebra $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ and the homomorphism $Z\colon \mathfrak{H}^0 = \mathbb{Q} \oplus x\mathfrak{H}y \to \mathbb{R}$.

We write $\mathfrak{P}$ for the $\mathbb{Q}$-vector space generated by all isomorphism classes of 2-labeled posets, and define a product on it by $[X] \cdot [Y] = [X \amalg Y]$. Then, the subspace $\mathfrak{P}^0$ generated by admissible 2-labeled posets is a subalgebra, and the map $I\colon \mathfrak{P}^0 \to \mathbb{R}$, defined by linearity, is indeed a $\mathbb{Q}$-algebra homomorphism.

In fact, there exists a surjective $\mathbb{Q}$-algebra homomorphism $W\colon \mathfrak{P} \to \mathfrak{H}$ satisfying $W(\mathfrak{P}^0) = \mathfrak{H}^0$ and $Z \circ W = I$. This is defined as the unique homomorphism whose kernel is generated by $[X] - [X_a^b] - [X_b^a]$ and which sends totally ordered $[X]$ to a monomial in $\mathfrak{H}$ encoding δ_X appropriately.

§ 3. Other examples

In this section, we consider some values representable by the integrals $I(X)$, other than MZVs and MZSVs.

§ 3.1. Arakawa-Kaneko zeta values

The Arakawa-Kaneko multiple zeta function [AK] is defined as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{Li_k(1-e^{-t})}{1-e^{-t}} e^{-t} t^{s-1} dt$$

for an integer $k > 0$, where $Li_k(x) = \sum_{n=1}^\infty \frac{x^n}{n^k}$ is the k-th polylogarithm function. By making the variable change $x = 1 - e^{-t}$, we can write it as

$$\xi_k(s) = \frac{1}{\Gamma(s)} \int_0^1 Li_k(x) \bigl(-\log(1-x)\bigr)^{s-1} \frac{dx}{x}.$$

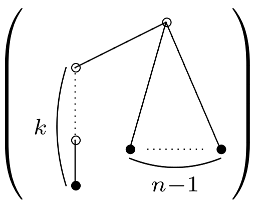
Now we use integral expressions

$$Li_k(x) = \int_{x>t_1>\cdots>t_k>0} \frac{dt_1}{t_1} \cdots \frac{dt_{k-1}}{t_{k-1}} \frac{dt_k}{1-t_k}$$

and

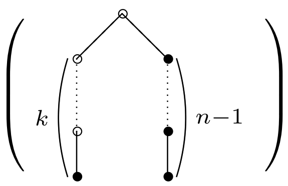
$$-\log(1-x) = \int_0^x \frac{du}{1-u},$$

to deduce for any integer $n > 0$

(3.1) $$\xi_k(n) = \frac{1}{(n-1)!} I$$ 

.

Moreover, there are exactly $(n-1)!$ ways to impose a total order on the $n-1$ black vertices. Thus we have the identity

$$\xi_k(n) = I$$ 

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Therefore, by (1.8), we obtain Ohno's relation [O, Theorem 2]

$$\xi_k(n) = \zeta^\star(k+1, \underbrace{1, \ldots, 1}_{n-1}).$$

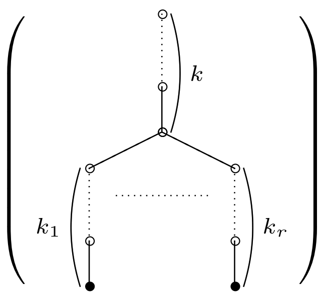
§ 3.2. Mordell-Tornheim zeta values

Next, we consider the values of the Mordell-Tornheim multiple zeta functions [M]

$$\zeta_{MT,r}(s_1, \ldots, s_r; s) = \sum_{m_1, \ldots, m_r > 0} \frac{1}{m_1^{s_1} \cdots m_r^{s_r} (m_1 + \cdots + m_r)^s}.$$

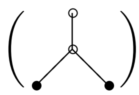
For positive integers $k_1, \ldots, k_r, k$, it is easy to show that

(3.2) $$\zeta_{MT,r}(k_1, \ldots, k_r; k) = I$$



For example,

$$I$$



$$\begin{aligned}
&= \int_{1>t_1>t_2>0} \frac{dt_1}{t_1} \frac{dt_2}{t_2} \left(\int_0^{t_2} \frac{du}{1-u} \right) \left(\int_0^{t_2} \frac{dv}{1-v} \right) \\
&= \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} \frac{dt_2}{t_2} \left(\sum_{m>0} \frac{t_2^m}{m} \right) \left(\sum_{n>0} \frac{t_2^n}{n} \right) \\
&= \sum_{m,n>0} \frac{1}{mn} \int_0^1 \frac{dt_1}{t_1} \int_0^{t_1} t_2^{m+n} \frac{dt_2}{t_2} \\
&= \sum_{m,n>0} \frac{1}{mn(m+n)} \int_0^1 t_1^{m+n} \frac{dt_1}{t_1} \\
&= \sum_{m,n>0} \frac{1}{mn(m+n)^2} = \zeta_{MT,2}(1,1;2).
\end{aligned}$$

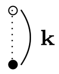
The identity (3.2) implies, in particular, the result by Bradley-Zhou [BZ] that the Mordell-Tornheim zeta value $\zeta_{MT,r}(k_1, \ldots, k_r; k)$ is expressed as a finite sum of MZVs of weight $k_1 + \cdots + k_r + k$ and depth r.

§ 3.3. Certain zeta values of root systems of type A

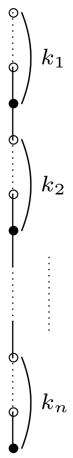
The third class of examples is a certain type of special values of zeta functions of root systems of type A_N, considered by Komori, Matsumoto and Tsumura [KMT] in a study of shuffle relations of MZVs. Explicitly, these values are written as

$$\zeta\left(p_1,\dots,p_a;\begin{matrix}q_1,\dots,q_b\\ r_1,\dots,r_c\end{matrix}\right)=\sum_{\substack{l_1>\cdots>l_a>m_1+n_1\\ m_1>\cdots>m_b>0\\ n_1>\cdots>n_c>0}}\frac{1}{l_1^{p_1}\cdots l_a^{p_a}m_1^{q_1}\cdots m_b^{q_b}n_1^{r_1}\cdots n_c^{r_c}},$$

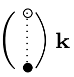
for three sequences $(p_1,\dots,p_a)$, $(q_1,\dots,q_b)$ and $(r_1,\dots,r_c)$ of positive integers. To describe the corresponding diagram, we introduce an abbreviation: For a sequence $\mathbf{k}=(k_1,\dots,k_n)$ of positive integers, we write



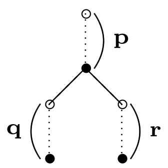
for the vertical diagram



so that

$$\zeta(\mathbf{k})=I\Bigg($$  $$\Bigg).$$

Using this notation, one can verify that

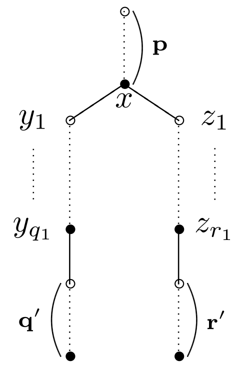
(3.3) $$\zeta(\mathbf{p};\mathbf{q};\mathbf{r})=I\Bigg($$  $$\Bigg)$$

for $\mathbf{p}=(p_1,\dots,p_a)$, $\mathbf{q}=(q_1,\dots,q_b)$ and $\mathbf{r}=(r_1,\dots,r_c)$.

In [KMT], the following relation plays an important role:

$$\begin{aligned}&\zeta\left(p_1,\dots,p_a;\begin{matrix}q_1,\dots,q_b\\ r_1,\dots,r_c\end{matrix}\right)\\ &=\sum_{j=0}^{q_1-1}\binom{r_1-1+j}{j}\zeta\left(p_1,\dots,p_a,r_1+j;\begin{matrix}q_1-j,q_2,\dots,q_b\\ r_2,\dots,r_c\end{matrix}\right)\\ &\quad+\sum_{j=0}^{r_1-1}\binom{q_1-1+j}{j}\zeta\left(p_1,\dots,p_a,q_1+j;\begin{matrix}q_2,\dots,q_b\\ r_1-j,r_2,\dots,r_c\end{matrix}\right).\end{aligned}\tag{3.4}$$

We point out that our expression (3.3) implies this relation quite naturally. To do this, we denote by X the 2-labeled poset indicated in (3.3), and name some vertices as follows:



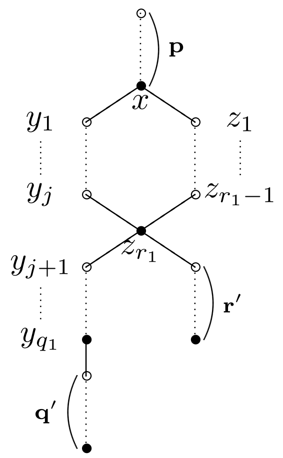
where $\mathbf{q}' = (q_2, \ldots, q_b)$ and $\mathbf{r}' = (r_2, \ldots, r_c)$. By (2.3), one has

$$I(X) = I(X_{y_{q_1}}^{z_{r_1}}) + I(X_{z_{r_1}}^{y_{q_1}}). \tag{3.5}$$

In $X_{y_{q_1}}^{z_{r_1}}$, the inequalities $x > z_{r_1} > y_{q_1}$ and $x > y_1 > \cdots > y_{q_1}$ hold, hence one can consider q_1 further refinements by imposing $y_j > z_{r_1} > y_{j+1}$ for $j = 0, \ldots, q_1 - 1$ (here we write $y_0 = x$). Thus the identity

$$I(X_{y_{q_1}}^{z_{r_1}}) = \sum_{j=0}^{q_1-1} I(X_j) \tag{3.6}$$

holds, where X_j is represented by the diagram



Moreover, since there are $\binom{r_1-1+j}{j}$ ways to impose a total order on the white vertices $y_1, \ldots, y_j$, $z_1, \ldots, z_{r_1-1}$, one has

$$I(X_j) = \binom{r_1 - 1 + j}{j} \zeta\left(p_1, \ldots, p_a, r_1 + j; \begin{matrix} q_1 - j, q_2, \ldots, q_b \\ r_2, \ldots, r_c \end{matrix}\right). \tag{3.7}$$

The identities (3.6) and (3.7) expresses the first term of (3.5) as desired in (3.4). The second is obtained in the same way.

Remark. In [KMT], using partial fraction decompositions, the identity (3.4) is proved with some variables (irrelevant to the decomposition) *complex valued*, not necessarily positive integral. It seems difficult to apply our method in this paper to such functional relations.

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