

Commutators and Iterated Commutators in Kleinian Groups

Harushi FURUSAWA

Kanazawa Gakuin University

古澤 治司 (金沢学院大学)

1. Introduction.

Let M denote the group of all Möbius transformations of the extended complex plane $\hat{C} = C \cup \{\infty\}$. We associate with each

$$f = \frac{az + b}{cz + d} \in M, ad - bc = 1,$$

the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, C).$$

We set $tr(f) = tr(A)$ where $tr(A)$ denotes the trace of A . Next for each f and g in M we let $[f, g]$ denote the commutator $fgf^{-1}g^{-1}$. We define the three complex numbers

$$(1) \quad \beta(f) = tr^2(f) - 4, \quad \beta(g) = tr^2(g) - 4, \quad \gamma(f, g) = tr([f, g]) - 2$$

the parameters of the two generator subgroup $\langle f, g \rangle$.

The following inequality [5][7] gives an important necessary condition for a two generator group $G = \langle f, g \rangle$ to be nonelementary and discrete.

Lemma 1 [5][7]. *If $\langle f, g \rangle$ is nonelementary and discrete, then*

$$(2) \quad |\gamma(f, g)| + |\beta(f)| \geq 1 \quad \text{and} \quad |\gamma(f, g) - \beta(f)| + |\beta(f)| \geq 1.$$

If $\langle f, g \rangle$ is a nonelementary discrete Fuchsian subgroup of M , then

$$(3) \quad |\gamma(f, g)| \geq 2 - 2 \cos(\pi/7).$$

See, [10].

When $\langle f, g \rangle$ is not Fuchsian, we can not obtain lower bounds for $|\gamma(f, g)|$ in a nonelementary discrete group $\langle f, g \rangle$. For example, Jørgensen has shown that if $1 < a < \infty$, then the transformations

$$f(z) = -a^2 z, \quad g(z) = \frac{(a^2 + a^{-2})z - 2}{2z - (a^2 + a^{-2})}$$

generate a nonelementary discrete subgroup of M with

$$\beta(f) = -(a + a^{-1})^2, \quad \beta(g) = -4, \quad \gamma(f, g) = 4(a - a^{-1})^{-2}.$$

Since $\gamma(f, g) \rightarrow 0$ as $a \rightarrow \infty$, (3) fails if $\langle f, g \rangle$ is not Fuchsian. But if we require some restrictive conditions for elements of generators, then we have the lower bounds for $|\gamma(f, g)|$. For example, if f or g is parabolic, then,

$$(4) \quad |\gamma(f, g)| \geq 1.$$

And further, Jørgensen proved in [8] that

$$(5) \quad |\gamma(f, g)| \geq \frac{1}{8},$$

if $\beta(f) = \beta(g)$

Gehring and Martin showed the following lemma 2.

Lemma 2[4]. *Suppose that $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0, \beta(f) = \beta(g) \neq -4$. Then*

$$(6) \quad |\gamma(f, g)| > 0.193.$$

The constant 0.193 is not sharp. Gehring and Martin considered the subgroup $\langle [f, g], f[f, g]f^{-1} \rangle$ of $\langle f, g \rangle$ and showed $|\gamma([f, g], f[f, g]f^{-1})| + |\beta[f, g]| < 1$ under the condition $|\gamma(f, g)| \leq 0.193$.

Lemma 3[4]. *Suppose that $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0, \beta(f) = \beta(g) \neq -4$, and*

$$\min\{|\beta(f)|, |\beta(fg)|, |\beta(fg^{-1})|\} \geq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}. \text{ Then}$$

$$(7) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7).$$

Proof. If A represents f , then by replacing A by $-A$ if necessary, we may assume that $\text{tr}(f) = \text{tr}(g)$. Let $a = \text{tr}(fg) - 2$, $b = \text{tr}(fg^{-1}) - 2$, $c = 2(\cos(2\pi/7) + \cos(\pi/7) - 1)$. Next by replacing g by g^{-1} we may assume that $|a| \leq |b|$. Then $\beta(f) = a + b$, $\beta(fg) = a^2 + 4a$, $\gamma(f, g) = -ab$ by Fricke's formula. Hence

$$(|a| + 2)^2 \geq |\beta(fg)| + 4 \geq c + 4 = (2\cos(2\pi/7) + 1)^2$$

by the assumption of Lemma 3 and $|a| \geq d$ where $d = 2\cos(2\pi/7) - 1$.

If $|a| \geq 0.5$, then $|\gamma(f, g)| = |ab| \geq |a|^2 \geq 0.25$ while if $|a| < 0.5$, then $|a| < c/2$ and

$$|\gamma(f, g)| = |a||\beta(f) - a| \geq d(c - d) = 2 - 2\cos(\pi/7)$$

by the assumption. This completes the proof of Lemma 3.

Remark 1. To show that (7) is sharp, let $\langle \phi, \psi \rangle$ be the $(2, 3, 7)$ triangle group with $\phi^2 = \psi^3 = (\phi\psi)^7 = \text{id}$. and set $f = [\phi, \psi]$, $h = \phi\psi$, $g = hf^{-1}h^{-1}$. Then $\beta(f) = \beta(g) = c$

where $c = 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$, $\beta(fg) = \beta(f, h)\{\beta(f, h) + 4\} = c$, $\beta(fg^{-1}) = \beta(fghf^{-1}) = \{\beta(f) - \gamma(f, h)\}\{\beta(f) - \gamma(f, h) + 4\} \geq c$ and $\gamma(f, g) = 2\cos(\pi/7) - 2$. And also, we have $\gamma(f, g) = \gamma(fg, g) = \gamma(fg^{-1}, g)$.

Lemma 4. Suppose that $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0$, $\beta(g) \neq -4$, and $|\beta(f)| \leq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. Then

$$(8) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7) \quad \text{or} \quad |\gamma(f, g) - \beta(f)| > 1.$$

The following Lemma 5 is a direct consequence of Lemma 4.

Lemma 5. Suppose that $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0$, $\beta(g) \neq -4$, and $|\beta(f)| \leq 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. Then

$$(9) \quad |\gamma(f, g)| \geq 2 - 2\cos(\pi/7) \quad \text{or} \quad |\gamma(f, gfg^{-1})| \geq 2 - 2\cos(\pi/7)$$

Proof. Suppose $|\gamma(f, g)| < 2 - 2\cos(\pi/7)$, then $|\beta(f) - \gamma(f, g)| > 1$ from Lemma 4. Next, $\langle [f, g], f \rangle$ is a discrete subgroup of $\langle f, g \rangle$ with $\gamma' = \gamma([f, g], f) = \gamma(\gamma - \beta) \neq 0$, $\beta' = \beta[f, g] = \gamma(\gamma + 4) \neq -4$ and $|\beta'| \leq 0.9 < c$ where $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$. Then we have $|\gamma'| \geq 2 - 2\cos(\pi/7)$ or $|\beta' - \gamma'| > 1$ from Lemma 4. But the second case does not yield since $|\beta' - \gamma'| = |\gamma\|\beta + 4| < 1$. This complete the proof.

It is easily seen that $\gamma(f, g) \neq 0$ if and only if $\text{fix}(f) \cap \text{fix}(g) = \emptyset$ where $\text{fix}(f)$ denotes the fixed point set of f in $\hat{C}$.

The triple $\beta(f), \beta(g), \gamma(f, g)$ with $\gamma(f, g) \neq 0$ and $\beta(f) \neq 0$ determine the two generator group $\langle f, g \rangle$ up to conjugacy. Since $\gamma(f, g) \neq 0$, f and g have no fixed points in common and there exists $\phi \in M$ such that $\phi f \phi^{-1}(0) = 0$, $\phi f \phi^{-1}(\infty) = \infty$ and $\phi g \phi^{-1}(1) = 1$. Then

$$\phi f \phi^{-1}(z) = kz, \quad \phi g \phi^{-1}(z) = \frac{az + b}{cz + d},$$

where $ad - bc = 1$, $a + b = c + d$, $bc \neq 0$ and $\arg(a + d) \in [0, \pi)$ if $a + d \neq 0$. Next by replacing ϕ by $\psi\phi$ where $\psi(z) = \frac{1}{z}$, we may arrange that $|\frac{b}{c}| \geq 1$ with $\text{Im}(\frac{b}{c}) \geq 0$ if $|\frac{b}{c}| = 1$. Next by replacing f by f^{-1} and g by g^{-1} if necessary, we may also assume that $|k| \geq 1$ with $\text{Im}(k) \geq 1$ if $|k| = 1$ and that $\arg(a - d) \in [0, \pi)$ or $\arg(b + c) \in [0, \pi)$ if $a = d$. Then the equations $\beta = k - 2 + \frac{1}{k}$, $\beta' = (a + d)^2 - 4 = (a - d)^2 + 4bc = (b + c)^2$, $\gamma = -bc\beta$, $\frac{\beta\beta'}{\gamma} = -(\frac{b}{c} + 2 + \frac{c}{b})$ determine $k, a, d, \frac{b}{c}, b + c$ and therefore b, c uniquely.

2. Preliminary.

We state two lemmas needed to establish theorems. These statements are easily obtained by operating matrices.

Lemma 6[4]. If f and g are in M with $\gamma(f, g) = \gamma$ and $\beta(f) = \beta$, then

$$(10) \quad \gamma(f, gfg^{-1}) = \gamma(f, [f, g]) = \gamma(\gamma - \beta) \quad \text{and} \quad \beta([f, g]) = \gamma(\gamma + 4).$$

Lemma 7[4]. *If $\langle f, g \rangle$ is an elementary discrete subgroup of M with $\gamma(f, g) \neq 0$, $\beta(f) = \beta(g) \neq -4$, then*

$$(11) \quad |\gamma(f, g)| \geq (3 - \sqrt{5})/2.$$

3. A lower bound for the commutator.

The following lemma gives a key tool to show Theorem 10.

Lemma 8[4]. *If $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0$, then*

$$(12) \quad |\beta(f) - 1| \leq |\gamma(f, g)|^2 / (1 - |\gamma(f, g)|) \quad \text{or} \quad |\beta(f) - 1| \geq 1 - |\gamma(f, g)|.$$

Proof. Suppose (12) does not hold and let $\gamma = \gamma(f, g)$ and $\beta = \beta(f)$. Then $\frac{|\gamma|^2}{1-|\gamma|} < |\beta - 1| < 1 - |\gamma|$ and hence $p(\gamma) = \{\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\}^2 \neq 0$ since otherwise we would have

$$|\beta - 1| = \left| \frac{\gamma^2}{\gamma + 1} \right| \leq \frac{|\gamma|^2}{1 - |\gamma|}$$

contradicting the first statement.

Let $\gamma_j = \gamma(f, g_j)$ where

$$g_{j+1} = g_j f^{-1} g_j^{-1} f g_j f g_j^{-1} f^{-1} g_j \quad (j = 1, 2, \dots), \quad g_1 = g.$$

Then $\gamma_1 = \gamma$, $|\gamma_{j+1}| = |\gamma_j| |p(\gamma_j)|$ and

$$|\gamma_j| + |\beta - 1| < 1, \quad p(\gamma_j) \neq 0$$

implies that

$$0 < |p(\gamma_j)| = |\gamma_j^2 - (\beta - 1)\gamma_j - (\beta - 1)|^2 < (|\gamma_j| + |\beta - 1|)^2 < 1.$$

Thus we see by induction that $0 < |\gamma_j| \leq |\gamma_1| (|\gamma_j| + |\beta - 1|)^{2j} \rightarrow 0$ as $j \rightarrow \infty$ contradicting Lemma 12. This completes the proof.

Lemma 9. *If $\langle f, g \rangle$ is a non-elementary discrete subgroup of M with $\gamma(f, g) \neq 0$, and $|\gamma(f, g)| < 2 - 2\cos(\pi/7)$, then*

$$(13) \quad |\beta(f) - 1| \geq 1 - |\gamma(f, g)|.$$

Proof. Suppose that $|\beta(f) - 1| \leq |\gamma(f, g)|^2 / (1 - |\gamma(f, g)|)$, and we lead the contradiction. Let $\delta = \inf\{|\gamma(f, g)|; \gamma \neq 0, \beta(f) = \beta(g), |\beta(f)| \leq c, \langle f, g \rangle \text{ discrete}\}$ where $c = 2\{\cos(2\pi/7) + \cos(\pi/7) - 1\}$. It is easily seen $0.193 < \delta < d$. For sufficiently small $\epsilon > 0$ with $0 < \epsilon < (d - \delta)/2 < d$, we have a discrete group $G = \langle f, g \rangle$ such that $|\gamma(f, g)| \leq \delta + \epsilon$, $|\beta(f)| \leq c$ and $\beta \neq -4$. If $\beta(f) - \gamma(f, g) = 1/\{\gamma(f, g) + 1\}$ for the given

discrete group G , we consider the subgroup $\langle A_1, A_0 A_1 A_0^{-1} \rangle$ of G where $A_0 = [f, g]$ and $A_1 = [A_0, f]$. Then we have

$$\delta \leq |\gamma(A_1, A_0 A_1 A_0^{-1})| = |\gamma^2(\gamma - \beta)^2 \gamma(\beta + 4)(\gamma + 2)^2| \leq |\gamma|^2 |1 + \frac{1}{\gamma + 1}|^2.$$

Since the function $|z| |1 + \frac{1}{z+1}|^2$ is subharmonic in the closed disk $\{z; |z| \leq d\}$, then we have the maximum value 1 of $|z| |1 + \frac{1}{z+1}|^2$ at $z = -d$. Therefore $0 < \max_{0.193 \leq |\gamma| \leq \delta + (d-\delta)/2} |z| |1 + \frac{1}{z+1}|^2 = k < 1$ and also we have $\delta \leq |\gamma(A_1, A_0 A_1 A_0^{-1})| \leq |\gamma| k \leq (\delta + \epsilon)k$. We remark that $|\beta(A_0 A_1 A_0^{-1})| = |\beta(A_1)| = |\gamma(\gamma - \beta)| |\gamma(\gamma - \beta) + 4| < c$. Thus if we select $\epsilon < \frac{(1-k)\delta}{k}$, this leads the contradiction. Therefore we have $\beta(f) - \gamma(f, g) \neq 1/\{\gamma(f, g) + 1\}$ for the given group G . Next we consider the subgroup $\langle f, h f h^{-1} \rangle$ where $h = g f^{-1} g^{-1} f g f g^{-1} f^{-1} g$, then

$$\gamma(f, h f h^{-1}) = \gamma\{\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\}^2 [\gamma\{\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\}^2 - \beta].$$

The above argument states $\gamma^2 - (\beta - 1)\gamma - (\beta - 1) \neq 0$. The assumption $|\gamma| < d$ and Jørgensen's inequality shows $|\beta| \geq 1 - d > 0.8$, then $|\gamma\{\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\}^2 - \beta| > 0.7$. Therefore we have $\gamma(f, h f h^{-1}) \neq 0$ but $|\gamma\{\gamma^2 - (\beta - 1)\gamma - (\beta - 1)\}^2| < 0.0125$ shows $0 < |\gamma(f, h f h^{-1})| < 0.013$, this contradicts to Lemma 2.

The following theorem is essentially based on Lemma 9 and we consider the subgroup $\langle f, h f h^{-1} \rangle$ of $\langle f, g \rangle$. By the analysis with the help of computer, we have the following theorem.

Theorem 10. Suppose that $\langle f, g \rangle$ is a discrete subgroup of M with $\gamma(f, g) \neq 0, \beta(f) = \beta(g) \neq -4$, then

$$(14) \quad |\gamma(f, g)| \geq 2 - 2 \cos(\pi/7).$$

The following Lemma 11 shows that Theorem 10 is applicable to the collar lemma. Let A_f be the axis of $f (\in M)$ in H^3 connecting with two fixed points in ∂H^3 .

Lemma 11[2][6]. Let f and g be nonparabolic elements in M and let μ be the complex distance between A_f and A_g . If $\gamma = \text{tr}(f g f^{-1} g^{-1}) - 2 \neq 0$, then

$$(15) \quad 4\gamma = \beta(f)\beta(g) \sinh^2(\mu).$$

If we consider the subgroup $\langle f, g f g^{-1} \rangle$ of $\langle f, g \rangle$ and suppose $\gamma(f, g f g^{-1}) \neq 0$, then $\gamma(f, g f g^{-1}) = \beta(f)^2 \sinh^2 \mu(A_f, g(A_f))$.

4. Iterative commutators.

Let β be a complex number. Let $R(z) = z(z - \beta)$, R^n denote the n -th iterate of

R and $R^0 = id.$ We define $g_0 = g, g_1 = gfg^{-1}$ and $g_{n+1} = g_nfg_n^{-1}$, inductively and consider a subgroup $\langle f, g_nfg_n^{-1} \rangle$ of a Kleinian group $\langle f, g \rangle$. Then we find $R^{n+1}(\gamma) = \gamma(f, g_n)\{\gamma(f, g_n) - \beta(f)\} = \gamma(f, g_{n+1})$ for any positive integer n , inductively.

Lemma 12[4]. *If f and g_j are elements of a Kleinian group G with $\gamma(f, g_j) \neq 0$ for $j = 1, 2, \dots$ and if f is not of order 2, then*

$$(16) \quad \liminf_{n \rightarrow \infty} |\gamma(f, g_j)| > 0$$

Proof. Let $\gamma_j = \gamma(f, g_j)$. If $\liminf_{n \rightarrow \infty} |\gamma(f, g_j)| = 0$, then we can choose j so that $2|\gamma_j| < \min \left[\frac{1}{|\beta|+1}, \frac{1}{|\gamma_j|+4} \right]$ and $\gamma_j \neq \beta$. Then $\gamma'_j = \gamma([f, g_j], f) = \gamma_j(\gamma_j - \beta)$ and $0 < |\gamma'_j| < 1/2$ and $\beta'_j = \beta([f, g_j]) = \gamma_j(\gamma_j + 4)$ and $0 < |\beta'_j| < 1/2$. Now $\gamma'_j \neq 0$ and $\gamma'_j \neq \beta'_j$ and the hypothesis that $\beta \neq -4$. Since $\langle f, [f, g_j] \rangle$ is a Kleinian subgroup of $\langle f, g \rangle$, $|\gamma'_j| + |\beta'_j| \geq 1$ and we have a contradiction.

Corollary 13. *Let $\langle f, g_{n-1}fg_{n-1}^{-1} \rangle$ be a subgroup of a Kleinian group $\langle f, g \rangle$, $R^n(\gamma) \neq 0, \beta(f) \neq -4$, then*

$$(17) \quad |R^n(\gamma)| \geq 2 - 2 \cos(\pi/7).$$

Lemma 14[9]. *Let $\langle f, g \rangle$ is Kleinian with $\gamma \neq 0$ and $R(\gamma) = 0$, then either f is elliptic of order 2, 3, 4 or 6 or g is elliptic of order 2.*

Let $\langle f, g \rangle$ be Kleinian with $|\beta(\beta + 2)| \leq 1$, and if $|2\gamma - \beta| < 1 + \sqrt{1 - |\beta(\beta + 2)|}$, then we have $|2R(\gamma) - \beta| < 1 + \sqrt{1 - |\beta(\beta + 2)|}$. Let D_r be the open disk centered at $\beta/2$ with the radius $r = \{1 + \sqrt{1 - |\beta(\beta + 2)|}\}/2$. Then, we have the following lemma.

Lemma 15. *Suppose that $\langle f, g \rangle$ is Kleinian with $|\beta(\beta + 2)| \leq 1$. If γ is in the open disk D_r stated above, then*

$$(18) \quad R^n(\gamma) \in D_r.$$

Lemma 16[5]. *Suppose that $\langle f, g \rangle$ is a non-elementary Kleinian group. If $|\beta(\beta + 2)| \leq 1$, $\gamma \neq \beta + 1$ and $\gamma \neq -1$, then*

$$(19) \quad |2\gamma - \beta| \geq 1 + \sqrt{1 - |\beta(\beta + 2)|}.$$

Proof. Suppose that $\langle f, g \rangle$ is non-elementary, Kleinian with $\gamma \neq \beta + 1$ and $\gamma \neq -1$. Suppose that $\gamma \in D_r$, and let $\gamma_j = \gamma(f, g_j)$ where $g_1 = g$ and $g_{j+1} = g_jfg_j^{-1}$ for $j = 1, 2, \dots$. If $z \in D_r$, then $|2R(z) - \beta| \leq r$. Thus $R^j(D_r) \subset D_r$, D_r lies in the Fatou set for $R(z)$ and $\gamma_{j+1} = R^j(\gamma) \in D_r$ for $j \geq 0$. If $\beta = 0$, then D_r is the unit disk, f is parabolic and

$|\gamma| \geq 1$, hence $\gamma \notin D_r$. Next if $\beta = -1$, then $D_r = \{z; |2z + 1| < 1\}$, f is of order 6, $\gamma_1 \neq 0$ and $\gamma_{j+1} = \gamma_j(\gamma_j - \beta - 1)^2 = \gamma_j^3 = \gamma^{3^j}$. Therefore $|\gamma| \geq 1$ by Lemma 12. Finally if $\beta = -2$, then $D_r = \{z; |z + 1| < 1\}$, $0 < |\gamma_{j+1} + 1| = |R(\gamma_j) + 1| = |\gamma_j + 1|^2 = |\gamma + 1|^{2^j}$ for $j \geq 0$, and $\gamma \notin D_r$ by Lemma 12. We have that $\beta \notin \{0, -1, -2\}$. Suppose that $|\beta| < 1$. Then 0 is an attracting fixed point for $R(z)$, $0 \in D_r$ and $\gamma_j \rightarrow 0$ as $j \rightarrow \infty$. Hence $\gamma_j = \gamma(f, g_j) = 0$ for some j by Lemma 12. Let k be the smallest such integer j . Since $\gamma \notin \{0, \beta\}$, $\gamma_1 = \gamma \neq 0$ and $\gamma_2 = R(\gamma) \neq 0$, therefore $k \geq 3$. Then $R(\gamma_{k-1}) = 0$, $\gamma_{k-1} \neq 0$ and $\gamma(f, g_{k-1}) = \gamma_{k-1} = \beta(g_{k-1}) \neq 0$. Thus f and g_{k-1} are elliptic of order 2, 3, 4, or 6 by Lemma 13 and $|\beta| \geq 1$, thus we have a contradiction. Finally suppose that $|\beta| \geq 1$. Then $|\beta + 2| < 1$ by the assumption and the fact that $\beta \notin \{-1, -2\}$, $\beta + 1$ is an attracting fixed point for $R(z)$ and $\beta \in D_r$. Thus as above, $\gamma_j = \beta + 1$ for some j by Lemma 12. Let k be the smallest such integer j . Because $\gamma_1 = \gamma \notin \{-1, \beta + 1\}$, $\gamma_1 \neq \beta + 1$ and $\gamma_2 = R(\gamma) \neq \beta + 1$, thus $k \geq 3$. Thus $\gamma_{k-2} \in R^{-1}(-1)$ and $|2\gamma_{k-2} - \beta| = \sqrt{|\beta^2 - 4|} > 1 + \sqrt{1 - |\beta(\beta + 2)|}$. In particular, $\gamma_{k-2} \notin D_r$ which contradict $\gamma_{j+1} \in D_r$. This completes the proof of Lemma 16.

Lemma 17. Suppose that $\langle f, g \rangle$ is non-elementary, Kleinian. If $|\beta(\beta + 2)| \leq 1$ and $\gamma \in D_r$, then

$$(20) \quad \gamma = \beta + 1 \quad \text{or} \quad \gamma = -1.$$

Lemma 18. Suppose that $\langle f, g \rangle$ is non-elementary, Kleinian. If f is elliptic of order $n \geq 6$, then

$$(21) \quad |2\gamma - \beta| \geq 1 + \sqrt{1 - |\beta(\beta + 2)|}.$$

Theorem 19. Suppose that $\langle f, g \rangle$ is Kleinian with $\gamma(f, g) \neq 0$ and $|\gamma - \beta/2| > |\beta^2/4 + \beta/2| + 1$, then $R^n(\gamma) \rightarrow \infty$ as $n \rightarrow \infty$.

Proof. Let $G(z) = z^2 + c$. If $|z| > |c| + 1$, then the orbit of z under $G(z)$ is not bounded. Since $|z^2 + c| \geq |z||z| - \frac{|c|}{|z|} > |z||c| + 1 - \frac{|c|}{|c|+1}$ for $|z| > |c| + 1$, we have $|G(z)| > r|z|$ where $r = 1 + |c|^2/(|c| + 1) > 1$. Let $H(z) = z - \beta/2$, then $HRH^{-1}(z) = z^2 - \beta^2/4 - \beta/2$. Therefore we have $|R^n(\gamma) - \beta/2| \geq r^n|\gamma - \beta/2|$ where $r > 1$, $n \geq 1$. Thus we have Theorem 19.

If $\langle f, g \rangle$ is non-elementary Kleinian group, then Jørgensen showed the inequalities $|\gamma| + |\beta| \geq 1$ and $|\gamma - \beta| + |\beta| \geq 1$. We say that a non-elementary Kleinian group $\langle f, g \rangle$ is extremal if one of these inequalities holds with equality. There exists many groups which are extremal for these inequalities; for example, the triangle groups with signatures $(2, 3, n)$ with $n \geq 7$ have this property.

Let $\langle f, g \rangle$ be a non-elementary Kleinian group where f is not parabolic. Then $|\beta| > 0$ and $|\gamma| > 0$. Let $U = \{z; |z| \leq 1 - |\beta|\}$, $U' = \{z; |z - \beta| \leq 1 - |\beta|\}$ and

$V = \{z; |z - \beta| \leq 1\}$. It is easy to show that $R(U) \subset U, R(U') \subset U, U \subset D_\beta$ where $D_\beta = \{z \in C; \sup |R^n(z)| < \infty\}$. Suppose that $\langle f, g \rangle$ is extremal in the inequality $|\gamma| + |\beta| \geq 1$. Then $|\beta| < 1, \gamma \in \partial U, R^n(\gamma) \in U$ and $\gamma \notin \text{int}(D_\beta)$ because Gehring and Martin proved in [6] that if $\langle f, g \rangle$ is non-elementary, Kleinian and $0 < |\beta| < 1$, then $\gamma \notin D_\beta$. The complete invariance of $\text{int}(D_\beta)$ under R implies that $R^n(\gamma) \in \partial U$ for $n \geq 0$. Therefore we have

$$1 - |\beta| = |R^n(\gamma)| = |R^{n-1}(\gamma)| |R^{n-1}(\gamma) - \beta| = (1 - |\beta|) |R^{n-1}(\gamma) - \beta|,$$

for $n \geq 1$. and we have $\gamma, R(\gamma) \in \partial U \cap \partial V$. Now $U \subset V, U \neq V$ and hence the circles $\partial U, \partial V$ meet in just one point. Thus $R(\gamma) = \gamma$ and $\gamma = \beta + 1$. Suppose next that $\langle f, g \rangle$ is extremal in the sense $|\gamma - \beta| + |\beta| = 1$. Then $\gamma \in \partial U', R^n(\gamma) \in U, \gamma \notin \text{int}(D_\beta)$ and $R^n(\gamma) \in \partial U$ for $n \geq 1$. Therefore $R(\gamma), R^2(\gamma) \in \partial U \cap \partial V$. Thus $R(\gamma) = \beta + 1$ and $\gamma = -1$ since $|\gamma - \beta| < 1$. Finally we have the followings.

Lemma 20[6]. *Suppose that $\langle f, g \rangle$ is non-elementary, Kleinian and f is not parabolic. If $\langle f, g \rangle$ satisfies the condition $|\gamma| + |\beta| = 1$, then $\gamma = \beta + 1$. And also if $\langle f, g \rangle$ satisfies $|\gamma - \beta| + |\beta| = 1$, then $\gamma = -1$.*

Therefore, if a non-elementary Kleinian group $\langle f, g \rangle$ is external, then $R(\gamma) = \gamma$ or $R^2(\gamma) = R(\gamma)$.

Remark 2. Let $\langle \phi, \psi \rangle$ be the $(2, 3, 7)$ triangle group stated in Remark 1. And also set $f = [\phi, \psi], h = \phi\psi, g = hf^{-1}h^{-1}$, then $\gamma(f, g) = 2 \cos(\pi/7) - 2, R(\gamma) = 2 \cos(2\pi/7) - 1$ and we have $R^2(\gamma) = \gamma$ for $\langle f, g \rangle$.

5. An arithmetic condition

Let $\langle f, g \rangle$ be a non-elementary, Kleinian and set $C = \{\gamma; \gamma(f', g') \neq 0, f', g' \in \langle f, g \rangle\}$. Applying chapter 4 to this, we have $R^n(\gamma) \in C$ if $\gamma \in C$. We establish here a description of the elements h in $\langle f, g \rangle$ for which $\gamma(f, h)$ can be expressed as a polynomial in $\gamma(f, g)$ and $\beta(f)$ with integer coefficients.

Lemma 21. *Suppose that r is an integer and $u \in C \setminus \{0\}$. Then*

$$(22) \quad u^{2r} + u^{-2r} = p_r((u - u^{-1})^2),$$

$$(23) \quad u^{2r+1} + u^{-(2r+1)} = (u + u^{-1})q_r((u - u^{-1})^2),$$

where p_r and q_r are polynomials with integer coefficients and where $p_r(0) = 2$ and $q_r(0) = 1$.

Theorem 22. *Let f and g are in M and that*

$$(24) \quad h = f^{r_1} g^{s_1} \dots f^{r_n} g^{s_n} f^{r_{n+1}}$$

Then $tr(h) = (tr(f))^\epsilon p(\gamma(f, g), \beta(f))$ where p is a polynomial in both variables with integer coefficients and where $\epsilon = 0$ or 1 depending on whether $r_1 + r_2 + \dots + r_{n+1}$ is even or odd.

Proof. If $r'_1 = r_1 + r_{n+1}$, then

$$tr(h) = tr(f^{r'_1} g^{s_1} \dots f^{r_n} g^{s_n}) = tr(f^{r_n} g^{s_n} f^{r'_1} g^{s_1} \dots f^{r_{n-1}} g^{s_{n-1}}).$$

Hence we may assume without loss of generality that $r_{n+1} = 0$ and that $s_j = (-1)^{j+1}$ for $j = 1, 2, \dots, n$. Suppose first that $n = 2$ and that f and g are represented by matrices

$$A = \begin{pmatrix} u & 0 \\ 0 & -u \end{pmatrix} \quad B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL(2, C)$ where we consider first f is nonparabolic. If f is parabolic, then we have a similar result so we omit that case. Then $tr(f) = (u + u^{-1})$, $\beta = (u - u^{-1})^2$, $\gamma = -bc(u - u^{-1})^2$ and we see that

$$tr(h) = (1 + bc)[u^{r_1+r_2} + u^{-(r_1+r_2)} - bc[u^{r_1-r_2} + u^{r_2-r_1}].$$

If $2r = r_1 + r_2$ is even, then so is $2s = r_1 - r_2$ and with above arguments we have

$$tr(h) = (1 + bc)p_r(\beta) - bcp_s(\beta) = p_r(\beta) - \gamma \frac{p_r(\beta) - p_s(\beta)}{\beta} = p(\gamma, \beta).$$

Then $p_r(0) - p_s(0) = 0$, hence p is a polynomial in γ and β with integer coefficients. If $2r + 1 = r_1 + r_2$ is odd, then so is $2s = r_1 - r_2$ thus we have

$$tr(h) = tr(f)[q_r(\beta) - \gamma \frac{q_r(\beta) - q_s(\beta)}{\beta}] = tr(f)p(\gamma, \beta).$$

Thus we have the proof of theorem for the case where $n = 2$. It is easy to show generally by inductive argument but we do not discuss here.

Lemma 23. If f and g are in M and

$$(25) \quad h = f^{r_1} g^{s_1} \dots f^{r_n} g^{s_n} f^{r_{n+1}}$$

where $s_1 = \pm 1$ and $s_j = (-1)^{j+1} s_1$ for $j = 1, 2, \dots, n$. Then

$$(26) \quad \gamma(f) = p(\gamma(f, g), \beta(f))$$

where p is a polynomial in both variables with integer coefficients. If, in addition, n is even, then $\beta(f) = q(\gamma(f, g), \beta(f))$ where q is a polynomial in both variables with integer coefficients.

Recently, Prof. Gehring announced that Chun Cao proved Theorem 10. We expect that his proof is not depend on computer and also this result has many applications.

References

- [1] A. F. Beardon, *The geometry of discrete groups*, Springer Verlag 1983.
- [2] H. Furusawa, A remark on the hyperbolic collar lemma, *Tôhoku Math. J.* 39 (1987) 291-298.
- [3] F. W. Gehring and G. J. Martin, Inequalities for Möbius transformations and discrete groups, *J. Reine Angew. Math.* 418(1991) 31-76.
- [4] F. W. Gehring and G. J. Martin, Some universal constraints for discrete Möbius groups, *Paul Halmos; Celebrating 50 Years of Mathematics*, Springer-Verlag, New York, (1991) 205-220.
- [5] F. W. Gehring and G. J. Martin, Iteration theory and inequalities for Kleinian groups, *Bull. (new series) of AMS* 21(1989) 57-63.
- [6] F. W. Gehring and G. J. Martin, Stability and extremality in Jørgensen's inequality, *Complex Variables*, 12(1987) 277-282.
- [7] T. Jørgensen, On discrete groups of Möbius transformations, *Amer. J. Math.* 98(1976) 739-749.
- [8] T. Jørgensen, Commutators in $SL(2, \mathbb{C})$, Riemann surfaces and related topics: Proceedings of the 1978 Stony Brook Conference, Princeton Univ. Press 1980, 301-303.
- [9] B. Maskit, Some special 2-generator Kleinian groups, *Proc. Amer. Math. Soc.* 106(1989) 175-186.
- [10] G. Rosenberger, All generating pairs of all two-generator Fuchsian groups, *Arch. Math.* 46(1986) 198-204.
- [11] C. R. Traina, Trace polynomial for two generator subgroup of $SL(2, \mathbb{C})$, *Proc. Amer. Math. Soc.* 79(1980) 369-372.