Summary

Dynamics of nonlinear Schrödinger equations with high-frequency oscillations and their averaged models

JUMPEI KAWAKAMI

Abstract

We examine two topics related to nonlinear Schrödinger equations (NLS). The first topic pertains to NLS with a strong magnetic field, which describes fermion pairs in the state of Bose-Einstein Condensation. Mathematically, this model is represented by NLS with a partial harmonic oscillator that induces high-frequency oscillations. By considering its average and the limit concerning the strong magnetic field, an averaged model was derived. We establish both local and global well-posedness for these models and provide local approximations for the original NLS using the averaged model. Furthermore, we clarify the global dynamics of the averaged model, involving scattering and blow-up phenomena. In our analysis, we introduce new tools to effectively utilize the harmonic Schrödinger propagators in nonlinearity. As an application of these tools, we explore the second topic: the dispersion-managed nonlinear Schrödinger equations (DMNLS), which describes signal propagation in a fiber optic cable where dispersion varies periodically. For the DMNLS, there is an averaged model in the setting of "strong dispersion management". For the averaged DMNLS, we demonstrate small and large data scattering. We also provide remarks on blow up.

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1 Nonlinear Schrödinger equations in a strong magnetic field

1.1 Introduction to NLS in a strong magnetic field

We consider the following 3-dimensional nonlinear Schrödinger equations (NLS):

$$i\partial_t \psi(t,x) = \frac{1}{\varepsilon^2} H\psi(t,x) - \partial_z^2 \psi(t,x) + V(z)\psi(t,x) + \lambda |\psi(t,x)|^{2\sigma}\psi(t,x). \tag{1.1}$$

$$i\partial_t \phi(t,x) = -\partial_z^2 \phi(t,x) + V(z)\phi(t,x) + \lambda F_{\mathrm{av}}(\phi(t,x)). \tag{1.2}$$

Where $x = (y,z) \in \mathbb{R}^2 \times \mathbb{R}$, $\sigma > 0$, $0 < \varepsilon \leq 1$ and H represents the harmonic oscillator in the y-direction:

$$H := -\Delta_y + |y|^2.$$

The nonlinear term $F_{\mathrm{av}}(\phi)$ is defined as

$$F_{\mathrm{av}}(\phi) := \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{i\theta H} (|e^{-i\theta H}\phi|^{2\sigma} e^{-i\theta H}\phi) d\theta. \tag{1.3}$$

Equations (1.1) and (1.2) are mathematically normalized and generalized models of NLS with a strong magnetic field. We will follow the derivation of these models, which is attributed to the work of Frank-Méhats-Sparber [22].

Derivation of the models

In [22], the authors initially considered the following 3-dimensional NLS-type model:

$$i\partial_t\psi = \frac{1}{2}(-i\nabla_x + A^\varepsilon(x))^2\psi + V(z)\psi + \beta^\varepsilon|\psi|^{2\sigma}\psi, \tag{1.4}$$

where $x = (y, z) \in \mathbb{R}^2 \times \mathbb{R}$, $\sigma \in \mathbb{N}$, and $\beta^\varepsilon \in \mathbb{R}$. The real-valued potential V is assumed to be smooth and sub-quadratic:

$$\frac{d^\alpha}{dz^\alpha}V(z) \in L^\infty(\mathbb{R}) \qquad \text{for any} \quad \alpha \geq 2. \tag{1.5}$$

For example, $V(z) = \pm z^2$. The vector potential A^ε is given by

$$A^\varepsilon(x) = \frac{1}{2\varepsilon^2}(-y_2, y_1, 0),$$

where $0 < \varepsilon \ll 1$ is a small parameter. This model macroscopically describes fermion pairs in the state of Bose-Einstein condensation.

Because A^ε is divergence free, $\nabla_x \cdot A^\varepsilon = 0$, one has

$$(-i\nabla_x + A^\varepsilon(x))^2 = -\Delta_x + \frac{1}{4\varepsilon^4}|y|^2 - \frac{i}{\varepsilon^2}(y_1\partial_{y_2} - y_2\partial_{y_1}).$$

The vector

$$B^\varepsilon = \nabla \times A^\varepsilon = \frac{1}{\varepsilon^2}(0, 0, 1) \in \mathbb{R}^3$$

is the corresponding (constant) magnetic field in the z-direction with field strength $|B^\varepsilon| = \frac{1}{\varepsilon^2} \gg 1$.

The objective of [22] is to analyze "the strong magnetic confinement limit" as $\varepsilon \to +0$. Then, the initial data for equation (1.4) is assumed of the form

$$\psi(0, x) = \varepsilon^{-1}\psi_0(\frac{y}{\varepsilon}, z).$$

This assumption implies that the initial wave function is already confined at the scale ε in the y-directions. Accordingly, we rescale

$$y' = \frac{y}{\varepsilon}, \quad z' = z, \quad \psi^\varepsilon(t, y', z') = \varepsilon\psi(t, \varepsilon y', z').$$

Finally, we rewrite $\beta^\varepsilon = \lambda\varepsilon^{2\sigma} \ll 1$ by a fixed constant $\lambda \in \mathbb{R}$. Thorough these procedures, equation (1.4) becomes

$$i\partial_t\psi^\varepsilon = \frac{\mathcal{H}}{\varepsilon^2}\psi^\varepsilon - \frac{\partial_z^2}{2}\psi^\varepsilon + V(z)\psi^\varepsilon + \lambda|\psi^\varepsilon|^{2\sigma}\psi^\varepsilon \tag{1.6}$$

with

$$\mathcal{H} = -\frac{\Delta_y}{2} + \frac{|y|^2}{8} - \frac{i}{2}(y_1\partial_{y_2} - y_2\partial_{y_1}).$$

The operator $\mathcal{H}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2) \subset L^2(\mathbb{R}^2)$ with the pure point spectrum given by

$$\text{spec}\mathcal{H} = \{n + \tfrac{1}{2} : n \in \mathbb{N}_0\}. \tag{1.7}$$

Based on equation (1.6), $\mathcal{H}$ induces high-frequency oscillations in time $\propto \mathcal{O}(\varepsilon^{-2})$ within the solution $\psi^\varepsilon(t)$. By filtering these oscillations with the profile $\phi^\varepsilon(t) := e^{it\mathcal{H}/\varepsilon^2}\psi^\varepsilon(t)$, they expected to observe the following limit

$$\phi^\varepsilon(t, x) = e^{it\frac{\mathcal{H}}{\varepsilon^2}}\psi^\varepsilon(t, x) \longrightarrow \exists\phi(t, x) \quad \text{as} \quad \varepsilon \to +0 \tag{1.8}$$

in a strong norm. This profile solves

$$i\partial_t\phi^\varepsilon(t) = -\frac{1}{2}\partial_z^2\phi^\varepsilon(t) + V(z)\phi^\varepsilon(t) + \lambda e^{it\frac{\mathcal{H}}{\varepsilon^2}}(|e^{-it\frac{\mathcal{H}}{\varepsilon^2}}\phi^\varepsilon(t)|^{2\sigma}e^{-it\frac{\mathcal{H}}{\varepsilon^2}}\phi^\varepsilon(t)). \tag{1.9}$$

Regarding nonlinearity, they introduced the nonlinear function

$$F(\theta, u) := e^{i\theta\mathcal{H}}(|e^{-i\theta\mathcal{H}}u|^{2\sigma}e^{-i\theta\mathcal{H}}u). \tag{1.10}$$

Because of (1.7), $F(\cdot, u)$ is 2π-periodic. Then, to study the behavior of $F(\frac{t}{\varepsilon^2}, u)$ as $\varepsilon \to +0$, they defined the average of $F(\cdot, u)$ as

$$F^*_{\text{av}}(u) := \lim_{T\to+\infty} \frac{1}{T}\int_0^T F(\theta, u)d\theta = \frac{1}{2\pi}\int_0^{2\pi} F(\theta, u)d\theta$$

and formally derived the limiting model:

$$i\partial_t \phi = -\frac{1}{2}\partial_z^2\phi + V(z)\phi + \lambda F^*_{\text{av}}(\phi). \tag{1.11}$$

This type of averaging method has often been used to derive the limiting model of NLS equations with strong confinement, cf. [5–7, 16, 39, 42].

In this paper, applying the transforms

$$u(t, y, z) \mapsto \tilde{u}(t, y, z) := (Te^{itL}u)(t, y, z)$$

with

$$L := -\frac{i}{2}(y_1\partial_{y_2} - y_2\partial_{y_1}), \qquad T : f(t, y, z) \mapsto 2^{\frac{1}{\sigma}} f(4t, \sqrt{2}y, \sqrt{2}z)$$

and $\tilde{V}(z) := V(\sqrt{2}z)$ to the solutions to (1.6) and (1.11), we normalize the coefficients and eliminate the angular momentum operator L to obtain (1.1) and (1.2) respectively. We remark that H is essentially self-adjoint on $C_0^\infty(\mathbb{R}^2)$ with pure point spectrum

$$\{2(n + 1) : n \in \mathbb{N}_0\},$$

and when we replace $\mathcal{H}$ with H, $F(\cdot, u)$ is $\frac{\pi}{2}$-periodic.

Conserved quantities

We return to equations (1.1) and (1.2). Equation (1.1) has the following conservative quantities.

- Mass:
$$M[\psi] := \frac{1}{2}\int_{\mathbb{R}^3} |\psi|^2 dx \tag{1.12}$$

- Hamiltonian:
$$\begin{aligned} E^\varepsilon[\psi] :=& \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^3} \Big(|\nabla_y\psi|^2 + |y\psi|^2\Big)dx + \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\partial_z\psi|^2 + V(z)|\psi|^2\Big)dx \\ &+ \frac{\lambda}{2\sigma + 2}\int_{\mathbb{R}^3} |\psi|^{2\sigma+2}dx \end{aligned} \tag{1.13}$$

- Momentum:
$$G^L[\phi] := \frac{i}{2}\int_{\mathbb{R}^3} \overline{\phi}(y_2\partial_{y_1} - y_1\partial_{y_2})\phi dx. \tag{1.14}$$

On the other hand, equation (1.2) contains the following conserved quantities.

- Mass:
$$M[\phi] := \frac{1}{2}\int_{\mathbb{R}^3} |\phi|^2 dx$$

- Hamiltonian:
$$E[\phi] := \int_{\mathbb{R}^3} \Big(\frac{1}{2}|\partial_z\phi|^2 + V(z)|\phi|^2\Big)dx + \frac{\lambda}{\pi(\sigma + 1)}\int_0^{\frac{\pi}{2}}\int_{\mathbb{R}^3} |e^{-i\theta H}\phi|^{2\sigma+2}dxd\theta \tag{1.15}$$

- Momentum:

$$G[\phi] := \operatorname{Im} \int_{\mathbb{R}^3} \overline{\phi} \partial_z \phi dx \tag{1.16}$$

- Kinetic energy:

$$K[\phi] := \frac{1}{2} \int_{\mathbb{R}^3} \Big(|\nabla_y \phi|^2 + |y\phi|^2 \Big) dx. \tag{1.17}$$

Although the conservation laws for M, E, and G are classical, the conservation law of K has been proven and used only when $\sigma \in \mathbb{N}$, cf. [A, 21, 25, 26]. However, by employing the representation (1.3), we can extend this conservation law to all $\sigma \geq \frac{1}{2}$.

Previous studies of NLS with a partial harmonic oscillator

In this subsection, we review previous works concerning NLS with a partial harmonic oscillator:

$$i\partial_t u = -\Delta_x u + |y|^2 u + \lambda |u|^{2\sigma} u, \qquad x = (y, z) \in \mathbb{R}^n \times \mathbb{R}^{d-n}, \quad 1 \leq n < d. \tag{1.18}$$

If $n = 0$, it is well-known that the linear term induces a large time dispersive effect, which leads to asymptotically linear behavior, that is, scattering. On the other hand, if $n = d$, the linear term corresponds to the isotropic harmonic oscillator and (1.18) no longer has the global dispersion. This case has been studied in [8, 31, 34]; however, the global behaviors are much less known than in the case $n < d$.

Our study is relevant to the case $1 \leq n < d$. In this case, the $d-n$-dimensional global dispersion arises in z. Recalling the classical scattering theory for the standard power-type NLS (cf. [9]), we expect the large solution to (1.18) to scatter in appropriate conditions. For small initial data in the weighted Sobolev space

$$\Sigma^1 := \{\varphi \in H^1(\mathbb{R}^d) : |x|\varphi \in L^2(\mathbb{R}^d)\},$$

Antonelli-Carles-Silva [3] proved scattering of (1.18) under the condition $1 \leq n < d$ and

$$\tfrac{2d}{(d+2)(d-n)} < \sigma < \tfrac{2}{(d-2)_+}. \tag{1.19}$$

Where, $a_+ := \max\{0, a\}$. They also prove the existence of wave operators under the same conditions. For the standard power-type NLS, small data scattering and the existence of the wave operator have been proven when

$$\min\{\tfrac{1}{d}, \tfrac{2}{d+2}\} < \sigma < \tfrac{2}{(d-2)_+},$$

(cf. [9]) and the lower limiting case for $d \geq 3$ was attained in Nakanishi-Ozawa [40]. Seeing (1.19), the expected lower bound $\frac{2}{d-n+2}$ is not reached when $d \geq 3$.

When we consider the large data scattering, one of the most simple situation is "mass-supercritical" for the spatial dimension $d - n$ and "energy-subcritical" for the spatial dimension d:

$$\tfrac{2}{d-n} < \sigma < \tfrac{2}{d-2}. \tag{1.20}$$

When $n = 1$, the condition (1.20) is not empty. In the defocusing case, Antonelli-Carles-Silva [3] proved the large data scattering under the conditions $1 = n < d \leq 4$ and (1.20) in Σ^1, employing " Anisotropic Morawetz estimates". We remark that we cannot obtain an effective "pseudoconformal conservation law" for (1.18) in the classical computation. In the focusing case, Ardila-Carles [4] studied (1.18) under the conditions $1 = n < d \leq 5$, $\sigma \geq \frac{1}{2}$ and (1.20) in the energy space

$$B^1 := \{\varphi \in H^1(\mathbb{R}^d) : |y|\varphi \in L^2(\mathbb{R}^d)\}.$$

They classified the global behavior (scattering or blow-up) of the solution below the ground state, inspired by Ibrahim-Masmoudi-Nakanishi [30].

Cheng-Guo-Guo-Liao-Shen [11] proved the large data scattering of (1.18) in B^1 under the condition $(d, n, \sigma) = (3, 1, 1)$, which is one of the lower endpoints of (1.20): $\sigma = 2/(d-n)$. They approximated (1.18) using the corresponding resonant system, which is the same as (1.2) except for spatial dimensions, and proved the global well-posedness and scattering of the resonant system in the space

$$L^2_z\Sigma^1_y := \{\varphi \in L^2(\mathbb{R}^d) : \nabla_y\varphi, |y|\varphi \in L^2(\mathbb{R}^d)\}.$$

The analysis of the resonant system was inspired by Dodson [17–19]. In [4, 11], the concentration compactness and rigidity argument developed by Kenig-Merle [33] was used.

When $n \geq 2$, the global dispersive effect becomes weaker than in the case $n = 1$ and the left side of (1.20) is equal to or greater than the right side. Hani-Thomann [26] studied (1.18) in the case $\sigma = 1$ and $d - n = 1$. They proved modified scattering and constructed modified wave operators for small initial and final data respectively, utilizing the resonant system to analyze (1.18). However, no explicit asymptotic formula has been obtained as in the case of the standard NLS (cf. [27]).

The same type of model as (1.2) is also used as the limiting model of NLS with strong magnetic confinement, cf. [A, 5–7, 22, 32]. The dynamics of (1.2) is closely related to many open problems concerning the dynamics of (1.18).

Equation (1.2) has not only been used to analyze (1.18), but has also been studied. This is one of the main topics of this paper.

1.2 Main results

This study aims to demonstrate the well-posedness and clarify the dynamics of equations (1.1) and (1.2). The following results are based on the works [A, B].

In all following theorems, we weaken the assumption (1.5) and assume that there exists V'' such that

$$V'' \in L^\infty(\mathbb{R}). \tag{1.21}$$

[A]-1. Well-posedness in Hermite Sobolev spaces

Theorem 1.1. *Let $\frac{1}{2} \leq \sigma \leq 2$, equation (1.1) is locally well-posed in Σ^1. Suppose that $[0, T^\varepsilon_{max})$ is the maximal lifespan of such solution.*

(i) In the case $\sigma < 2$.

- *If $\lambda = +1$, it holds $T^\varepsilon_{max} = +\infty$.*
- *If $\lambda = -1$ and V is bounded below, there exists $\varepsilon_* = \varepsilon_*(\|\psi_0\|_{\Sigma^1}) > 0$ such that for all $\varepsilon \in (0, \varepsilon_*]$, $T^\varepsilon_{max} = +\infty$ holds.*
- *In the other cases, it holds $\lim_{\varepsilon\to+0} T^\varepsilon_{max} = +\infty$.*

(ii) In the case $\sigma = 2$ and $\lambda = +1$, it holds $\lim_{\varepsilon\to+0} T^\varepsilon_{max} = +\infty$.

Theorem 1.2. *Let $\frac{1}{2} \leq \sigma \leq 4$, (1.2) is locally well-posed in Σ^1.*

(i) When $\sigma < 2$, all solutions exist globally.

(ii) When $2 \leq \sigma < 4$ and $\lambda = +1$, all solutions exist globally.

If we assume $V \equiv 0$, the same statement holds in B^1 and (1.2) is left invariant by the scaling

$$\phi(t, y, z) \longmapsto \phi_\mu(t, y, z) := \mu^{\frac{1}{\sigma}}\phi(\mu^2 t, y, \mu z), \qquad \forall\mu > 0. \tag{1.22}$$

This scaling leaves neither $K[\phi_\mu(t)]$ nor $E[\phi_\mu(t)]$ invariant, but conserves

$$K[\phi_\mu(t)]^{\sigma+2}E[\phi_\mu(t)]^{\sigma-2},$$

where E and K are defined in (1.15) and (1.17), respectively.

Moreover, the case $\sigma = 4$ is energy-critical. In particular, there exists $\delta > 0$ such that for any $\phi_0 \in B^1$ satisfying the scale invariant condition $\|\phi_0\|^3_{L^2_z\Sigma^1_x}\|\partial_z\phi_0\|_{L^2} \leq \delta$, the solution $\phi(t)$ to (1.2) with the initial data ϕ_0 exists globally and scatters; there exist functions $\phi_\pm \in B^1$ such that

$$\lim_{t\to\pm\infty} \|\phi(t) - e^{\pm it\frac{\partial_z^2}{2}}\phi_\pm\|_{B^1} = 0.$$

When $2 < \sigma$, equation (1.18) become energy-supercritical and the global existence and dynamics for large solution are unknown. We expect (1.2) will be useful in clarifying the dynamics of (1.18) under the energy-supercritical condition $2 < \sigma \leq 4$.

[A]-2. Local approximations

Next, we present the convergence of (1.8).

Theorem 1.3.

(i) Let $\sigma \in [\frac{1}{2}, 2]$, $\psi_0 \in \Sigma^1$, and $\phi \in C([0, T_{max}), \Sigma^1)$ be the maximal solution to (1.2) with $\phi(0) = \psi_0$. Then, for any sufficiently small $\varepsilon > 0$ and any $T \in (0, T_{max})$, there exists the solution $\psi^\varepsilon \in C([0,T], \Sigma^1)$ to (1.1) with $\psi^\varepsilon(0) = \psi_0$ which satisfies

$$\lim_{\varepsilon \to +0} \|e^{it\frac{H}{\varepsilon^2}}\psi^\varepsilon - \phi\|_{L^\infty([0,T],\Sigma^1)} = 0. \tag{1.23}$$

Furthermore, there exist $\varepsilon_0 = \varepsilon_0(\|\psi_0\|_{\Sigma^1}, T) > 0$ and $C = C(\|\psi_0\|_{\Sigma^1}, T) > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$, it holds

$$\|e^{it\frac{H}{\varepsilon^2}}\psi^\varepsilon - \phi\|_{L^\infty([0,T],L^2)} \leq C\varepsilon. \tag{1.24}$$

(ii) Let $\sigma \geq 1$, $\psi_0 \in \Sigma^2$, and $\phi \in C([0, T_{max}), \Sigma^2)$ be the maximal solution to (1.2) with $\phi(0) = \psi_0 \in \Sigma^2$. Then, for any sufficiently small $\varepsilon > 0$ and any $T \in (0, T_{max})$, there exists the solution $\psi^\varepsilon \in C([0,T], \Sigma^2)$ to (1.1) with $\psi^\varepsilon(0) = \psi_0$ for any sufficiently small $\varepsilon > 0$, which satisfies

$$\lim_{\varepsilon \to +0} \|e^{it\frac{H}{\varepsilon^2}}\psi^\varepsilon - \phi\|_{L^\infty([0,T],\Sigma^2)} = 0. \tag{1.25}$$

When we obtain the global solution to (1.2), the convergence result holds over any compact time interval.

On the right-hand side of (1.24), the convergence rate ε is due to the difference of the first-order regularity between the spaces Σ^1 and L^2. We do not know whether this rate is optimal.

[B]. Global dynamics of the averaged magnetic NLS in the super-quintic case

In this subsection, we consider the global behavior of the solution to (1.2) in B^1 with the condition $V \equiv 0$:

$$i\partial_t \phi(t,x) = -\partial_z^2 \phi(t,x) + \lambda F_{\mathrm{av}}(\phi(t,x)). \tag{1.26}$$

In the focusing case, we demonstrate the existence of a ground-state solution and clarify the dynamics of the solution to (1.26) below the ground state. We introduce a standing wave of (1.26). First, we perform the following transformation

$$\psi^*(t) := e^{-itH}\phi(t),$$

where $\phi(t)$ represents the solution to (1.26). Then, $\psi^*(t)$ solves the following equation (see Remark **??**):

$$i\partial_t \psi^*(t) = H\psi^*(t) - \partial_z^2 \psi^*(t) + \lambda F_{\mathrm{av}}(\psi^*(t)). \tag{1.27}$$

In the focusing case, assuming of the form $\psi^*(t,x) = e^{it}\varphi(x)$, an elliptic problem

$$H\varphi - \partial_z^2 \varphi + \varphi - F_{\mathrm{av}}(\varphi) = 0 \tag{1.28}$$

arises. This computation is equivalent to $\phi(t,x) = e^{it}e^{itH}\varphi(x)$. Then we introduce the action

$$\begin{aligned} S[\varphi] :=& K[\varphi] + M[\varphi] + E[\varphi] \\ =& \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla_x \varphi|^2 + |y\varphi|^2 + |\varphi|^2\right) dx - \frac{1}{\pi(\sigma+1)} \int_0^{\frac{\pi}{2}} \int_{\mathbb{R}^3} |e^{-i\theta H}\varphi|^{2\sigma+2} dx d\theta, \end{aligned}$$

and refer to Q as a ground state solution if it satisfies

$$S[Q] = \inf\{S[\varphi] : \varphi \neq 0 \text{ and } \varphi \text{ solves } (1.28)\}.$$

To construct the ground state solution, we consider the variational problem

$$\inf\{S[\varphi] : \varphi \in B^1\backslash\{0\} \text{ and } I[\varphi] = 0\}, \tag{1.29}$$

where

$$I[\varphi] := \int_{\mathbb{R}^3} \big(|\nabla_x \varphi|^2 + |y\varphi|^2 + |\varphi|^2\big) dx - \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \int_{\mathbb{R}^3} |e^{-i\theta H}\varphi|^{2\sigma+2} dx d\theta.$$

Then, we can demonstrate that (1.29) is finite and a minimizer exists, which ensures the existence of the ground state solution. In fact, a minimizer Q is an extremizer of the Strichartz estimate

$$\Big(\int_0^{\frac{\pi}{2}} \int_{\mathbb{R}^3} |e^{-i\theta H}\varphi|^{2\sigma+2} dx d\theta \Big)^{\frac{1}{2\sigma+2}} \lesssim \Big(\int_{\mathbb{R}^3} \big(|\nabla_x \varphi|^2 + |y\varphi|^2 + |\varphi|^2\big) dx \Big)^{\frac{1}{2}}. \tag{1.30}$$

We next study the global dynamics of the solutions to (1.26) below the ground state for (1.29). Our strategy is based on the work of Ardila-Carles [4]. We introduce the functional

$$P[\varphi] := 2 \int_{\mathbb{R}^3} |\partial_z \varphi|^2 dx - \frac{2\sigma}{\pi(\sigma+1)} \int_0^{\frac{\pi}{2}} \int_{\mathbb{R}^3} |e^{-i\theta H}\varphi|^{2\sigma+2} dx d\theta.$$

We also define the following subsets of B^1,

$$\begin{aligned} \mathcal{K}^+ &:= \{\varphi \in B^1 : S[\varphi] < S[Q], \quad P[\varphi] \geq 0\}, \\ \mathcal{K}^- &:= \{\varphi \in B^1 : S[\varphi] < S[Q], \quad P[\varphi] < 0\}, \end{aligned} \tag{1.31}$$

where Q is a minimizer of (1.29). Note that $S[\cdot]$ is the conserved quantity of the solutions to (1.26) and (1.27). We prove that all solutions whose initial data belong to $\mathcal{K}^+$ exist globally and scatter, and those whose initial data belong to $\mathcal{K}^-$ blow up at finite or infinite time.

We will present main results of this subsection.

Theorem 1.4. *Let $\lambda = -1$ and $\sigma < 4$. Then, a minimizer Q of* (1.29) *exists. Moreover, this is an extremizer of the Strichartz estimate* (1.30).

Theorem 1.5. *Let $\lambda = -1$ and $2 \leq \sigma < 4$. Let $\phi \in C((T_-, T_+), B^1)$ be the solution to* (1.26) *with $\phi(0) = \phi_0 \in B^1$.*

(i) If $\phi_0 \in \mathcal{K}^+$, then the corresponding solution $\phi(t)$ exists globally. Moreover, if $2 < \sigma < 4$, $\phi(t)$ scatters in B^1; there exist $\phi^\pm \in B^1$ such that

$$\lim_{t \to \pm\infty} \|\phi(t) - e^{it\partial_z^2}\phi^\pm\|_{B^1} = 0.$$

(ii) If $\phi_0 \in \mathcal{K}^-$, one of the following two cases occurs.

- *The corresponding solution $\phi(t)$ blows up in positive time. That is, $T_+ < \infty$ and*

$$\lim_{t \to T_+} \|\partial_z \phi(t)\|_{L^2} = \infty.$$

- *The corresponding solution $\phi(t)$ blows up at infinite positive time. That is, $T_+ = \infty$ and*

$$\limsup_{t \to \infty} \|\partial_z \phi(t)\|_{L^2} = \infty.$$

The same statement holds for negative time. Moreover, if ϕ_0 satisfies $z\phi_0 \in L^2(\mathbb{R}^3)$, the corresponding solution blows up in finite time.

Our work also yields the scattering result in the defocusing case.

Corollary 1.6. *Let $\lambda = +1$, $2 < \sigma < 4$, and $\phi \in C(\mathbb{R}, B^1)$ be the global solution to* (1.26), *$\phi(t)$ scatters in B^1.*

The scattering result is based on Kenig–Merle's concentration compactness and rigidity arguments [33]. As mentioned in [4], because the model contains partial harmonic oscillator, the method developed in [20, 28] could not be applied. They employed a variational approach based on the work of Ibrahim-Masmoudi-Nakanishi [30]. We also follow the strategy.

We replace the condition (1.20) in our case. In equation (1.26), the global dispersive effect arises in one dimension, and as mentioned in Theorem 1.2, equation (1.26) becomes energy-critical when $\sigma = 4$. Thus, to obtain the corresponding results, we assume that

$$2 < \sigma < 4.$$

However, we include the case $\sigma = 2$ except for the scattering result.

In proving scattering, the condition $\sigma > 2$ presents some technical difficulties. NLS equations with a standard power-type nonlinearity, such as (1.1) and (1.18), are energy-supercritical when $\sigma > 2$. Therefore, in the analysis of the averaged nonlinearity, we need to employ Strichartz's estimate for the propagator $e^{-i\theta H}$ (on $[0, \frac{\pi}{2}] \times \mathbb{R}^2_y$), as well as the global-in-time Strichartz estimate for $e^{it\partial_z^2}$ (on $\mathbb{R}_t \times \mathbb{R}_z$). This approach requires the utilization of at least one anisotropic spatial norm, and we must carefully consider the balance of integrability among the four variables: t, θ, y, and z. On the other hand, we would prefer to minimize the use of anisotropic norms in constructing a critical element that is essential for the concentration of compactness and rigidity arguments. Then, we introduce new space-time norms and derive new global-in-time Strichartz's estimates. By using these tools, we can effectively employ the two propagators and overcome the difficulties.

2 Dispersion-managed nonlinear Schrödinger equations

All the contents concerning the dispersion-managed NLS are based on the work [C].

2.1 Introduction to the dispersion-managed NLS

Our interest in this work lies in the scattering theory for the averaged dispersion-managed nonlinear Schrödinger equations of the form

$$i\partial_t u = -\Delta_x u + \lambda \int_0^1 e^{-i\theta\Delta_x}(|e^{i\theta\Delta_x}u|^{2\sigma} e^{i\theta\Delta_x}u)d\theta, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^d. \tag{2.1}$$

Where, $\lambda = \pm 1$. This model arises in the field of nonlinear optics, particularly in the setting of 'dispersion management' and in the investigation of optical solitons. The particular model under consideration is sometimes called the 'Gabitov–Turitsyn' equation, and arises in the *strong dispersion management* regime. The relevant literature is quite extensive by now, but we would like to refer the reader to [1, 13, 16, 23, 29, 35, 37, 41, 45] for a representative sample of relevant results. In the following subsection, we will follow [13] to introduce the derivation of the particular equation (2.1) from the more general dispersion-managed model (see also [16]).

Derivation of the equations

Nonlinear Schrödinger equations also arise in the context of nonlinear fiber optics. We consider NLS with a periodically varying dispersion coefficient:

$$i\partial_t u + d(t)\partial_x^2 u + |u|^{2\sigma}u = 0, \quad (t,x) \in \mathbb{R}. \tag{2.2}$$

We assume that $d(t)$ is a periodic function of the form

$$d(t) = d_{\text{av}} + d_0\big(\tfrac{t}{\varepsilon}\big).$$

Where d_0 is a periodic function of mean zero and $d_{\text{av}} \in \mathbb{R}$ is the average of $d(t)$ over one period. This model arises in the setting of laser light propagating down a fiber optics cable in which the dispersion varies periodically. Here, the variable t represents the distance along the fiber and x denotes the time. This model has been extensively studied both mathematically and physically. In particular, many studies involve the strong dispersion management regime:

$$d(t) = d_{\text{av}} + \tfrac{1}{\varepsilon}d_0\big(\tfrac{t}{\varepsilon}\big). \tag{2.3}$$

Where $\varepsilon > 0$ is a small parameter which affects the period and the amplitude of the dispersion $d(t)$. This setting is based on the idea that by varying strong dispersion over a short period to achieve an average close to zero, one can mitigate the undesirable effects of dispersion on signal propagation and stabilize pulse transmission. The technique of dispersion management was introduced in Lin-Cogelnik-Cohen [36] and proved to be highly effective in generating stable soliton-like pulses. See also the reviews [43, 44]. For equation (2.2) with (2.3), the averaging process with respect to the limit $\varepsilon \to +0$ has been considered. First, Gabitov-Turitsyn [23, 24] found a good approximation for small ε. Let $D(t) := \int_0^t d_0(s)ds$, the profile $v^\varepsilon(t) := e^{-iD(t/\varepsilon)\partial_x^2}u(t)$ solves

$$i\partial_t v^\varepsilon(t) + d_{\rm av}\partial_x^2 v^\varepsilon(t) + e^{-iD(\frac{t}{\varepsilon})\partial_x^2}\left(\left|e^{iD(\frac{t}{\varepsilon})\partial_x^2}v^\varepsilon(t)\right|^{2\sigma} e^{iD(\frac{t}{\varepsilon})\partial_x^2}v^\varepsilon(t)\right). \tag{2.4}$$

Let $L > 0$ is the period of $d_0(t)$, $D(\frac{t}{\varepsilon})$ is also periodic with period εL. When ε is small, $e^{iD(\frac{t}{\varepsilon})\partial_x^2}$ induces fast rotating oscillations. Thus, averaging the nonlinear team over one period, we formally derive the following averaged model:

$$\begin{aligned} i\partial_t v(t) &= -d_{\rm av}\partial_x^2 v(t) - \frac{1}{\varepsilon L}\int_0^{\varepsilon L} e^{-iD(\frac{\tau}{\varepsilon})\partial_x^2}\left(\left|e^{iD(\frac{\tau}{\varepsilon})\partial_x^2}v(t)\right|^{2\sigma} e^{iD(\frac{\tau}{\varepsilon})\partial_x^2}v(t)\right)d\tau \\ &= -d_{\rm av}\partial_x^2 v(t) - \frac{1}{L}\int_0^{L} e^{-iD(\tau)\partial_x^2}\left(\left|e^{iD(\tau)\partial_x^2}v(t)\right|^{2\sigma} e^{iD(\tau)\partial_x^2}v(t)\right)d\tau. \end{aligned}$$

In this paper, we consider the most typical case:

$$d_0(t) = \begin{cases} 1 & t \in [0,1) \\ -1 & t \in [1,2) \end{cases}$$

over one period. Then the averaged model is given by

$$i\partial_t v + d_{\rm av}\partial_x^2 v + \int_0^1 e^{-i\tau\partial_x^2}(|e^{i\tau\partial_x^2}v|^{2\sigma} e^{i\tau\partial_x^2}v)d\tau = 0. \tag{2.5}$$

Note that the case $d_{\rm av} > 0$ corresponds to focusing, while the case $d_{\rm av} < 0$ is defocusing. The case $d_{\rm av} = 0$ is a singular limit. The Cauchy problem for equation (2.5) is locally well-posed in $H^1(\mathbb{R})$ when $d_{\rm av} \neq 0$ and $0 < \sigma$, or $d_{\rm av} = 0$ and $0 < \sigma \le 2$. Moreover, the H^1-solution exists globally when $0 < \sigma$ for $d_{\rm av} < 0$; $0 < \sigma < 4$ for $d_{\rm av} > 0$; $0 < \sigma \le 2$ for $d_{\rm av} = 0$. See [2, 13]. The case $\sigma = 4$ is mass-critical. Following these results, Choi-Hong-Lee [12] identified the global versus blow-up criteria for the H^1-solution to (2.5) in the case $d_{\rm av} > 0$ and $\sigma > 4$. In this result, the asymptotic behavior of the global solution is unknown. On the other hand, the averaging process (the limit $\varepsilon \to +0$) was mathematically verified in [14, 16, 45].

In this paper, we examine the averaged DMNLS in general dimensions, excluding the case $d_{\rm av} = 0$. Therefore, we rewrite equation (2.5) as (2.1).

2.2 Main results

Small data scattering

Our interest in this work lies in the scattering theory for the averaged dispersion-managed NLS. Our first main result concerns the small-data global well-posedness and scattering for (2.1). In this regime the sign of the nonlinearity in (2.1) is irrelevant. The results may be stated as follows:

Theorem 2.1 (Small-data scattering, intercritical case). *Let $d \ge 1$ and $\sigma \in [\frac{2}{d}, \frac{2}{d-2}]$ (with $\sigma \in [\frac{2}{d}, \infty)$ in dimensions $d \in \{1, 2\}$). Let $u_0 \in H^{s_c}(\mathbb{R}^d)$, where*

$$s_c = \tfrac{d}{2} - \tfrac{1}{\sigma} \in [0, 1].$$

If $\||\nabla|^{s_c}u_0\|_{L^2}$ is sufficiently small, then there exists a unique, global-in-time solution $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ to (2.1) with $u|_{t=0} = u_0$. Moreover, u scatters *in H^{s_c}. That is, there exist unique $u_\pm \in H^{s_c}$ such that*

$$\lim_{t\to\pm\infty}\|u(t) - e^{it\Delta}u_\pm\|_{H^{s_c}(\mathbb{R}^d)} = 0.$$

Theorem 2.2 (Small-data scattering, mass-subcritical case)**.** *Let $d \geq 1$ and $\sigma \in (\frac{1}{d}, \frac{2}{d}) \cap [\frac{2}{d+2}, \frac{2}{d})$. Let $u_0 \in \mathcal{F}H^\gamma(\mathbb{R}^d)$, where*

$$\gamma = \tfrac{1}{\sigma} - \tfrac{d}{2} \in (0, 1].$$

If $\||x|^\gamma u_0\|_{L^2}$ is sufficiently small, then there exists a unique, global-in-time solution $u : \mathbb{R}\times\mathbb{R}^d \to \mathbb{C}$ to (2.1) *with $u|_{t=0} = u_0$. Moreover, u* scatters *in $\mathcal{F}H^\gamma$. That is, there exist unique $u_\pm \in \mathcal{F}H^\gamma$ such that*

$$\lim_{t\to\pm\infty} \|e^{-it\Delta}u(t) - u_\pm\|_{\mathcal{F}H^\gamma(\mathbb{R}^d)} = 0.$$

Theorems 2.1 and 2.2 parallel sharp scattering results for the standard power-type NLS (see e.g. [10]). The parameters s_c, γ are the critical regularity/weight associated with the NLS with nonlinearity $|u|^{2\sigma}u$, obtained by considering the L^2-based spaces of data whose norms are invariant under the rescaling that preserves the class of solutions. Theorems 2.1 and 2.2 give small-data scattering in the appropriate space for critical regularity/weight taking values between 0 and 1. The additional constraint $\sigma > \frac{1}{d}$ in Theorem 2.2 corresponds to the 'short-range' case for the standard NLS; we are also able to prove that scattering fails below this exponent (see [C, Theorem 1.4]).

We were inspired to consider the small-data scattering problem for (2.1) by the recent work of Choi-Lee-Lee [15]. In that work, the authors establish scattering for $\sigma = \frac{2}{d}$ (the 'mass-critical exponent' for the standard NLS) in dimensions $d = 1, 2$ for small data in L^2. Their strategy was to employ the U^p, V^p atomic spaces, observing that these spaces are compatible with the structure of the DMNLS nonlinearity. At a technical level, the key ingredient in [15] was a bilinear Strichartz estimate for linear solutions (extended to the U^p space by the transference principle).

On the other hand, we demonstrate that a straightforward modification of the standard approach is sufficient to recover a suitable small-data scattering theory. To employ the strange nonlinearity, we prove $e^{i\theta\Delta}u \in L^\infty_\theta L^q_t L^r_x([0,1] \times \mathbb{R} \times \mathbb{R}^d)$. To handle the effect of $e^{i\theta\Delta}$ in the estimates, we use a slight modification of the standard Strichartz estimate (inspired by [B]), which we refer to as a 'shifted Strichartz estimate'.

Large data scattering

Our next main result concerns the large-data scattering theory for (2.1) in the defocusing regime. We prove scattering for intercritical powers (including $\sigma = \frac{2}{d-2}$).

Theorem 2.3 (Large data scattering)**.** *Let $\lambda = +1$ and $\frac{2}{d} \leq \sigma \leq \frac{2}{d-2}$ (with $\sigma < \infty$ in in dimensions $d \in \{1, 2\}$). For any $u_0 \in \Sigma^1$, there exists a unique, global-in-time solution $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}$ to* (2.1) *with $u|_{t=0} = u_0$. Moreover, if σ satisfies*

$$\begin{cases} \frac{3+\sqrt{5}}{2} < \sigma < \infty & d = 1, \\ 1 < \sigma < \infty & d = 2, \\ \frac{2}{d} < \sigma \leq \frac{2}{d-2} & d \geq 3, \end{cases}$$

then u scatters in Σ^1 as $t \to \pm\infty$. That is, there exist unique $u_\pm \in \Sigma^1$ such that

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - u_\pm\|_{\Sigma^1} = 0.$$

Global well-posedness for the range of powers considered in Theorem 2.3 follows from the standard contraction mapping argument, incorporating a Strichartz estimate in the averaged nonlinearity. We find that for the full range $\frac{2}{d} \leq \sigma \leq \frac{2}{d-2}$, we obtain a local existence time that depends only on the H^1-norm of the initial data (even for $\sigma = \frac{2}{d-2}$). In particular, it seems that the averaging effect of the nonlinearity is helpful for short times. Global well-posedness then follows from the conservation of mass and energy (in the defocusing case).

For the proof of scattering, we adapt the pseudoconformal energy estimate used for the standard NLS (see e.g. [9, 38]).

The pseudoconformal energy estimate is essentially based on calculating the time derivative of the quantity $\|Ju\|^2_{L^2}$ (where $J(t) = x + 2it\nabla$ is the Galilean vector field introduced above), utilizing the conservation of energy and the Gronwall inequality. For power-type nonlinearities, one obtains a decay estimate on the $L_x^{2\sigma+2}$-norm of the solution for any power $2\sigma > 0$. These estimates imply critical space-time bounds (which then yield scattering) exactly when 2σ exceeds the 'Strauss exponent' $\alpha_0(d) = \frac{2-d+\sqrt{d^2+12d+4}}{2d}$. We refer the reader to [38] for more details.

In the setting of (2.1), one can also obtain an analogue of the pseudoconformal energy identity, albeit with some key differences.

Remarks on blowup

Finally, we would like to make some remarks concerning blowup in the focusing case related to the recent work [12]. In that work, the authors considered the focusing DMNLS in one space dimension, i.e.

$$(i\partial_t + \Delta)u + \int_0^1 e^{-i\theta\Delta}(|e^{i\theta\Delta}u|^{2\sigma}e^{i\theta\Delta}u)\,d\theta = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{2.6}$$

They defined a 'ground state threshold' given in terms of the optimal constant in the global Strichartz–Gagliardo–Nirenberg inequality

$$\|e^{i\theta\Delta}\varphi\|^{2\sigma+2}_{L^{2\sigma+2}_{\theta,x}(\mathbb{R}\times\mathbb{R})} \leq C_0\|\varphi\|^{\sigma+4}_{L^2_x}\|\partial_x\varphi\|^{\sigma-2}_{L^2_x},$$

which they proved admits an optimizer $Q \in H^1$ satisfying

$$-Q + \Delta Q + \int_{\mathbb{R}} e^{-i\theta\Delta}(|e^{i\theta\Delta}Q|^{2\sigma}e^{i\theta\Delta}Q)\,d\theta = 0.$$

While uniqueness of such optimizers is not yet known, the sharp constant C_0 can be described purely in terms of the mass/energy of any such optimizer. Defining the conserved mass by $M(u) = \|u\|^2_{L^2}$ and the energies

$$E_I(u) = \tfrac{1}{2}\int_{\mathbb{R}}|\nabla u|^2\,dx - \int_I\int_{\mathbb{R}}|e^{i\theta\Delta}u|^{2\sigma+2}\,dx\,d\theta, \quad I \subset \mathbb{R},$$

the main result of [12] may be stated as follows:

Theorem (Main result of [12])**.** *Let $\sigma > 4$. Suppose $u_0 \in H^1(\mathbb{R})$ satisfies*

$$M(u_0)^{\frac{\sigma+4}{\sigma-4}}E_{[0,1]}(u_0) < M(Q)^{\frac{\sigma+4}{\sigma-4}}E_{\mathbb{R}}(Q). \tag{2.7}$$

Let $u : (-T_-, T_+)\times\mathbb{R} \to \mathbb{C}$ denote the corresponding maximal-lifespan solution to (2.6)*.*

(i) If

$$\|u_0\|^{\frac{\sigma+4}{\sigma-4}}_{L^2}\|\partial_x u_0\|_{L^2} < \|Q\|^{\frac{\sigma+4}{\sigma-4}}_{L^2}\|\partial_x Q\|_{L^2},$$

then $T_\pm = \infty$ and $u \in L_t^\infty H_x^1(\mathbb{R}\times\mathbb{R})$.

(ii) If $|x|u_0 \in L^2(\mathbb{R})$ and

$$\|u_0\|^{\frac{\sigma+4}{\sigma-4}}_{L^2}\|\partial_x u_0\|_{L^2} > \|Q\|^{\frac{\sigma+4}{\sigma-4}}_{L^2}\|\partial_x Q\|_{L^2},$$

then u blows up backward in time, i.e. $T_- < \infty$.

The proof in [12] was based on a virial identity for (2.6). Similar the case of the pseudoconformal energy estimate described above, one encounters terms that only have a 'good sign' in one time direction. It is for this reason that the blowup in the above theorem was only obtained in one time direction.

In this section, we demonstrate how a modified version of time reversal symmetry may be applied to obtain blowup in both time directions:

Theorem 2.4. *Under the same assumptions of Main result of [12]-(ii), the solution u blows up in both time directions, i.e. $T_\pm < \infty$.*

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