

Phase-space structure of collisionless dark matter halos from particle dynamics

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Abstract

Unveiling the nature of dark matter remains one of the greatest unsolved problems in modern physics. In the current concordant model of our universe, Λ Cold Dark Matter (Λ CDM) model, dark matter consists of 80% of matter component and is indispensable to facilitating structure formation. While cold dark matter (CDM) is the most plausible model in terms of observational facts, other dark matter candidates are still under consideration as alternatives to CDM. In this point, dark matter halos, which are gravitationally collapsed objects of dark matter, play a key role in unveiling the nature of dark matter. In other words, in testing dark matter models on small scales like halos, a quantitative understanding of the internal structure of dark matter halos is indispensable.

However, our current theoretical understanding of the structure of halos is far from comprehensive. From cosmological N -body simulations, the CDM halos exhibit universal features such as density profiles and pseudo-density profiles. The physical origin of these universalities is not yet fully understood. Additionally, CDM halos possess the characteristic multi-stream structure of the three-dimensional sheet-like CDM distribution folded repeatedly within halos in six-dimensional phase space. This multi-stream region likely holds the secret of the origin of the universalities above, but a quantitative understanding of it is also still missing. Therefore, detailed investigations of the phase-space structure would provide a clearer description of the structure of halos, which would, in turn, be a critical foundation for the observational validation of dark matter models.

In this thesis, we first investigate the phase-space structure of the multi-stream region in CDM halos. To characterize and discretize individual streams, we classify dark matter particles in cosmological N -body simulations based on the number of apocenter passages. Our analysis reveals that the radial density profiles of these classified particles follow a double power-law behavior, with a slope of -1 in the central regions and -8 in the outskirts, independent of halo mass. By comparing these density profiles with the self-similar solutions of the spherical collapse model, we identified that the multi-stream regions of N -body halos exhibit a broader angular momentum distribution than predicted by analytic solutions.

We also conducted simulations for the warm dark matter (WDM) models with particle masses of 1 keV and 0.5 keV to explore the potential of distinguishing between dark matter models using this universal structure. Our analyses confirmed the presence of the same universal structure within halos across all models. Additionally, we focused on the radial action variable J_r , a quantity more closely related to observable parameters, to classify simulation particles and analyze their density profiles. The results demonstrated that while the inner slope of -1 is preserved across all dark matter models, the density amplitude varies between models.

Finally, a similar analysis on different N -body simulations of the first-formed halos also uncovered the -1 inner slope. We discuss the origin of this universal slope in relation to the properties of the distribution of action variables.

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Chapter 1

Introduction

1.1 Phase-space structure of dark matter halos

Dark matter is widely regarded as a critical component of the universe, constituting approximately 80% of the matter content. Its significance lies in its pivotal role in cosmic structure formation, driven by gravitational instability. Specifically, dark matter is assumed to be non-relativistic with an extremely small velocity dispersion, characteristics that facilitate the early growth of cosmic density fields following the recombination epoch—a framework known as the cold dark matter (CDM) model.

During cosmic structure formation, the gravitational collapse of cold dark matter leads to the emergence of dark matter halos—self-gravitating systems composed of dark matter. These halos develop potential wells capable of accumulating substantial amounts of baryonic matter, making them prime sites for galaxy and star formation (Rees & Ostriker (1977); White & Rees (1978), see also Mo et al. (2010); Wechsler & Tinker (2018) as recent textbook and review). Consequently, dark matter halos are fundamental to explaining the observed large-scale structure of the universe.

The structural properties of dark matter halos are highly sensitive to their formation and merging histories, offering an invaluable framework for testing structure formation scenarios (e.g., Navarro et al. (1997); Wechsler et al. (2002); Zhao et al. (2003a); Vera-Ciro et al. (2011); Wang et al. (2011); Ludlow et al. (2014)). Numerous studies have explored these processes by utilizing cosmological N -body simulations, drawing insights from observed small-scale structures (for detailed reviews, see Bullock & Boylan-Kolchin, 2017).

From a theoretical perspective, CDM halos are treated as collisionless bound systems whose evolved structures are generally influenced by their initial conditions. Despite this, early numerical simulations revealed that the radial density profile of halos, $\rho(r)$, exhibits a nearly universal shape, often characterized by the Navarro-Frenk-White (NFW) profile (Navarro et al., 1997). A defining feature of this profile is its shallow cusp, with a density slope of -1 , i.e., $\rho(r) \propto r^{-1}$. While subsequent studies have suggested that the Einasto profile (Einasto, 1965) provides a more precise description of the density profile of halos (Navarro et al., 2004, 2010; Dutton & Macciò, 2014; Wang et al., 2020), the cuspy structure of CDM halos remains a robust feature in cosmological N -body simulations, with its physical origin still unresolved.

Observational evidence supports the hypothesis that CDM initially possesses a negligible velocity dispersion. This implies that, in a six-dimensional phase space (comprising three spatial and three velocity dimensions), the initial dark matter distribution forms a

three-dimensional sheet. Due to its collisionless nature, the topology of this phase-space structure remains intact throughout the formation and evolution of halos. Consequently, the phase-space distribution of halos is still described by a three-dimensional sheet, albeit with a complex, folded structure that presents multi-valued velocity flows at given spatial locations (White & Vogelsberger, 2009; Vogelsberger et al., 2009; Vogelsberger & White, 2011; Colombi, 2021). This multi-stream structure is a hallmark of CDM and could serve as a key distinguishing feature to differentiate CDM from other dark matter candidates.

It is reasonable to hypothesize that the universal density profiles observed in configuration space are linked to underlying universal features in the phase-space structure. Investigating the phase-space structure of CDM halos in detail could therefore yield significant insights. Recent interest has focused on the multi-stream nature of halos, particularly the outer boundary of the multi-stream region, known as the splashback radius (Diemer & Kravtsov, 2014; Adhikari et al., 2014). Numerous studies have explored this splashback radius (More et al., 2016; Baxter et al., 2017; Contigiani et al., 2019; Khakaj et al., 2020; Shin et al., 2021), further highlighting its importance in understanding halo dynamics.

Motivated by these considerations, our study focuses on exploring the phase-space structure and dynamical properties of CDM halos through cosmological N -body simulations. Although many prior studies have examined the structure of CDM halos in configuration space, investigations into their phase-space structures remain comparatively scarce. Notably, Sugiura et al. (2020) pioneered such efforts by comparing the radial phase-space structures of halos with the self-similar solution derived from the spherical collapse model in the EdS universe. Their findings revealed that in the outer regions of approximately 30% of the CDM halos analyzed, the radial phase-space structure aligns well with the self-similar solution. However, a detailed understanding of the inner halo structure was lacking.

Building upon and extending their work, our aim is to provide a quantitative analysis of the inner regions of halos' phase-space structures. Additionally, we simulate warm dark matter (WDM) models—an alternative to CDM—and examine the potential for distinguishing dark matter models based on the phase-space characteristics of halos.

Theoretical studies of halo structures are indispensable as they lay the groundwork for observational tests of dark matter models. Despite the abundance of research on this topic, the theoretical understanding of dark matter halos has often relied on empirical observations on N -body simulations. The origins of the universal NFW profile and other structural universalities remain elusive. By investigating the phase-space structures of simulated halos, we aspire to make significant strides toward the observational validation of dark matter models and unraveling the mysteries behind these universal properties.

1.2 The structure of this thesis

Chapter 2

We overview the previous works on the structure of dark matter halos. We start with providing the theoretical background of our universe in Section 2.1, and the linear-order perturbation theory in Section 2.2. After introducing analytic works of the non-linear evolution of our universe in Section 2.3, we start to briefly describe the results from N -body simulations in the CDM model in Section 2.4, and those in WDM model in Section 2.5.

Chapter 3

In this chapter, we present the discussion of our paper Enomoto et al. (2023) and Enomoto et al. (2024a). In these papers, extending the method performed in Sugiura et al. (2020), we investigate the phase-space structure of CDM halos using the number of apocenter passages of particles as a dynamical indicator of multistream flows in the halos. In Sections 3.2 and 3.3, we introduce our data and method respectively. Then we present the main results in Section 3.4, and discuss the dependence of the results on halo samples in Section 3.5. We compare our findings with the self-similar solutions in Section 3.6, and also discuss the origin of the universality we found in Section 3.7. We summarize our results in this chapter in Section 3.8.

Chapter 4

Building upon the results in Chapter 3, we then apply the method to the warm dark matter simulation. Furthermore, we change the indicator of multistream dynamics of halos from the number of apocenter passages to the radial action and show the phase-space of halos exhibits universal structure in both CDM and WDM simulations. In Section 4.2, we present sets of the simulations used in this chapter and introduce the methods estimating radial actions in Section 4.3. We show that the density profiles of particles classified radial action follow double-power law irrespective of halo mass and dark matter models in Section 4.4, and discuss the connection between the number of apocenter passages and the radial action in Section 4.5. Then we wrap up our results and comment on some outlooks in Section 4.6. Note that the results of this section are written in Enomoto et al. (2024b), which was accepted from PASJ on January 9th and is to be published.

Chapter 5

In this chapter, we focus on the inner structure of the prompt cusp, which is the first dark matter halo in the universe created from the collapse of a smooth density field around the free-streaming scale of the dark matter model. We start by presenting the data simulating the formation of prompt cusps in Section 5.2, and show the results in Section 5.3.

Chapter 6

We discuss the connection between the density slope of profiles and the distribution function of action variables to obtain a physical explanation of the origin of universality we presented in previous chapters. After showing our formulae in Section 6.3, we apply them to N -body data to discuss the asymptotic slope of density profiles of particles classified by radial action in Section 6.4. While it is not fully understood, we can at least show that the asymptotic slope of the classified profiles is determined solely by the shape of the distribution function. Note that the results and discussions presented in Chapters 5 and 6 offer further investigations building upon those in Chapters 3 and 4, which are to be put together in a paper and published.

Chapter 7

We conclude and sum up our thesis in this chapter.

Chapter 2

Overview: Dark matter halos

In this Chapter, we briefly overview the structure formation of the universe paying special attention to the formation of dark matter halos. We start by overiewing Λ CDM model and basic background equations in Section 2.1, and present linear perturbation theory in Section 2.2. Section 2.2.1 is devoted to basic equations of linear perturbation theory, and Section 2.2.2 discusses the necessity of dark matter as a main driver of the structure formation of our universe. Section 2.2.3 introduces free-streaming damping of density perturbation, which is the key property of the WDM universe. From section 2.3, we focus on the non-linear evolution of density fluctuation starting with introducing a basic top-hat spherical collapse model in Section 2.3.1. We describe the concept of the virial equilibrium of the self-gravitating system in the top-hat collapse model in Section 2.3.2, and derive the mass function of such virialized object by extrapolating linear perturbation theory in Section 2.3.3. Then, we describe the theoretical description of the evolution of spherical density fluctuation after its first shell crossing (collapse) in Section 2.3.4.

In Section 2.4, we show the results from previous studies on the structure of dark matter halos utilizing numerical simulations. We start by introducing a method for implementing cosmological N -body simulations in Section 2.4.1 and for identifying halos in those simulations in Section 2.4.2. We discuss the structure of CDM halos in Section 2.4.3, and introduce its universal behavior along with the universal behavior of pseudo-phase-space density introduced in Section 2.4.4. We also present the evolution and emergence of the prompt cusps in first-formed halos in Section 2.4.5. In Section 2.4.6, we introduce results from Sugiura et al. (2020) as a previous study on the phase-space structure of halos, which is closely related to our study. Our results in Chapter 3 are based on the method provided in Sugiura et al. (2020). Finally, in Section 2.5, we overview previous studies on alternative dark matter models.

The discussions in this section are mainly based on those presented in Matsubara (2010) (Sections 2.2 and 2.4.1, and overall), Mo et al. (2010) (Section 2.3.1, and overall), Kitayama (2020) (Section 2.3.3), Binney & Tremaine (2008) (Section 2.3.2), and Dodelson & Schmidt (2020) (overall).

2.1 Friedmann universe and Λ CDM model

As a framework to describe the structure formation of our universe, we start by presenting the Friedmann-Lemaître-Robertson-Walker (FLRW) metric and the Friedmann equation, which provide the general relativistic background of spatially uniform and isotropic universe.

It is now considered reasonable to assume that the universe is spatially uniform and isotropic on a very large scale typically larger than 10 Mpc. These properties, combined with the time invariance of physical laws, constitute the cosmological principle. Under this principle, the dynamics of spacetime can be described using general relativity, providing a framework for constructing large-scale models of the universe by providing FLRW metric. A metric that satisfies the cosmological principle takes the following FLRW form, utilizing a time coordinate defined in this way:

$$ds^2 = -dt^2 + a^2(t) \left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2) \right]. \quad (2.1)$$

Here, $a(t)$ represents the scale factor, normalized such that $a(t_0) = 1$ at the present time t_0 . The parameter K denotes the curvature of the universe: $K < 0$ corresponds to an open universe, while $K > 0$ indicates a closed universe. The fact that the metric depends solely on the time coordinate, and not on spatial coordinates, arises from the requirement of uniformity and isotropy in spacetime. Observational evidence suggests that $K \simeq 0$ (e.g., (Planck Collaboration et al., 2020)), and for simplicity, we will consider a flat universe with $K = 0$ in the following discussion.

The size of the universe at time t is represented by the scale factor $a(t)$. The time Δt required for the universe to expand by the same factor is given by:

$$\frac{da}{dt} \Delta t = a(t) \Leftrightarrow \Delta t = \frac{a}{\dot{a}}. \quad (2.2)$$

This Δt is known as the Hubble time, expressed as $\Delta t = H^{-1}(t)$, where $H(t) = \frac{\dot{a}}{a}$ is the Hubble parameter, representing the rate of expansion of the universe. The distance light travels during the Hubble time is called the horizon scale, which defines the causal connection range in comoving coordinates at that time. Since this study adopts a unit system where the speed of light is set to 1, the horizon scale is equivalent to the Hubble time, $H^{-1}(t)$.

In the following, tensor indices are denoted as follows: 0 represents the time component, 1 $\sim$ 3 represents spatial components, Greek letters such as μ and ν range over 0 $\sim$ 4, and Roman letters like i and j represent 1 $\sim$ 3. From the metric, the Einstein tensor components for four-dimensional spacetime are calculated as:

$$\begin{aligned} G^0_0 &= -3H^2, \\ G^i_j &= -2 \left(\frac{\ddot{a}}{a} + \frac{1}{2} H^2 \right) \delta^i_j, \\ G^i_0 &= G^0_j = 0. \end{aligned} \quad (2.3)$$

The Einstein equation is given as:

$$G^\mu_\nu = 8\pi G T^\mu_\nu \quad (2.4)$$

Using equation (2.3), it can be deduced that T^μ_ν has only diagonal components. Such a T^μ_ν takes the same form as the energy-momentum tensor for an ideal fluid. Representing the energy density as ρ and the pressure as p , it can be expressed as:

$$T^\mu_\nu = \begin{pmatrix} -\rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix}. \quad (2.5)$$

Physical quantities that induce anisotropic effects, such as momentum flow, energy flux, and viscosity, are absent here due to the assumptions of uniformity and isotropy. However, these quantities play a crucial role in the development of anisotropy and the formation of cosmic structures, as discussed in subsequent sections. The specific values of ρ and p are determined by the types and proportions of matter and energy present in the universe. Consequently, the Einstein equations, combined with equations (2.3), (2.5), and (2.4), lead to the Friedmann equations:

$$\begin{aligned} H^2 &= \frac{8\pi G}{3}\rho, \\ \frac{\ddot{a}}{a} &= -\frac{4\pi G}{3}(\rho + 3p). \end{aligned} \quad (2.6)$$

Furthermore, the energy-momentum conservation law $T^\mu_{\nu;\mu} = 0$ yields:

$$\dot{\rho} = -3H(\rho + p). \quad (2.7)$$

A universe whose spacetime dynamics are governed by the Friedmann equations is referred to as a Friedmann universe.

Here, particles such as photons that are massless, or relativistic particles whose average energy at a given time exceeds their rest mass energy, are collectively referred to as "radiation". Conversely, particles with average kinetic energy smaller than their rest mass energy are referred to as "matter". Let (ρ_r, p_r) and (ρ_m, p_m) represent the energy density and pressure of radiation and matter, respectively. From the properties of photon gas in equilibrium systems and non-relativistic matter, it follows that $p_r = \rho_r/3$ and $p_m = 0$. Substituting these into equation (2.7), we obtain

$$\rho_r \propto a^{-4}, \quad \rho_m \propto a^{-3}. \quad (2.8)$$

If the Einstein field equations include a cosmological constant Λ , an additional term $\Lambda\delta^\mu_\nu$ is introduced on the left-hand side of equation (2.4). This is equivalent to adding an energy-momentum tensor component to T^μ_ν as follows,

$$\rho_\Lambda = -p_\Lambda = \frac{\Lambda}{8\pi G} = \text{const.} \quad (2.9)$$

As we shall see, the current universe is dominated by energy components with negative pressure, such as the cosmological constant, which drives accelerated expansion. Hypothetical energy components that accelerate the universe's expansion, including those beyond the standard model like scalar fields, are collectively referred to as dark energy. While a general dark energy equation of state $p = w\rho$ allows w to take values other than -1 , this study assumes dark energy satisfies $p = -\rho$, akin to the cosmological constant.

Next, let us consider a universe dominated by radiation, matter, or dark energy. Substituting $\rho = (\rho_r \propto a^{-4}, \rho_m \propto a^{-3}, \rho_\Lambda \propto \text{const})$ into equation (2.6), we derive:

$$\begin{aligned} \text{Radiation-dominated universe: } a(t) &\propto t^{\frac{1}{2}}, \quad H(t) = \frac{1}{2t}, \\ \text{Matter-dominated universe: } a(t) &\propto t^{\frac{2}{3}}, \quad H(t) = \frac{2}{3t}, \\ \text{Dark energy-dominated universe: } a(t) &\propto e^{\lambda t}, \quad H(t) = \lambda^2 = \text{const.} \end{aligned} \quad (2.10)$$

This indicates that the universe decelerates during the radiation- and matter-dominated eras but undergoes accelerated expansion during the dark energy-dominated era. The matter-dominated universe is referred to as the Einstein–de Sitter (EdS) universe.

The temperature fluctuations of the Cosmic Microwave Background (CMB), as measured by WMAP (2001) and Planck (2009), have demonstrated that the power spectrum of these fluctuations can be adequately explained using linear perturbation theory (see Section 2.2) when appropriate cosmological parameters are employed. Using the obtained power spectrum, various cosmological parameters such as the quantities of dark matter and dark energy, the curvature of the universe, and the spectral index of primordial fluctuations have been inferred. The spectral index of primordial fluctuations specifies the wavenumber dependence of the power spectrum of density fluctuations on large scales.

The Planck satellite has provided the most precise determination of these cosmological parameters from CMB temperature fluctuations. These results represent the standard cosmological parameters within the currently prevailing Λ CDM model, which is considered the standard cosmological model. The current expansion rate of the universe, $H_0 (= H(t_0))$, is determined to be

$$H_0 = 100h \text{ km s}^{-1} \text{ Mpc}^{-1} = 67.66 \pm 0.42 \text{ km s}^{-1} \text{ Mpc}^{-1} \quad (2.11)$$

at a 68% confidence interval (TT,TE,EE+lowE+lensing+BAO) (Planck Collaboration et al., 2020).

The universe is believed to comprise three energy components: radiation, matter, and dark energy. The amounts of these components, modulated through the scale factor a , are critical parameters influencing the universe's overall evolution (see equations (2.6), (2.7), and (2.10)). The critical density of the universe is defined as

$$\rho_{\text{crit}} = \frac{8\pi G}{3H^2(t_0)}. \quad (2.12)$$

At the present time t_0 , the energy densities for radiation, matter, and dark energy can be expressed as $\rho_r(t_0) = \rho_{\text{crit}}\Omega_r$, $\rho_m(t_0) = \rho_{\text{crit}}\Omega_m$, and $\rho_\Lambda(t_0) = \rho_{\text{crit}}\Omega_\Lambda$, respectively. The following values have been obtained:

$$\Omega_\Lambda = 0.6889 \pm 0.0056, \quad \Omega_m = 0.3111 \pm 0.0056 \quad (2.13)$$

(TT,TE,EE+lowE+lensing+BAO) (Planck Collaboration et al., 2020). The radiation density parameter Ω_r , derived from the CMB temperature $T(t_0) = 2.725 \text{ K}$, is

$$\Omega_r = 4.155 \times 10^{-5} h^{-2}. \quad (2.14)$$

This value is approximately four orders of magnitude smaller than Ω_Λ and Ω_m . The spectral index of primordial fluctuations, n_s , is determined to be

$$n_s = 0.9665 \pm 0.0038. \quad (2.15)$$

Using these cosmological parameters, it can be demonstrated that the universe has undergone the dominant phases of radiation, matter, and dark energy, as described by equation (2.10). The transition time between the radiation-dominated and matter-dominated eras is called the epoch of equality, and the corresponding scale factor a_{eq} is given by equation (2.1) as $a_{\text{eq}} = \Omega_r/\Omega_m$. Consequently, the redshift at equality is:

$$z_{\text{eq}} = \frac{1}{a_{\text{eq}}} - 1 = \frac{\Omega_m}{\Omega_r} - 1 \sim 3000.$$

In the Λ CDM model, the horizon scale $H^{-1}(t)$ at the epoch of equality a_{eq} , the epoch of recombination a_{dec} , and the present time $a(t_0) = 1$ are approximately:

$$H^{-1}(a_{\text{eq}}) \sim 100 \text{ Mpc}, \quad H^{-1}(a_{\text{dec}}) \sim 200 \text{ Mpc}, \quad H^{-1}(a = 1) \sim 4.3 \text{ Gpc}. \quad (2.16)$$

To address issues related to the initial conditions of the universe, such as the horizon problem and the flatness problem, Katsuhiko Sato and A. H. Guth proposed in 1981 the concept of inflation—a rapid expansion of the universe during its earliest stages. The spectrum of quantum fluctuations generated during inflation is consistent with the spectral index of primordial fluctuations, n_s , as discussed above. This consistency lends significant credibility to the presence of inflation within the framework of the Λ CDM model. In summary, the Λ CDM model is characterized by the following key features:

1. The Friedmann universe
2. The existence of inflation and the Big Bang
3. The presence of unknown energy components, namely dark matter and dark energy
4. A spectral index of primordial fluctuations, n_s , close to unity

These attributes make the Λ CDM model the most consistent cosmological model with current observational evidence.

2.2 Linear perturbation theory

2.2.1 Basic equations

Dark matter halos are formed through the nonlinear growth of dark matter density fluctuations. Before entering the nonlinear growth regime, the density distribution can be described as uniform and isotropic with slight perturbations. These perturbations evolve over time and can be analyzed using perturbation theory to understand their evolution process. This section introduces the theory of linear structure formation, providing an overview of the fundamental framework of cosmological structure formation.

While Section 2.1 discussed a homogeneous and isotropic universe, the actual universe contains numerous galaxies and stars, demonstrating that it is not perfectly homogeneous or isotropic. If the assumption of homogeneity and isotropy were strictly maintained, no non-uniform structures could emerge. However, if there were slight density fluctuations at some point in the past, gravity would cause matter to accumulate in slightly overdense regions, initiating the formation of structures. This gravitational instability-driven evolution of structure is the fundamental framework for understanding the development and evolution of cosmic structures.

As previously mentioned, the Cosmic Microwave Background (CMB), which exhibits nearly uniform temperatures across the whole sky, contains slight temperature fluctuations. Baryonic fluctuations, up until the equality time a_{eq} , were coupled with photon fluctuations through Thomson scattering, causing baryons to move in tandem with these photon fluctuations. Consequently, baryon density fluctuations were linked to the CMB's temperature fluctuations. However, as will be discussed below, the growth of baryon

fluctuations driven solely by their self-gravity is insufficient to account for the current large-scale structures. Instead, after decoupling with photons, baryon fluctuations grow under the influence of dark matter fluctuations.

In this section, we treat a homogeneous and isotropic energy component in the universe as a fluid and describe the evolution of its density fluctuations using classical fluid equations. When approximating a certain substance as a fluid, the physical quantities to be determined are the mass density field $\rho(\mathbf{r}, t)$, the velocity field $\mathbf{v}(\mathbf{r}, t)$, and the pressure field $p(\mathbf{r}, t)$. Here, $\mathbf{r}$ is the coordinate fixed in space and not the comoving coordinate. These three physical quantities obey the following continuity equation, Euler equation, and Poisson equation:

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} &= -\frac{\nabla p}{\rho} - \nabla \phi \\ \nabla^2 \phi &= 4\pi G \rho_{\text{tot}}\end{aligned}\tag{2.17}$$

Here, ρ_{tot} represents the mass density of all the energy components in the universe, and ρ and ρ_{tot} do not need to be uniform and isotropic. When $\rho = \rho_{\text{tot}}$, the equations describe the self-gravitating system.

First, we transform the coordinate to the comoving coordinate to express the effect of cosmic expansion. The relationship between the $\mathbf{r}$ coordinate and the comoving coordinate $\mathbf{x}$ is given by $\mathbf{r} = a\mathbf{x}$, so $\mathbf{v} = \dot{a}\mathbf{x} + a\dot{\mathbf{x}}$. The material's intrinsic velocity in the comoving coordinate is expressed as $\tilde{\mathbf{v}} = a\dot{\mathbf{x}}$. The time coordinate does not require a transformation. The partial derivatives, when changing from $(\mathbf{r}, t)$ to $(\mathbf{x}, t)$, are:

$$\begin{aligned}\left. \frac{\partial}{\partial t} \right|_{\mathbf{r}} &= \left. \frac{\partial t}{\partial t} \right|_{\mathbf{r}} \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}} + \left. \frac{\partial \mathbf{x}}{\partial t} \right|_{\mathbf{r}} \left. \frac{\partial}{\partial \mathbf{x}} \right|_t = \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}} - H\mathbf{x} \cdot \left. \frac{\partial}{\partial \mathbf{x}} \right|_t, \\ \left. \frac{\partial}{\partial \mathbf{r}} \right|_t &= \left. \frac{\partial t}{\partial \mathbf{r}} \right|_t \left. \frac{\partial}{\partial t} \right|_{\mathbf{x}} + \left. \frac{\partial \mathbf{x}}{\partial \mathbf{r}} \right|_t \left. \frac{\partial}{\partial \mathbf{x}} \right|_t = \frac{1}{a} \left. \frac{\partial}{\partial \mathbf{x}} \right|_t,\end{aligned}\tag{2.18}$$

Using these, equation (2.17) in the comoving coordinate becomes:

$$\frac{\partial \rho}{\partial t} + 3H\rho + \frac{1}{a} \nabla \cdot (\rho \tilde{\mathbf{v}}) = 0\tag{2.19}$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + H\tilde{\mathbf{v}} + \frac{1}{a} (\tilde{\mathbf{v}} \cdot \nabla) \tilde{\mathbf{v}} = -\frac{\nabla p}{a\rho} - \frac{\nabla \phi}{a} - \ddot{a}\mathbf{x}\tag{2.20}$$

$$\nabla^2 \phi = 4\pi G a^2 \rho_{\text{tot}}\tag{2.21}$$

In the above equations, $\nabla \equiv \frac{\partial}{\partial \mathbf{x}}$ represents the spatial derivative in the comoving frame. The right-hand side of equation (2.20), $-\ddot{a}\mathbf{x}$, represents the non-inertial force due to a coordinate transformation. The gravitational potential in the comoving coordinate is redefined as $\Phi = \phi + \frac{1}{2}a\ddot{a}|\mathbf{x}|^2$. Assuming that the energy of the uniform and isotropic component of the universe is dominated by non-relativistic matter with zero pressure, and expressing its energy density as $\bar{\rho}_{\text{tot}}$, the scale factor a follows the Friedmann equation in a uniform isotropic background spacetime:

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3} \bar{\rho}_{\text{tot}}\tag{2.22}$$

Thus, the Poisson equation for Φ can be derived from equations (2.21) and (2.22) as:

$$\nabla^2 \Phi = 4\pi G a^2 \delta_{\text{tot}} \bar{\rho}_{\text{tot}} \quad (2.23)$$

Here, the spatial non-homogeneous component of the total matter is given by:

$$\delta_{\text{tot}} = \frac{\rho_{\text{tot}}}{\bar{\rho}_{\text{tot}}} - 1. \quad (2.24)$$

From the above equation, it follows that in the comoving coordinate, only the spatial non-homogeneous component δ contributes to the potential, while the uniform isotropic component $\bar{\rho}_{\text{tot}}$ does not contribute to the potential and does not accelerate the matter in the comoving coordinate.

We assume that the spatially inhomogeneous component of the substance under consideration, δ is sufficiently small compared to the homogeneous component

$$\delta = \frac{\rho}{\bar{\rho}} - 1 \ll \bar{\rho}. \quad (2.25)$$

Moreover, the velocity $\tilde{\mathbf{v}}$ in the comoving coordinate is also assumed to be of the same order as these small quantities. In this case, the equation governing the evolution of the inhomogeneous component of the substance is obtained by extending equations (2.19) and (2.20) to first order in perturbation:

$$\frac{\partial \delta}{\partial t} + \frac{1}{a} (\nabla \cdot \tilde{\mathbf{v}}) \quad (2.26)$$

$$\frac{\partial \tilde{\mathbf{v}}}{\partial t} + H \tilde{\mathbf{v}} = -\frac{\nabla \delta p}{a \bar{\rho}} - \frac{1}{a} \nabla \Phi. \quad (2.27)$$

By eliminating $\tilde{\mathbf{v}}$ using the spatial divergence of equation (2.26), the time derivative of equation (2.27), and equation (2.23), we obtain:

$$\ddot{\delta} + 2H\dot{\delta} - \frac{\nabla^2 \delta p}{a^2 \bar{\rho}} - 4\pi G \bar{\rho}_{\text{tot}} \delta_{\text{tot}} = 0. \quad (2.28)$$

Here, the equation of state for the substance under consideration is written in terms of the density ρ and entropy S of the fluid element as $p = p(\rho, S)$. In this case, we have:

$$\delta p = \left(\frac{\partial p}{\partial \rho} \right) \Big|_S \delta \rho + \left(\frac{\partial p}{\partial S} \right) \Big|_\rho \delta S. \quad (2.29)$$

Here, the sound speed is defined as $c_s^2 = \left(\frac{\partial p}{\partial \rho} \right) \Big|_S$. On the other hand, for the entropy fluctuation $\delta S = S - \bar{S}$, in the absence of thermal conduction, the entropy is conserved along the fluid element, and for the uniform isotropic component, it does not change with time:

$$\frac{dS}{dt} = \frac{d\bar{S}}{dt} = 0. \quad (2.30)$$

Thus, $\delta S = S - \bar{S}$ does not change with time:

$$\frac{d\delta S}{dt} = 0. \quad (2.31)$$

Therefore, if the fluctuations of the substance in focus initially did not have any entropy fluctuations (adiabatic fluctuations), no entropy fluctuations will be created afterward, and $\delta S = 0$ will always hold. From equation (2.29), we obtain:

$$\delta p = c_s^2 \bar{\rho} \delta. \quad (2.32)$$

However, if the substance under consideration interacts with other components, entropy may arise due to differences in velocity, but we assume this does not occur here. The density fluctuation of substances other than the one in focus, δ_{ext} , is defined as:

$$\bar{\rho}_{\text{ext}}\delta_{\text{ext}} = \bar{\rho}_{\text{tot}}\delta_{\text{tot}} - \bar{\rho}\delta. \quad (2.33)$$

Thus, equation (2.28) becomes, from equation (2.32),

$$\ddot{\delta} + 2H\dot{\delta} - \left(4\pi G\bar{\rho}\delta + \frac{c_s^2}{a^2}\nabla^2\delta\right) = 4\pi G\bar{\rho}_{\text{ext}}\delta_{\text{ext}} \quad (2.34)$$

By performing a Fourier transform on the density fluctuation, we obtain

$$\delta(\mathbf{k}, t) = \int d^3x e^{-i\mathbf{k}\cdot\mathbf{x}}\delta(\mathbf{x}, t), \quad (2.35)$$

and equation (2.34) becomes

$$\ddot{\delta} + 2H\dot{\delta} - \left(4\pi G\bar{\rho}\delta - \frac{c_s^2 k^2}{a^2}\delta\right) = 4\pi G\bar{\rho}_{\text{ext}}\delta_{\text{ext}}. \quad (2.36)$$

In the case of a self-gravitating system, the right-hand side of equation (2.36) becomes 0, and we obtain

$$\ddot{\delta} + 2H\dot{\delta} + \left(\frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}\right)\delta = 0. \quad (2.37)$$

This is the same equation as that of a harmonic oscillator with a time-dependent angular frequency and a friction term. Therefore, the time evolution of the density fluctuation δ is determined by whether the angular frequency is complex or real. For example, when

$$4\pi G\bar{\rho} > \frac{c_s^2 k^2}{a^2}, \quad (2.38)$$

the frequency $\omega^2 = \frac{c_s^2 k^2}{a^2} - 4\pi G\bar{\rho}$ becomes complex, with a non-zero imaginary part, meaning the density fluctuation can grow. This indicates that gravitational contraction exceeds the pressure resistance, causing the fluctuation to grow. In the opposite case, the frequency is real, and the density fluctuation undergoes damped oscillations due to the friction term.

The scale that separates the two solutions is called the Jeans wavelength λ_J , which is defined along with the Jeans wavenumber k_J as follows:

$$k_J \equiv \frac{a}{c_s}\sqrt{4\pi G\bar{\rho}} \quad (2.39)$$

$$\lambda_J \equiv \frac{2\pi a}{k_J} = c_s\sqrt{\frac{\pi}{G\bar{\rho}}}. \quad (2.40)$$

On scales smaller than the Jeans wavelength, pressure exceeds gravity, and the fluctuation cannot grow. As we will see later, while baryons are coupled with photons, the Jeans wavelength is large due to photon pressure, and fluctuations can only grow after the decoupling of photons during the epoch of reionization. On the other hand, for non-relativistic matter such as dark matter, which has no self-interaction other than gravity and does not interact with radiation, we have $\delta p \approx 0$, so equation (2.37) becomes:

$$\ddot{\delta} + 2H\dot{\delta} - 4\pi G\bar{\rho}\delta = 0. \quad (2.41)$$

Since the universe is filled with non-relativistic matter, from (2.10) we have $H = \frac{2}{3t}$ and $\bar{\rho} = \frac{1}{6\pi G t^2}$, so equation (2.41) becomes

$$\ddot{\delta} + \frac{4}{3t}\dot{\delta} - \frac{2}{3t^2}\delta = 0. \quad (2.42)$$

Substituting $\delta \propto t^\nu$, we obtain

$$\nu = \frac{2}{3}, -1, \quad (2.43)$$

and

$$\delta(t) \propto t^{\frac{2}{3}}, t^{-1}. \quad (2.44)$$

The decaying mode $\delta \propto t^{-1}$ decays over time and can be neglected after a sufficiently long time. The growing mode $\delta \propto t^{\frac{2}{3}} \propto a(t)$ is the dominant mode for structure formation, growing with the expansion of the universe. In general, when considering the contributions of the cosmological constant and curvature in the background spacetime, there exists a growing mode or a mode that tends to a constant value. This mode is expressed as $D_+(t)$ and is called the linear growth factor. In a matter-dominant universe, we have $D_+(t) = a(t)$.

2.2.2 Dark matter as a main driver of structure formation

Next, we discuss the growth of baryon density fluctuations. Baryons in the universe are primarily protons and neutrons, which are already non-relativistic during the matter-dominated era. Assuming that the universe's matter content is primarily baryons, the baryon density fluctuation δ_b follows equation (2.37).

Between the equality time and the epoch of reionization, baryons and photons are strongly coupled via Thomson scattering, and baryon-photon fluctuations behave as a single fluid. In this case, the sound speed is close to the sound speed in the photon fluid, which can be expressed as:

$$c_s^2 \sim \frac{c}{\sqrt{3}} \quad (2.45)$$

At this time, the Jeans wavelength is:

$$\lambda_J \sim \frac{c}{\sqrt{\bar{\rho}}} \propto \frac{c}{H} \quad (2.46)$$

which is of the order of the horizon scale. On sub-horizon scales, fluctuations cannot grow. The growth of baryon fluctuations occurs only after decoupling from photons, when $a > a_{dec}$. After photon decoupling, baryon density fluctuations grow according to the linear growth factor. If the baryon density fluctuation at decoupling is of the same order as the temperature fluctuations observed in the CMB, then the current value of the baryon density fluctuation $\delta_b(t_0)$ is:

$$\delta_b(t_0) = \frac{D_+(t_0)}{D_+(t_{dec})} \delta_b(t_{dec}) \sim \frac{1}{a_{dec}} \delta_b(t_{dec}) \sim 10^3 \cdot 10^{-5} \sim 10^{-2} \quad (2.47)$$

On the other hand, in the current universe, on scales smaller than $\mathcal{O}(1)$ Mpc, $\delta_b > 1$, which contradicts observations.

Therefore, if the main component of the universe is not baryons but dark matter, which interacts only gravitationally, dark matter can grow according to the linear growth factor after the equality time. Baryon density fluctuations grow after photon decoupling due to the potential created by the earlier-growing dark matter density fluctuations.

In this case, δ_b follows equation (2.36) with the dark matter density and its fluctuations represented as $\bar{\rho}_{\text{dm}}$ and δ_{dm} , respectively:

$$\ddot{\delta}_b + 2H\dot{\delta}_b = 4\pi G\bar{\rho}_{\text{dm}}\delta_{\text{dm}} \quad (2.48)$$

By performing the change of variable $y = \frac{a}{a_{\text{dec}}}$ and using the Friedmann equation $H^2 = \frac{8\pi G\bar{\rho}_{\text{tot}}}{3} \approx \frac{8\pi G\bar{\rho}_{\text{dm}}}{3}$, equation (2.48) becomes:

$$y^2 \frac{d^2 \delta_b}{dy^2} + \frac{3}{2} y \frac{d\delta_b}{dy} = \frac{3}{2} \delta_{\text{dm}} \quad (2.49)$$

In a matter-dominated universe, $\delta_{\text{dm}} \propto y$, and using the initial conditions $\delta_b|_{a_{\text{dec}}} = 0$, $\left. \frac{d\delta_b}{dy} \right|_{a_{\text{dec}}} = 0$, the solution is:

$$\delta_b = \delta_{\text{dm}} \left(1 + 2y^{-\frac{3}{2}} - 3y^{-1} \right) \quad (2.50)$$

Thus, after photon decoupling, baryon density fluctuations asymptotically approach the earlier-growing dark matter fluctuations. This phenomenon is called baryon catch-up, and due to the presence of dark matter, baryon density fluctuations that lead to the structure observed in the current universe can be generated. The existence of baryon catch-up is one of the main reasons for the necessity of dark matter in cosmological structure formation theory.

2.2.3 Free-streaming of dark matter

In the case where dark matter particles do not interact with each other, if the velocity dispersion of the dark matter particles is large, there is an effect where the density fluctuations are smoothed out due to their free motion. This is known as the free-streaming damping of dark matter density fluctuation. If the dark matter particles are relativistic, their velocity dispersion is close to the speed of light, and their density fluctuations smaller than the horizon scale at that time are damped due to their free motion.

Suppose dark matter particles behaved as relativistic particles in the early universe, and then, as the universe expanded and the temperature dropped, they became non-relativistic at some time a_{nr} . If $a_{\text{nr}} > a_{\text{eq}}$, from equation (2.1), fluctuations smaller than 100 Mpc at the equality time would be smoothed out. However, as discussed in Section 2.2.2, if the density fluctuations of dark matter have not grown by this time, the growth of baryon fluctuations is delayed. Since 100 Mpc is larger than the scale of galaxy clusters, no structure, including galaxy clusters, would form in the universe. Therefore, in the bottom-up scenario of structure formation, we must have $a_{\text{nr}} < a_{\text{eq}}$.

However, even if no structures smaller than 100 Mpc exist at the equality time, it is possible that fluctuations larger than 100 Mpc grow first and form structures, which then fragment to produce smaller scale structures. This is called the top-down structure formation scenario, and dark matter in this scenario is referred to as hot dark matter (HDM). However, it has been determined that galaxies form before galaxy clusters, so this scenario is not considered realistic.

Cold dark matter (CDM) refers to dark matter particles that have a velocity dispersion of zero, meaning there is no free-streaming damping. CDM is able to explain the structure of the current universe well. This is why CDM is considered the most plausible form of dark matter.

Warm dark matter (WDM) refers to dark matter that satisfies $a_{\text{nr}} < a_{\text{eq}}$, and has free-streaming damping that is effective on scales smaller than the galaxy scale. The cosmological model with WDM has the same large-scale structure as CDM, but it alters the structure on scales where free-streaming damping is effective. Therefore, the WDM model is gaining attention as a solution to the discrepancies between the Λ CDM model and observations at small scales. This will be discussed in Section 2.5.

2.3 Non-linear evolution of density fluctuation I: analytic results

2.3.1 Top hat spherical collapse

The density fluctuations of dark matter in the early stages of the matter-dominant era can be described by the linear theory discussed in the previous section. However, as the density fluctuations grow significantly in accordance with the linear growth factor D_+ and approach $\delta \sim \mathcal{O}(1)$, the linear theory breaks down, and nonlinear growth becomes prominent. Alongside the nonlinear growth of density fluctuations, high-density regions of dark matter begin to form. These high-density regions, where dark matter particles are gravitationally bound by their own self-gravity, are referred to as dark matter halos (hereafter, halos).

As shown in Section 2.1, dark matter consists of 85% of the total matter in the universe and constitutes the most dominant gravitational source in the universe. Galaxies and galaxy clusters are believed to form through the aggregation of baryons within the gravitational potential wells created by dark matter halos. Understanding the structure of dark matter halos—particularly the evolution of their shapes, density distributions, and velocity distributions—is essential for unraveling the formation history of galaxies and galaxy clusters.

The linear theory previously described cannot quantitatively account for the evolution of fluctuations leading to the formation and growth of dark matter halos. To accurately study this process, numerical simulations, specifically the N -body simulations discussed later, serve as the primary theoretical method. Nonetheless, it is also important to formulate the structure of dark matter halos using simplified models, such as those assuming spherical symmetry.

In this section, we describe a simplified model to analyze the collapse of density fluctuations in the nonlinear regime. Specifically, we focus on a spherically symmetric system where a shell with mass M collapses under its own gravity. This model provides a framework to understand the gravitational contraction and collapse of such fluctuations.

Here, we consider a matter-dominant universe as the background spacetime. Let M denote the mass enclosed within a spherical shell, and $r(t)$ represents the radius of the shell. The equation of motion for $r(t)$ and the energy conservation law derived by

integrating the equation of motion are given as follows:

$$\begin{aligned}\frac{d^2 r}{dt^2} &= -\frac{GM}{r} \\ \frac{1}{2} \left(\frac{dr}{dt} \right)^2 - \frac{GM}{r} &= E = \text{const.}\end{aligned}\tag{2.51}$$

Here, $r(t)$ is the radius of the spherical shell containing mass M , where M is assumed to be constant.

The solutions for gravitational collapse with $E < 0$ are expressed in the following cycloid form, parameterized by θ ($0 < \theta < 2\pi$) and determined by initial conditions involving constants (A, B) :

$$\begin{aligned}r(\theta) &= A(1 - \cos \theta) \\ t(\theta) &= B(\theta - \sin \theta).\end{aligned}\tag{2.52}$$

Here, $\theta = \pi$ corresponds to the turning point of the shell, while $\theta = 2\pi$ marks the collapse. If the radius at the turning point is denoted as r_{ta} , then $r_{\text{ta}} = 2A$. From this solution and equation (2.51), the relationships among (A, B) and E can be derived as

$$\begin{aligned}\frac{dr}{dt} &= \frac{d\theta}{dt} \frac{dr}{d\theta} = \frac{A \sin \theta}{B(1 - \cos \theta)} \\ \frac{d^2 r}{dt^2} &= \frac{d\theta}{dt} \frac{d}{d\theta} \left(\frac{dr}{dt} \right) = \frac{-A}{B^2(1 - \cos^2 \theta)^2}.\end{aligned}\tag{2.53}$$

Substituting these expressions into equation (2.51), we obtain

$$B^2 = \frac{A^3}{GM}, \quad E = -\frac{GM}{2A}.\tag{2.54}$$

Since the background spacetime is matter-dominant, the mean density of the universe is given by $\bar{\rho} = \frac{1}{6\pi G t^2}$. Thus, the density contrast of the spherical shell relative to the background spacetime, $\delta(t)$, is

$$\delta(t) = \frac{\rho}{\bar{\rho}} - 1 = \frac{M}{\frac{4}{3}\pi r^3(t)} \times 6\pi G t^2 - 1 = \frac{9(\theta - \sin \theta)^2}{2(1 - \cos \theta)^3} - 1.\tag{2.55}$$

At the turning point $t = t_{\text{ta}}$ and the collapse point $t = t_{\text{coll}}$, the density contrast is

$$\delta(t_{\text{ta}}) = \delta(\pi) = \frac{9\pi^2}{16} - 1 \simeq 4.55\tag{2.56}$$

$$\delta(t_{\text{coll}}) = \delta(2\pi) \rightarrow \infty.\tag{2.57}$$

At $\theta \ll 1$, we find $\delta(t) \simeq \frac{3}{20}\theta^2 + O(\theta^4)$ and $t \simeq \frac{B\theta^3}{6} + O(\theta^5)$. Ignoring terms of $O(\theta^4)$ and higher, we have

$$\delta(t) = \frac{3}{20} \left(\frac{6t}{B} \right)^{\frac{2}{3}} \propto a(t)\tag{2.58}$$

This result matches the linear growth solution described in 2.2.1. While the linear growth solution is valid only for $\delta(t) \ll 1$, it allows for estimating the amplitude of linear perturbations at the turning and collapse times. Setting $\delta_L(t) = \frac{3}{20} \left(\frac{6t}{B} \right)^{\frac{2}{3}}$ and substituting

$t_{\text{ta}} = \pi B$ and $t_{\text{coll}} = 2\pi B$, we obtain:

$$\begin{aligned}\delta_L(t_{\text{ta}}) &= \frac{3}{20} \left(\frac{6\pi B}{B} \right)^{\frac{2}{3}} \simeq 1.06, \\ \delta_L(t_{\text{coll}}) &= \frac{3}{20} \left(\frac{12\pi B}{B} \right)^{\frac{2}{3}} \simeq 1.69\end{aligned}\tag{2.59}$$

Thus, when the amplitude of perturbations grown by linear theory exceeds 1.69, the corresponding region is considered to undergo gravitational collapse.

The time required for a system to turn around and collapse is referred to as the free-fall time, which provides a typical timescale for gravitational collapse in self-gravitating systems. In this case, the free-fall time t_{ff} can be expressed using (2.54) and $r_{\text{ta}} = 2A$ as follows:

$$t_{\text{ff}} = t_{\text{coll}} - t_{\text{ta}} = \pi B = \pi \sqrt{\frac{A^3}{GM}} = \sqrt{\frac{3\pi}{32G\rho_{\text{ta}}}}\tag{2.60}$$

Here, the density of the spherical shell at the turnaround point is given by $\rho_{\text{ta}} = M/\frac{4}{3}\pi r_{\text{ta}}^3$. Thus, the density at the turnaround point determines the characteristic timescale for gravitational collapse.

2.3.2 Collisionless Boltzmann equation and virial equilibrium

The top-hat spherical collapse model discussed in the last section describes the gravitational process of a multi-particle system composed of dark matter to some extent. However, in actual systems, the mass does not strictly concentrate at a single point due to tidal torque from a larger scale, and angular momentum induced by particle velocity dispersion or tidal torque during gravitational collapse. In reality, an isolated self-gravitating dark matter system is considered to approach a state of dynamical equilibrium known as virial equilibrium¹. In the following, we derive the relations of physical quantities in such a system under virial equilibrium, assuming that cosmic expansion can be neglected.

If there are no self-interactions between dark matter particles, the time evolution of the distribution of dark matter particles is described by the collisionless Boltzmann equation for the six-dimensional phase-space distribution function $f(\mathbf{x}, \mathbf{v}, t)$:

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{d\mathbf{x}}{dt} \frac{\partial f}{\partial \mathbf{x}} + \frac{d\mathbf{v}}{dt} \frac{\partial f}{\partial \mathbf{v}} = \frac{\partial f}{\partial t} + \mathbf{v} \frac{\partial f}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \frac{\partial f}{\partial \mathbf{v}} = 0\tag{2.61}$$

Here, it is assumed that individual particles move under the influence of gravitational interaction only. Using the gravitational potential $\Phi(\mathbf{x}, t)$, the equations of motion for particles are expressed as $\frac{d\mathbf{x}}{dt} = \mathbf{v}$ and $\frac{d\mathbf{v}}{dt} = -\frac{\partial \Phi}{\partial \mathbf{x}}$. In the self-gravitating system, we also have the Poisson equation

$$\nabla^2 \Phi(\mathbf{r}, t) = -4\pi G \int f(\mathbf{x}, \mathbf{v}, t) d^3v,\tag{2.62}$$

and equations (2.61) and (2.62) are called the Vlasov-Poisson equations as whole.

¹In N -body simulations, most of the massive halos are experiencing accretion from outside and they never complete virial equilibrium or thermal equilibrium. In general, these equilibrium states are conceptual but provide us theoretical backbone of the structure of halos.

The density distribution $\rho(\mathbf{x}, t)$ and velocity distribution $\bar{\mathbf{v}}(\mathbf{x}, t)$ of dark matter particles are given by

$$\begin{aligned}\rho(\mathbf{x}, t) &= \int d^3v f \\ \bar{\mathbf{v}}(\mathbf{x}, t) &= \frac{1}{\rho} \int d^3v \mathbf{v} f\end{aligned}\tag{2.63}$$

Integrating equation (2.61) over velocity space yields

$$\int d^3v \frac{df}{dt} = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \mathbf{x}} (\rho \bar{\mathbf{v}}) = 0\tag{2.64}$$

which gives the continuity equation. By multiplying the i -th component of velocity v_i ($i = 1, 2, 3$) with the collisionless Boltzmann equation (2.61) and integrating over velocity space, we obtain

$$\int d^3v v_i \frac{df}{dt} = \frac{\partial}{\partial t} (\rho \bar{v}_i) + \frac{\partial}{\partial x_j} (\rho \overline{v_i v_j}) + \rho \frac{\partial \Phi}{\partial x_i} = 0\tag{2.65}$$

Here, $\overline{v_i v_j}$ is defined as:

$$\overline{v_i v_j} = \frac{1}{\rho} \int d^3v v_i v_j f\tag{2.66}$$

The velocity dispersion tensor $\sigma_{ij}^2(\mathbf{x}, t)$ is defined as:

$$\begin{aligned}\sigma_{ij}^2 &= \frac{1}{\rho} \int d^3v (v_i - \bar{v}_i) (v_j - \bar{v}_j) \\ &= \overline{v_i v_j} - \bar{v}_i \bar{v}_j\end{aligned}\tag{2.67}$$

By transforming equation (2.65) using (2.64), we obtain the Euler equation:

$$\rho \frac{\partial \bar{v}_i}{\partial t} + \bar{v}_j \frac{\partial}{\partial x_j} (\rho \bar{v}_i) = -\rho \frac{\partial \Phi}{\partial x_i} - \frac{\partial}{\partial x_j} (\rho \sigma_{ij}^2)\tag{2.68}$$

Next, the following quantities W_{ij} , K_{ij} , I_{ij} are defined to represent the global properties of the system:

$$W_{ij} = - \int d^3x \rho x_i \frac{\partial \Phi}{\partial x_j}\tag{2.69}$$

$$K_{ij} = \frac{1}{2} \int d^3x \rho \overline{v_i v_j}\tag{2.70}$$

$$I_{ij} = \int d^3x \rho x_i x_j\tag{2.71}$$

The integrals are taken over a sufficiently large volume containing the isolated multi-particle system. W_{ij} is referred to as the potential energy tensor, K_{ij} as the kinetic energy tensor, and I_{ij} as the moment of inertia tensor. From their definitions, all tensors except W_{ij} are trivially symmetric. For W_{ij} , using the expression for the gravitational potential

$$\Phi(\mathbf{x}) = -G \int d^3x' \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|},\tag{2.72}$$

we can write

$$\begin{aligned}
W_{ij} &= - \int d^3x \rho x_i \frac{\partial \Phi}{\partial x_j} = G \int d^3x d^3x' \rho(\mathbf{x}) \rho(\mathbf{x}') x_i \frac{\partial}{\partial x_j} \frac{1}{|\mathbf{x}' - \mathbf{x}|} \\
&= G \int d^3x d^3x' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}') x_i (x'_j - x_j)}{|\mathbf{x}' - \mathbf{x}|^3} = G \int d^3x d^3x' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}') x'_i (x_j - x'_j)}{|\mathbf{x}' - \mathbf{x}|^3} \quad (2.73) \\
&= -\frac{G}{2} \int d^3x d^3x' \rho(\mathbf{x}) \rho(\mathbf{x}') \frac{(x'_i - x_i)(x'_j - x_j)}{|\mathbf{x}' - \mathbf{x}|^3},
\end{aligned}$$

which confirms that W_{ij} is also a symmetric tensor. By multiplying equation (2.65) by x_k and integrating over space, we obtain

$$\frac{d}{dt} \int d^3x \rho x_i \bar{v}_j = W_{ij} + 2K_{ij}. \quad (2.74)$$

Since the right-hand side is a symmetric tensor, the left-hand side can be expressed as

$$\frac{d}{dt} \int d^3x \rho x_i \bar{v}_j = \frac{1}{2} \frac{d}{dt} \int d^3x \rho (x_i \bar{v}_j + x_j \bar{v}_i). \quad (2.75)$$

On the other hand, the time derivative of the moment of inertia tensor I_{ij} can be written using the continuity equation (2.64) and integration by parts as

$$\frac{dI_{ij}}{dt} = \int d^3x \frac{\partial \rho}{\partial t} x_i x_j = - \int d^3x \frac{\partial}{\partial x_k} (\rho \bar{v}_k) x_i x_j = \int d^3x \rho (x_i \bar{v}_j + x_j \bar{v}_i) \quad (2.76)$$

Thus, equation (2.74) reduces to the following tensor virial theorem,

$$\frac{1}{2} \frac{d^2 I_{ij}}{dt^2} = W_{ij} + 2K_{ij}. \quad (2.77)$$

If the system under consideration reaches dynamical equilibrium, the shape of the system no longer changes, so the left-hand side of equation (2.77) can be assumed to vanish. In this case, the potential energy tensor and kinetic energy tensor satisfy

$$W_{ij} = -2K_{ij}. \quad (2.78)$$

When a multi-particle system satisfies equation (2.78), it is said to be in a state of virial equilibrium. Taking the trace of equation (2.78), we obtain

$$K = K_{ii} = \frac{1}{2} \int d^3x \rho \bar{v}_i \bar{v}_i = \frac{1}{2} \int d^3x d^3v v^2 f \quad (2.79)$$

$$W = W_{ii} = -\frac{G}{2} \int d^3x d^3x' \frac{\rho(\mathbf{x}) \rho(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \frac{1}{2} \int d^3x \rho \Phi. \quad (2.80)$$

Thus, K is the total kinetic energy of the system, and W is the total potential energy. In other words, when the shape of the system no longer changes, the following scalar virial theorem holds regardless of the velocity distribution or shape of the system

$$W = -2K. \quad (2.81)$$

The total energy of the system is given by $E = W + K$, and under virial equilibrium, we have

$$E = -K = \frac{1}{2} W. \quad (2.82)$$

Now, referring to the previously discussed top-hat spherical collapse model, the radius and density of the system when it reaches virial equilibrium can be determined as follows. For simplicity, we consider a model in which the system contracts while maintaining uniform density. The total potential energy W of the system when its radius is R is, as before, given by the mass within a spherical shell M , i.e.,

$$W = - \int_0^R d^3r \frac{GM(r)}{r} \rho(r) = - \frac{3GM^2}{R^6} \int dr r^5 = - \frac{3GM^2}{5R}. \quad (2.83)$$

The total energy E of the system is given by the energy at the turnaround radius r_{ta} , so from the above expression, we have $E = -\frac{3GM^2}{5r_{\text{ta}}}$. Thus, when the system reaches virial equilibrium, using equation (2.82), we find that the radius R_{vir} at virial equilibrium is

$$-\frac{1}{2} \frac{3GM^2}{5R_{\text{vir}}} = -\frac{3GM^2}{5r_{\text{ta}}} \Leftrightarrow R_{\text{vir}} = \frac{1}{2} r_{\text{ta}} \quad (= A). \quad (2.84)$$

Thus, the system reaches virial equilibrium when its radius is half the value of the turnaround radius.

Assuming that the system reaches virial equilibrium at time $t = t_{\text{coll}} = 2\pi B$, the density contrast Δ_{vir} at virial equilibrium is given by

$$\Delta_{\text{vir}} = \frac{\rho_{\text{vir}}}{\bar{\rho}(t_{\text{coll}})} - 1 = \frac{3M}{4\pi A^3} \times 6\pi G (2\pi B)^2 - 1 = 18\pi^2 - 1 \simeq 177. \quad (2.85)$$

If the dark matter halo is a self-gravitating dark matter system in virial equilibrium, then roughly speaking, the region with a density contrast of Δ_{vir} that includes the center of the over-dense region of dark matter can be considered the dark matter halo. The radius and mass of such a sphere are referred to as the virial radius R_{vir} and virial mass M_{vir} , respectively, and they are widely used as indicators of the "size" of the halo.

2.3.3 Halo mass function

In this section, we will demonstrate that the mass function of dark matter halos can be derived analytically using the nonlinear growth of density fluctuations described by the top-hat spherical collapse model discussed in Section 2.3.1 and the linear power spectrum. First, we consider the physical quantity known as the fluctuation averaged over a certain spatial scale.

In Section 2.2.1, we discussed density fluctuations defined as fields at each point in space. However, when actual halos form from density fluctuations, it is necessary for certain regions to be overdense with a certain characteristic size. Therefore, we consider the density fluctuation averaged over a certain spatial scale as follows

$$\delta_R(\mathbf{x}) = \int d^3y W_R(|\mathbf{x} - \mathbf{y}|) \delta(\mathbf{y}). \quad (2.86)$$

Here, $W_R(|\mathbf{x}|)$ is a function called the top-hat window function defined as

$$W_R(|\mathbf{x}|) = \frac{3}{4\pi R^3} \Theta(R - |\mathbf{x}|) \quad (2.87)$$

$$\Theta(x) = \begin{cases} 1, & (x \geq 0) \\ 0, & (x < 0), \end{cases} \quad (2.88)$$

which averages the density fluctuation over a scale R . The Fourier components are given by

$$W_R(k) = \frac{3}{k^3 R^3} (\sin kR - kR \cos kR), \quad (2.89)$$

and

$$\delta_R(\mathbf{k}) = W_R(k) \delta(\mathbf{k}). \quad (2.90)$$

Note that $W_R(k)$ primarily extracts fluctuations on scales $k < 1/R$, and decays for $k > 1/R$.

The variance of the averaged density fluctuation σ_R^2 is given by:

$$\begin{aligned} \sigma_R^2 &= \langle \delta_R^2(\mathbf{x}) \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} \langle \delta_R(\mathbf{k}) \delta_R(\mathbf{k}') \rangle e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} W_R(k) W_R(k') \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{d^3k'}{(2\pi)^3} (2\pi)^3 \delta_D^{(3)}(\mathbf{k} + \mathbf{k}') P(k) e^{-i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{x}} W_R(k) W_R(k') \\ &= \frac{1}{(2\pi)^3} \int d^3k |W_R(k)|^2 P(k) \end{aligned} \quad (2.91)$$

Using $\delta(\mathbf{k})$ and $P(k)$, and employing the linear power spectrum, from the top-hat spherical collapse model, the density at the collapse time (equation (2.59)) leads us to conclude that the regions where $\sigma_R > 1.06$ are the regions where fluctuations become nonlinear, and halos start to form. The mass M of the overdense region formed at this time can be approximated by

$$M = \frac{4\pi}{3} \bar{\rho}_m(t) R^3 = \frac{4\pi}{3} \Omega_m \rho_{cr} r^3 \simeq 2 \times 10^{14} M_\odot \left(\frac{\Omega_m h^2}{0.15} \right) \left(\frac{r}{10 \text{ Mpc}} \right). \quad (2.92)$$

where $\bar{\rho}_m(t)$ is the background matter density and $r = R/a(t)$ is the comoving distance corresponding to the scale R . Thus, the mass M of the overdense region and the comoving distance r corresponding to the scale over which the density fluctuations are averaged are in a one-to-one correspondence.

Therefore, it is often convenient to write σ_M and $\delta_M(\mathbf{x})$ instead of σ_R and $\delta_R(\mathbf{x})$. The time dependence of σ_M and δ_M within the framework of linear theory is given by the following expressions. If their values at a time t_i are $\sigma_M(t_i)$ and $\delta_M(t_i)$, then

$$\sigma_M(t) = \sigma_M(t_i) \frac{D_+(t)}{D_+(t_i)}, \quad \delta_M(t) = \delta_M(t_i) \frac{D_+(t)}{D_+(t_i)}, \quad (2.93)$$

where $D_+(t)$ is the growth factor at time t . The linear density fluctuation peak $\nu(M, t)$ is defined as

$$\nu(M, t) \equiv \frac{\delta_L(t_{\text{coll}})}{\sigma(M, t)} = \frac{1.686}{\sigma(M, t)}. \quad (2.94)$$

This represents the approximate mass of the halo that has formed at time t , where the density fluctuation associated with the mass M becomes nonlinear. Specifically, if $\nu(M, t) < 1$, then the density fluctuation for that mass M has collapsed into a halo before time t . Thus, $\nu(M, t)$ is an important indicator of how close a density fluctuation on a given scale is to collapse and form a halo.

The observed value of σ_R is a crucial quantity that represents the scale at which nonlinear growth of density fluctuations begins in the real universe. Conventionally, the

current value of $\sigma_8 = \sigma(8h^{-1} \text{Mpc})$ is often used. From the observations of the CMB temperature fluctuations (TT, TE, EE+lowE+lensing+BAO), it is found that:

$$\sigma_8 = 0.8102 \pm 0.0060 \quad (2.95)$$

(Planck Collaboration et al., 2020).

The mass function of a certain type of object refers to the number of such objects as a function of their mass. Below, we show that the mass function of dark matter halos can be analytically derived using the linear power spectrum and the spherical collapse model, based on several assumptions:

1. The randomness of the density fluctuations in the early universe follows Gaussian statistics.
2. The nonlinear growth of density fluctuations can be described by the spherical collapse model, with the state in virial equilibrium (see Section 2.3.2) corresponding to dark matter halos.
3. The mass of dark matter halos increases monotonically with time.
4. The spatial correlation of density fluctuations is neglected.

From the assumption 1, the probability that the averaged density fluctuation δ_M in a region containing mass M at the initial universe time t_i has a value in the range $\delta_M \sim \delta_M + d\delta_M$ is given by

$$P_G[\delta_M(t_i)]d\delta_M = \frac{1}{\sqrt{2\pi\sigma_M^2(t_i)}} \exp\left(-\frac{\delta_M^2(t_i)}{2\sigma_M^2(t_i)}\right). \quad (2.96)$$

From equation (2.93) and the assumptions 2 and 3, at time t , the condition

$$\delta_M(t) = \delta_M(t_i) \frac{D_+(t)}{D_+(t_i)} \geq \delta_L(t_{\text{coll}}) \quad (2.97)$$

indicates that the fluctuation on this scale has collapsed into a dark matter halo of mass M by time t . The fraction $F(M, t)$ of dark matter in the universe contained in such regions is given by

$$F(M, t) = \int_{\delta_L(t_{\text{coll}}) \frac{D_+(t_i)}{D_+(t)}}^{\infty} P_G[\delta_M(t_i)]d\delta_M = \frac{1}{2} \text{erfc}\left[\frac{\delta_L(t_{\text{coll}})D_+(t_i)}{\sqrt{2}\sigma_M(t_i)D_+(t)}\right]. \quad (2.98)$$

where $\text{erfc}(x)$ is the complementary error function

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-u^2} du. \quad (2.99)$$

From equation (2.93), it can be seen that the right-hand side of this expression is independent of t_i , so t_i can be any arbitrary time. Equation (2.97) is a sufficient condition for the formation of a dark matter halo of mass M by time t at any point in the universe, but not a necessary condition. The necessary and sufficient condition is that there exists a $\tilde{M} > M$ such that

$$\delta_{\tilde{M}}(t) = \delta_L(t_{\text{coll}}). \quad (2.100)$$

Let the probability that event A, which satisfies equation (2.97), occurs be $P(A)$, and the probability that event B, which satisfies equation (2.100), occurs be $P(B)$. By Bayes' theorem,

$$P(B) = \frac{P(A)P(B|A)}{P(A|B)}. \quad (2.101)$$

Since equation (2.97) is a sufficient condition for equation (2.100), we have $P(B|A) = 1$. On the other hand, from assumption 4, the value of $\delta_{\tilde{M}}(t) = \delta_L(t_{\text{coll}})$ can be either larger or smaller than $\delta_M(t)$, and the actual probability depends on the stochastic process. However, based on assumption 4, we set $P(A|B) = \frac{1}{2}$, so equation (2.101) becomes:

$$P(B) = 2P(A) \quad (2.102)$$

Let the number density of halos with mass in the range $M \sim M + dM$ be $n(M)dM$, where $n(M)$ is the mass function of dark matter halos. The total mass density is assumed to be entirely contained within halos, so:

$$\int_0^\infty M n(M) dM = \Omega_m \rho_{\text{cr}}$$

Using this, from equation (2.102), we obtain

$$\begin{aligned} M n(M) dM &= \Omega_m \rho_{\text{cr}} \times 2 (F(M, t) - F(M + dM, t)) \\ \Leftrightarrow n(M) &= -\sqrt{\frac{2}{\pi}} \frac{\Omega_m \rho_{\text{cr}}}{M^2} \frac{d \log \sigma_M(t)}{d \log M} \frac{\delta_L(t_{\text{coll}})}{\sigma_M(t)} \exp \left(-\frac{\delta_L^2(t_{\text{coll}})}{2\sigma_M^2(t)} \right). \end{aligned} \quad (2.103)$$

This is the Press-Schechter mass function (Press & Schechter, 1974). Using the linear density fluctuation peak $\nu(M, t)$, we can express the mass function as

$$\begin{aligned} n(M) dM &= \frac{\Omega_m \rho_{\text{cr}}}{M} f_{MF}(\nu) \frac{d\nu}{\nu} \\ f_{MF}(\nu) &= \sqrt{\frac{2}{\pi}} \nu e^{-\frac{\nu^2}{2}}, \end{aligned} \quad (2.104)$$

where $f_{MF}(\nu)$ is called the multiplicity function. Although the Press-Schechter mass function is a phenomenological mass function derived by extrapolating the spherical collapse model and linear theory, it is known to qualitatively match the mass function obtained from cosmological N -body simulations.

In Bond et al. (1991), they derived this Press-Schechter formula from the excursion-set theory of the averaged density field $\delta_R(\mathbf{x})$ by adopting several filters including the top-hat filter above. For example, when a sharp- k filter is used, the averaged density field $\delta_R(\mathbf{x})$ performs a Brownian random walk as a function of smoothing scale R , enabling the derivation of the Press-Schechter mass function. Lacey & Cole (1993) expanded on this framework, deriving additional quantities beyond the mass function, such as the merger rate, the distributions of halo formation epochs, and survival times, all expressed as functions of halo mass. This extension is widely regarded as a landmark achievement, providing an analytical description of the statistical properties of non-linear structure formation of the universe.

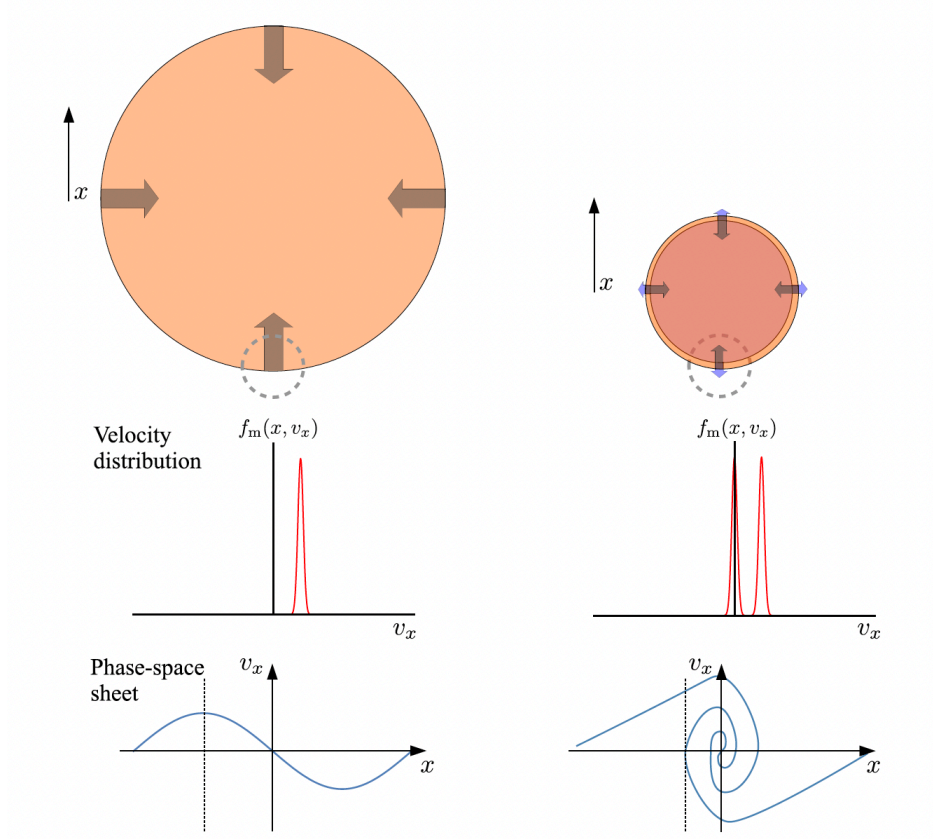


Figure 2.1: The collapse of spherical dark matter shells in early time (left), and late time after shell-crossing (right). The upper panel provides the sketch of the shells in the configuration space. The middle panel shows the velocity distribution of the shells at the radii indicated by the grey-dashed circle in the upper panel. The lower panel shows the phase-space distribution of the shells along the x -axis. This figure is taken from Dodelson & Schmidt (2020).

2.3.4 Multi-stream region and self-similar solutions

Let us examine the spherical collapse model after the shell reaches its turnaround point. In Section 2.3.2, we explore the concept of virial equilibrium and deduce that a collapsed spherical shell stabilizes at half the turnaround radius. If the dark matter distribution is described as a superposition of collisionless spherical shells collapsing toward the center after their turnaround, their behavior is governed by the Vlasov-Poisson equation (2.61). Notably, these shells will return to their apocenter after passing through the center. Therefore, as a shell moves outward from the center, it encounters other collapsing shells. Consequently, its internal mass is no longer constant at M but decreases after crossing inward-moving shells.

Figure 2.1 illustrates this shell-crossing process. The left-bottom panel presents a phase-space distribution, showing that all shells within the inner region collapse toward the center. After collapsing, a shell returns to its apocenter, leading to shell crossing, as depicted in the right panel. By examining a specific radius, represented by the grey-dashed circle in the upper-right panel, we observe that the velocity distribution at that radius is multi-valued by the shell moving outward and inward (middle and lower panels). Moreover, within this radius (indicated by the black-dotted line in the lower-right panel), frequent encounters between collapsed shells occur, forming the so-called multi-stream region. Investigating the universal characteristics of this region is a primary focus of this thesis and is introduced in Section 2.4.

This shell-crossing fundamentally alters the assumptions of the top-hat spherical collapse model outlined in Section 2.3.1, where the internal mass M remains constant during the collapse. Thus, a more realistic model that accounts for shell-crossing is necessary. While this is ultimately achieved through N -body simulations, it is also possible to derive a self-similar solution for such a system. This section is dedicated to deriving the self-similar solution, including angular momentum effects, following the works of Fillmore & Goldreich (1984); Nusser (2001); Zukin & Bertschinger (2010).

The system we are considering is the collapsing spherical overdensity in the EdS universe (halo), which is described by the superposition of spherical shells going outward at first following Hubble flow, and then going inward after their turnaround. This system setup allows us to find a self-similar solution because there is no specific scale in the system. Here, self-similarity implies that the properties of the halo remain invariant under scaling by the turnaround radius $r_{\text{ta}}(t)$ at time t . This radius represents the location where a shell is currently turning around. Similarly, the internal mass $M_{\text{ta}}(t)$ enclosed within the turnaround radius also serves as a scaling factor for maintaining this invariance. We start by deriving $r_{\text{ta}}(t)$ and $M_{\text{ta}}(t)$.

Consider the i -th shell with initial radius r_i and initial overdensity δ_i at time t_0 . Because we are considering the EdS universe, the initial internal mass inside r_i is

$$M(< r_i, t_0) \equiv M_i(1 + \delta_i) = \frac{4\pi}{3} r_i^3 \rho_b(t_0)(1 + \delta_i(r_i)), \quad (2.105)$$

where $M_i = \frac{4\pi}{3} r_i^3 \rho_b(t_0) = 2r_i^3/9t_0^2$. Suppose that the initial velocity of this shell is Hubble flow, i.e., $v_i = r_i H_i = 2r_i/3t_0$ (see equation (2.10)), we obtain its energy E_i as

$$E_i = \frac{1}{2} v_i^2 - \frac{GM_i(1 + \delta_i)}{r_i} = -\frac{GM_i \delta_i}{r_i}. \quad (2.106)$$

Here, we assume that the initial overdensity is distributed in a scale-free manner, i.e.,

$$\delta_i = \left(\frac{M_i}{M_0} \right)^{-\epsilon}, \quad (2.107)$$

where M_0 is referential mass and $0 \leq \epsilon \leq 1$ determines the initial density fluctuation. Equation (2.107) can be interpreted as the relation between r_i and δ_i as

$$r_i = \mathcal{R}_c \delta_i^{-1/3\epsilon}, \quad \mathcal{R}_c \equiv \left(\frac{3M_0}{4\pi\rho_b(t_0)} \right)^{1/3}, \quad (2.108)$$

where the constant $\mathcal{R}_c$ represents the referential length and does not affect the dynamics below.

If we define $r_{\text{ta},i}$ and $t_{\text{ta},i}$ as the turnaround radius where the radial velocity become 0 and its time of the i -th shell, $r_{\text{ta},i}$ can be calculated from the energy conservation

$$E_i = -\frac{GM_i(1 + \delta_i)}{r_{\text{ta},i}}, \quad (2.109)$$

and obtain

$$r_{\text{ta},i} = \delta_i^{-\frac{1+3\epsilon}{3\epsilon}} \mathcal{R}_c. \quad (2.110)$$

Because, its trajectory can be described by the top-hat spherical collapse model until its turnaround, $t_{\text{ta},i}$ can be calculated as

$$t_{\text{ta},i} = \pi \sqrt{\frac{r_{\text{ta},i}^3}{8GM_i}} = \frac{3\pi}{4} \delta_i^{-\frac{3}{2}} t_0 \quad (2.111)$$

from equation (2.52). The current turnaround radius is defined as the turnaround radius of the shell with $t_{\text{ta},i} = t$, therefore it can be calculated from the following simultaneous equations

$$\begin{cases} r_{\text{ta}}(t) = \delta^{-\frac{1+3\epsilon}{3\epsilon}} \mathcal{R}_c \\ t = \frac{3\pi}{4} \delta^{-\frac{3}{2}} t_0 \end{cases}, \quad (2.112)$$

and obtain

$$r_{\text{ta}}(t) = \left(\frac{4t}{3\pi t_0} \right)^{\frac{2+6\epsilon}{9\epsilon}} \mathcal{R}_c, \quad M_{\text{ta}}(t) = \frac{2}{9G} \left(\frac{3\pi}{4} \right)^2 \frac{r_{\text{ta}}^3(t)}{t^2}. \quad (2.113)$$

Now, consider the mass profile at a given time t , $M(< r, t)$, after the evolution of the spherical shells. Due to the self-similarity of the system, the enclosed mass can be expressed as

$$M(< r, t) = M_{\text{ta}}(t) \mathcal{M}(r/r_{\text{ta}}(t)), \quad (2.114)$$

where $\mathcal{M}(r/r_{\text{ta}}(t))$ denotes the normalized internal mass distribution, satisfying $\mathcal{M}(1) = 1$. By regarding the subscript i as the Lagrange coordinate of the shells, $M(< r, t)$ can be also written as

$$M(< r, t) = \int_{M'_i=0}^{M'_i=M_{\text{ta}}(t)} dM'_i H[r - r(t, r'_i)], \quad (2.115)$$

where $r(t, r_i)$ denotes the current radius of the i -th shell and $H[u]$ is the Heaviside step function satisfying $H[u] = 1$ for $u > 0$. The righthand side of this equation is just summing up contributions from the shells whose current radius is inside r . Noting that $r_{\text{ta},i}$ also acts as a Lagrange coordinate for the shells, we change the integration variable

from M_i to $r_{\text{ta},i}$ and derive

$$\begin{aligned}
\mathcal{M}(r/r_{\text{ta}}(t)) &= \frac{1}{M_{\text{ta}}(t)} \int_{r'_{\text{ta},i}=0}^{r'_{\text{ta},i}=r_{\text{ta}}(t)} dM'_i(r'_{\text{ta},i}) H[r - r(t, r'_{\text{ta},i})] \\
&= \frac{1}{M_{\text{ta}}(t)} \int_{u=\infty}^{u=1} dM'_i(u) H[r/r_{\text{ta}}(t) - r(t, r'_{\text{ta},i})/r_{\text{ta}}(t)] \\
&= \frac{2}{3\epsilon} \int_{u=1}^{u=\infty} \frac{du}{u^{1+\frac{2}{3\epsilon}}} H[r/r_{\text{ta}}(t) - r(t, r_{\text{ta}}(t)/u)/r_{\text{ta}}(t)], \tag{2.116}
\end{aligned}$$

where we introduced the new variable $u \equiv r_{\text{ta}}(t)/r_{\text{ta},i}$ in transitioning from the first to the second equation, and utilized the relationship between M_i and $r_{\text{ta},i}$ from equations (2.108) and (2.110).

The remaining problem is solving the equation of motion (EoM) to determine the trajectory $r(t, r_{\text{ta}}(t)/u)$. Once this is achieved, the integral in equation (2.116) can be computed numerically. We begin by introducing the EoM for the i -th shell

$$\frac{d^2 r(t, r_{\text{ta},i})}{dt^2} = -\frac{GM(< r, t)}{r} + \frac{L^2(r_{\text{ta},i})}{r^3}, \tag{2.117}$$

where $r_{\text{ta},i}$ serves as the Lagrange coordinate for the shells, and we assume that the angular momentum varies as a function of $r_{\text{ta},i}$ for each shell. For simplicity, we assume $L(r_{\text{ta},i}) = \mathcal{L}\sqrt{GM_i r_{\text{ta},i}}$, where $\mathcal{L}$ spans the range $0 \leq \mathcal{L} \leq 1$. To facilitate calculations, we normalize the Lagrange coordinates and define new variables $\lambda \equiv r_{\text{ta},i}/r_{\text{ta}}(t)$ and $\tau \equiv \log(t/t_0)^2$. Additionally, we introduce $\beta \equiv (2 + 6\epsilon)/9\epsilon$, which allows us to express $r_{\text{ta}}(t) \propto t^\beta$. Using these variables, the terms in equation (2.117) can be rewritten as

$$\begin{aligned}
\frac{d^2 r(t, r_{\text{ta},i})}{dt^2} &= \frac{r_{\text{ta}}(t)}{t^2} \left\{ \frac{d^2 \lambda}{d\tau^2} + (2\beta - 1) \frac{d\lambda}{d\tau} + \beta(\beta - 1)\lambda \right\}, \\
-\frac{GM(< r, t)}{r} &= -\frac{2}{9} \left(\frac{3\pi}{4} \right)^2 \frac{r_{\text{ta}}(t)}{t^2} \mathcal{M}(\lambda), \\
\frac{L^2(r_{\text{ta},i})}{r^3} &= \mathcal{L}^2 \frac{r_{\text{ta}}(t)}{t^2} \frac{1}{\lambda^3} e^{2\tau(1-2\beta)}. \tag{2.118}
\end{aligned}$$

From these, we derive the nondimensional EoM

$$\frac{d^2 \lambda}{d\tau^2} + (2\beta - 1) \frac{d\lambda}{d\tau} + \beta(\beta - 1)\lambda = -\frac{2}{9} \left(\frac{3\pi}{4} \right)^2 \mathcal{M}(\lambda) + \frac{\mathcal{L}^2}{\lambda^3} e^{2\tau(1-2\beta)}. \tag{2.119}$$

We can also express the nondimensional mass distribution from equation (2.116) as

$$\mathcal{M}(\lambda) = \frac{2}{3\epsilon} \int_1^\infty \frac{du}{u^{1+\frac{2}{3\epsilon}}} H[\lambda - \lambda(u)]. \tag{2.120}$$

By numerically solving equations (2.119) and (2.120) simultaneously with initial conditions $\lambda(\tau = 0) = 1$ and $d\lambda/d\tau(\tau = 0) = -\beta$, and iterating until convergence, we can obtain the mass distribution of the self-similar solution, $\mathcal{M}(\lambda)$. The density distribution and the phase space are shown in Fig. 3.25 in Section 3.6. The asymptotic slope of $\mathcal{M}(\lambda)$ is given by $3/(1 + 3\epsilon)$ for any value of $\mathcal{L}$ (Nusser, 2001).

²Here is the essence of the self-similarity. $\lambda(\tau)$ can be either interpreted as labeling the location of a particular i -th shell at different times, or labeling the location of all mass shells at a particular time t (Zukin & Bertschinger, 2010).

2.4 Non-linear evolution of density fluctuation II: Cosmological N -body simulations in CDM case

This chapter presents the results of previous studies on the structure of CDM halos mainly based on N -body simulations. It consists of three subsections as follows.

We begin by outlining the methodology for implementing cosmological N -body simulations in Section 2.4.1, followed by a discussion on halo identification techniques in Section 2.4.2. The structural properties of CDM halos are addressed in Section 2.4.3, where we also introduce their universal characteristics, alongside the universal behavior of pseudo-phase-space density described in Section 2.4.4. In Section 2.4.5, we examine the emergence and evolution of prompt cusps in the first-formed halos. Lastly, Section 2.4.6 presents the findings of Sugiura et al. (2020), a prior study on the phase-space structure of halos that closely relates to our work.

2.4.1 Methods of cosmological N -body simulation

Up to this point, we have reviewed methods for describing the structure and mass function of dark matter halos using spherically symmetric models and the Press-Schechter theory. However, neither of these theories can accurately describe the process by which dark matter particles densify due to gravitational interactions and form dark matter halos. To more realistically describe this process, cosmological N -body simulations are used.

In cosmological N -body simulations, the density field of dark matter is represented by the distribution of N finite simulation particles. The evolution of these particles due to gravitational interactions is numerically calculated and simulated. Focusing solely on the evolution of the dark matter density distribution, without considering the gravitational contribution of baryons, and assuming that Newtonian gravity can be used, the time evolution of each simulation particle (hereafter referred to simply as "particle") is governed by Newtonian gravity. The motion of the particles is calculated within a simulation box of size L^3 under periodic boundary conditions.

Let the mass of a particle be m , the comoving coordinates of the i -th particle be $\mathbf{x}_i$, and the physical velocity be $\mathbf{v}_i$. The velocity $\mathbf{u}_i = \frac{d\mathbf{x}_i}{dt}$ in comoving coordinates is given by $\mathbf{u}_i = \frac{\mathbf{v}_i}{a}$. The equation of motion for the particle is given by replacing $\tilde{\mathbf{v}}$ in equation (2.20) with $\mathbf{v}$ and ignoring the pressure p that existed in the fluid approximation. Thus, we have

$$\frac{d\mathbf{v}_i}{dt} + H\mathbf{v}_i = -\frac{1}{a}\nabla\Phi \Leftrightarrow \frac{d\mathbf{u}_i}{dt} + 2H\mathbf{u}_i = -\frac{1}{a^2}\nabla\Phi. \quad (2.121)$$

Here, the gravitational potential $\Phi(\mathbf{x}, t)$ is obtained from equation (2.23) as

$$\Phi(\mathbf{x}, t) = -Ga^2\nabla \int d^3x' \frac{\rho(\mathbf{x}', t) - \bar{\rho}(t)}{|\mathbf{x} - \mathbf{x}'|} = Ga^2 \int d^3x' \rho(\mathbf{x}', t) \frac{\mathbf{x} - \mathbf{x}'}{|\mathbf{x} - \mathbf{x}'|^3}, \quad (2.122)$$

where $\bar{\rho}(t)$ is a uniform isotropic energy component and does not contribute to the integral due to its lack of spatial dependence. $\rho(\mathbf{x}, t)$ is the density distribution of point particles, given by

$$\rho(\mathbf{x}, t) = \frac{m}{a^3} \sum_i \delta_D^3(\mathbf{x} - \mathbf{x}_i(t)). \quad (2.123)$$

From equations (2.121), (2.122), and (2.123), we obtain

$$\frac{d\mathbf{u}_i}{dt} + 2H\mathbf{u}_i = \frac{Gm}{a^3} \sum_{j \neq i} \frac{\mathbf{x}_i - \mathbf{x}_j}{|\mathbf{x}_i - \mathbf{x}_j|^3}. \quad (2.124)$$

The right-hand side of the above equation diverges when $\mathbf{x}_i \sim \mathbf{x}_j$. This divergence arises from the two-body interaction due to the discretization of the density distribution and does not occur in the original continuous distribution. To handle this, we introduce $\epsilon > 0$ and modify the equation as

$$\frac{d\mathbf{u}_i}{dt} + 2H\mathbf{u}_i = \frac{Gm}{a^3} \sum_{j \neq i} \frac{\mathbf{x}_i - \mathbf{x}'_j}{(|\mathbf{x}_i - \mathbf{x}'_j|^2 + \epsilon^2)^{\frac{3}{2}}}. \quad (2.125)$$

This modification eliminates the large accelerations due to two-body interactions. The parameter ϵ is known as the softening length, and in N -body simulations, structures on scales smaller than ϵ cannot be accurately calculated. Generally, ϵ is chosen to be less than 5% of the number of particles per dimension, $N^{\frac{1}{3}}/L$.

Using equation (2.125), we can calculate the positions of the N particles at each time step, but this calculation is not a trivial task. First, calculating the gravitational acceleration on the right-hand side of equation (2.125) requires $\mathcal{O}(N^2)$ operations. This is known as the direct method, and current N -body simulations typically have $N \gtrsim 10^8$, making this approach impractical in terms of computation time. To address this, a method is employed where the gravitational forces from nearby particles are computed using the direct method, while the forces from distant particles are averaged.

For example, the simulation box is divided into a grid, and the density distribution at each grid point is calculated. Then, the gravitational acceleration at each grid point is determined based on this density distribution. The method of substituting the gravitational acceleration from distant particles with that calculated at these grid points is called the Particle-Particle-Particle-Mesh method (P³M method). The computation of gravitational acceleration at the grid points requires, intuitively, $\mathcal{O}(N_g^2)$ operations for N_g grid points, and increasing the grid resolution seems to make the computation time similar to that of the direct method. However, by performing a discrete Fourier transform and using the Fast Fourier Transform (FFT) during the transformation, the computation time for gravitational acceleration at the grid points can be reduced to $\mathcal{O}(N_g \log N_g)$. The method of approximating the gravitational force from nearby particles using grid points is called the Particle-Mesh method (PM method).

In the Tree method, the simulation box is first divided into 8 cubes of equal size, and each of these cubes is further divided into 8 smaller cubes. The division stops when no particles remain in a cube. Through this process, the entire simulation box takes on a hierarchical structure of cubes with parent-child relationships. For a given simulation particle, the gravitational force from the particles in the cubes within a specified solid angle is approximated by assuming that all the mass is concentrated at the center of mass of the particles in the cube. This reduces the computation time to $\mathcal{O}(N \log N)$, significantly decreasing the computational time compared to the direct method. Additionally, there are methods such as the Tree-PM method, where the gravitational force from nearby particles is computed using the Tree method and the gravitational force from distant particles is computed using the grid division method, and the Adaptive Particle-Particle-Particle-Mesh (AP³M) method, which refines the grid in high-density regions in the P³M method.

2.4.2 Halo finders

To analyze the structure of dark matter halos, it is necessary to determine which regions of the dark matter density distribution calculated by cosmological N -body simulations

correspond to the dark matter halo. Since dark matter halos are self-gravitating systems, if a particle is bound by the gravitational potential created by the halo, then that particle is considered part of the halo. However, in order to calculate this gravitational potential, we first need to determine which particles constitute the halo. Therefore, the first step is to identify the halo particles without using potential information.

The classical methods used for this purpose include the Spherical Overdensity (SO) method and the Friends of Friends (FOF) method. The principle of the SO method is to first locate the density peaks within the simulation box and then consider a sphere of radius R centered at these peaks. As the radius R is increased, when the density ρ inside the sphere reaches a certain ratio Δ relative to the average density $\bar{\rho} = N/L^3$ of the simulation particles ($\rho/\bar{\rho} = \Delta$), the particles inside the sphere are considered to be part of the halo, and this radius is denoted as R_{vir} . By setting $\Delta = \Delta_{\text{vir}}$ (equation (2.85)), halos corresponding to virial equilibrium can be defined (principle SO method).

Although the principle of the SO method is to consider the region with virial overdensity as the halo, there are issues such as the arbitrariness in determining the density peaks and the overlap of halos. Therefore, more detailed methods for defining halos are used, as described below (Lacey & Cole, 1994).

For a given particle, the N_s -th nearest particle is identified, and the distance to this particle is denoted as r_{N_s} . The density peak at this point is calculated as $3(N+1)/(4\pi r_{N_s}^3)$. First, the principle SO method is applied around the point with the largest density peak to determine the tentative halo particles. Then, the center of mass of these tentative particles is found, and the principle SO method is applied again. This process is repeated until convergence, and the final halo particles are determined. Afterward, the largest density peak from the other particles is identified, and the same procedure is repeated. By repeating this process, the most massive halo among the overlapping halos is chosen as the final halo, and the particles from the smaller halo are added to it. However, this method does not guarantee that the particles identified as part of the halo are gravitationally bound. To address this, the gravitational potential is calculated using the particles determined by the SO method, and the particles that are bound to the potential are considered the true halo particles.

The Friends of Friends (FOF) method is as follows. The average particle spacing is given by $\lambda = L/N^{1/3}$, and for a chosen parameter b , two particles are connected if their distance is less than $b\lambda$. This process is repeated to form clusters of connected particles (FOF clusters), which are considered halos in the FOF method. Let the density of the connected particle group be ρ , and using b and the average particle density $\bar{\rho} = N/L^3$, the overdensity of the particle group is given by

$$\frac{\rho}{\bar{\rho}} \sim \frac{3}{2\pi b^3}. \quad (2.126)$$

It is known that when $b = 0.2$, this overdensity becomes close to the virial overdensity, and this value is commonly used. However, the FOF method can create bridge-like structures between different halos, leading to the identification of unrealistic halo shapes. By reducing b , such structures can be diminished, but this also results in the selection of halos with fewer particles, which are more overdense. To address this, composite methods are used, such as determining the tentative halo particles with a smaller b in the FOF method, and then performing the principle SO method with the center of mass as the density peak (Jing & Suto, 2002).

Thus, defining halos in cosmological N -body simulations is not a trivial procedure, and in recent years, specialized programs called halo finders are often used. The ROCKSTAR

halo finder (Behroozi et al., 2013), which was used in this study, improves the FOF method and determines halos using phase space.

The ROCKSTAR halo finder determines halo constituent particles using the following method:

1. First, determine the tentative constituent particles using the FOF method.
2. Calculate the average velocity $\bar{\mathbf{v}}$, velocity dispersion σ_v , center of mass $\bar{\mathbf{x}}$, and distance dispersion σ_x of these constituent particles.
3. Define the "distance" d_p of each constituent particle as follows:

$$d_p(\mathbf{x}_i, \mathbf{v}_i) = \left(\frac{|\mathbf{x}_i - \bar{\mathbf{x}}|^2}{\sigma_x^2} + \frac{|\mathbf{v}_i - \bar{\mathbf{v}}|^2}{\sigma_v^2} \right)^{\frac{1}{2}}. \quad (2.127)$$

4. Choose a distance d such that 70% of the particles defined in step 1 are connected, and connect particles whose distance from the others is less than or equal to d (FOF method in phase space).
5. Using the particles connected in the previous step, determine the new "distance" in phase space according to step 3.
6. Choose a new d such that 70% of the connected particles are joined, and connect particles whose new distance is less than or equal to d (repeat steps 5 and 6).
7. Repeat steps 5 and 6, and when the number of connected particles drops to 10 or fewer, define them as seed halos and end the process.
8. If there is only one seed halo in the hierarchical FOF clusters defined in step 7, assign all particles in that hierarchy to the seed halo's constituent particles. For hierarchies with two or more seed halos, calculate the "distance" to halos in lower hierarchies and assign each particle to the nearest halo.
9. Once all particles in the FOF clusters defined in step 1 have been assigned to halos, the process ends.

Subhalos and host halos are close in distance, so they would normally be connected in the FOF method. However, because their velocity distributions differ, they can be separated by the "distance" in phase space, which is the key strength of the ROCKSTAR halo finder. However, the output halo catalog does not explicitly mark whether a halo is a subhalo, so if one is interested in focusing only on subhalos (or vice versa), separate processing must be done. In this thesis, we focus only on host halos and develop a method to separate subhalos for this purpose (see Section 3.3).

2.4.3 Density profile of CDM halos

Cosmological N -body simulations enable us to elucidate how density fluctuations predicted by linear theory evolve nonlinearly to form various structures. Large-scale simulations within cosmological boxes of several Mpc scales have become feasible since the 1990s, with pioneering works including Navarro et al. (1996) and Fukushige & Makino (1997).

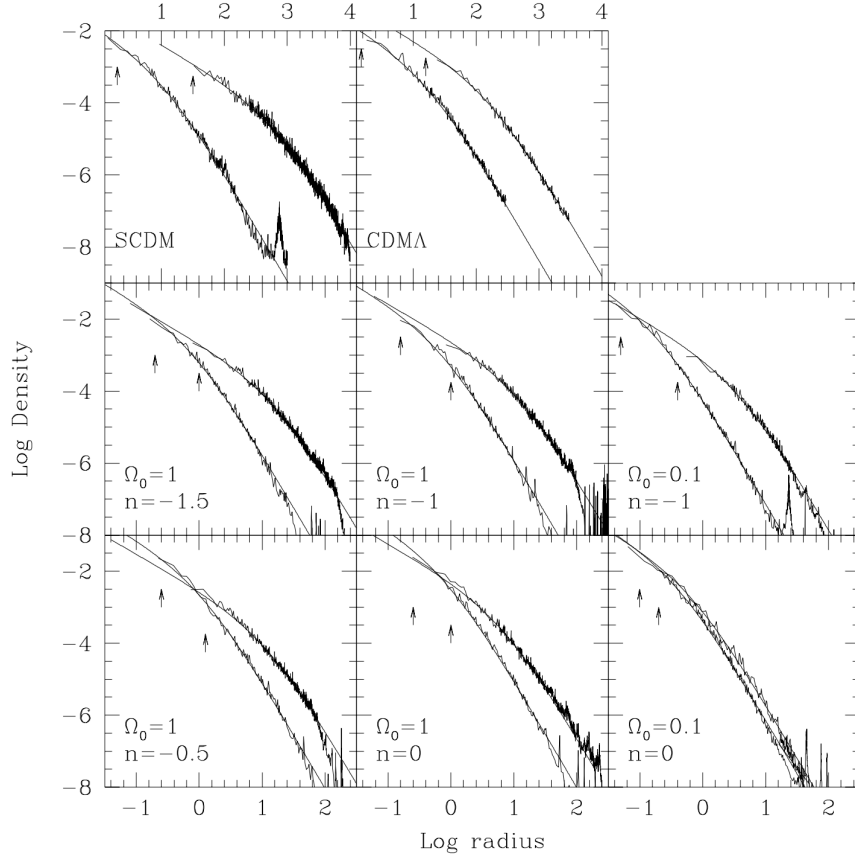


Figure 2.2: Radial density profiles of halos obtained by varying the initial conditions and cosmological parameters. Here, n represents the power-law index of the initial power spectrum, and $\Omega_0 = 1$ corresponds to an Einstein-de Sitter (EdS) universe, while $\Omega_0 = 0.1$ represents an open universe. SCDM and CDMA are models where the initial power spectrum remains unchanged, but the cosmological parameters are set for the EdS universe and Λ CDM, respectively. The two density profiles shown in each panel correspond to the lightest halo (left) and the heaviest halo (right) from each simulation. The fitting curves follow the form of Equation (2.128). Adapted from Navarro et al. (1997).

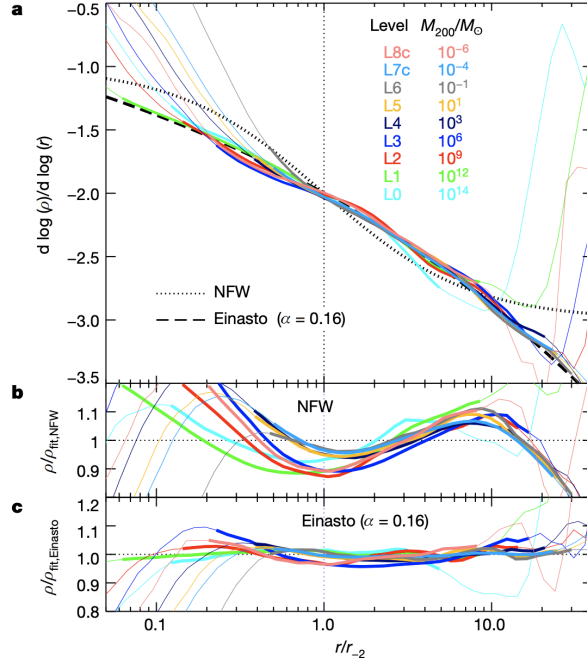


Figure 2.3: Radial density profiles of halos in the mass range $10^{-6}M_\odot < M_{200} < 10^{14}M_\odot$ (top panel). In the middle and the bottom panels, the ratio between the profiles and the best-fit NFW and Einasto profiles is shown. This figure is taken from Wang et al. (2020).

Dark matter density fluctuations grow and undergo gravitational collapse starting from smaller scales, leading to the formation of small halos. These smaller halos subsequently merge to form progressively larger halos. This evolutionary process of halos is referred to as the hierarchical clustering model. The structure of halos at the present time depends on the mass of the halo itself (e.g., Fukushige & Makino (1997); Moore et al. (1999); Jing & Suto (2000); Navarro et al. (2010)), the size and timing of merged halos (e.g., Bullock et al. (2001); Wechsler et al. (2002); Zhao et al. (2003a, 2009); Wang et al. (2011)), as well as cosmological parameters that determine the amount of dark matter and the shape of the initial power spectrum (e.g., Macciò et al. (2013); Ludlow et al. (2014); Klypin et al. (2016); Ludlow et al. (2016); Diemer & Joyce (2019); Brown et al. (2020)).

While it is now understood that these factors indeed influence the structure of halos, historically, the first key finding was that the radial density distribution of dark matter halos at the present time follows a universal Navarro-Frenk-White (NFW) profile, independent of the halo mass, the shape of the power spectrum, or cosmological parameters, as revealed by results from cosmological N -body simulations (Navarro et al., 1997).

Figure 2.2 shows the radial density profiles of halos for each model, with the curves representing the fitting results based on the density distribution:

$$\rho(r) = \frac{\rho_0}{\left(1 + \frac{r}{r_s}\right)^2}, \quad (2.128)$$

where ρ_0 and r_s are constants, and r_s denotes the scale radius. Equation (2.128) is referred to as the NFW profile (Navarro et al., 1996, 1997). As shown in Fig. 2.2, the radial density profiles of dark matter halos are well-fitted by equation (2.128), regardless of the cosmological model or the initial power spectrum.

Due to this universality, as well as the simplicity and practicality of the functional

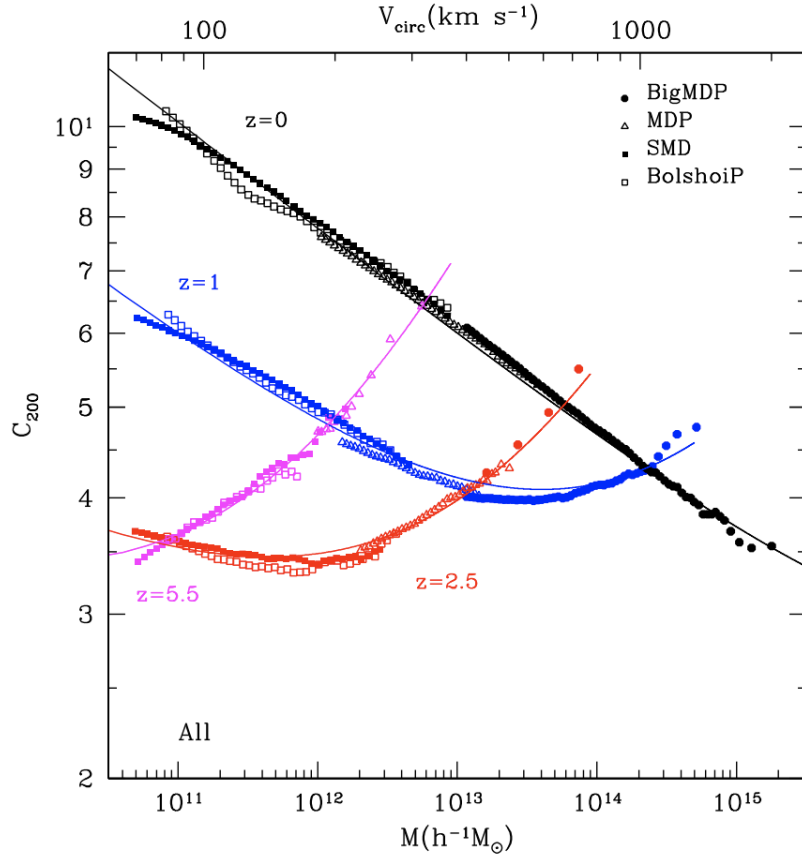


Figure 2.4: The concentration-mass relation at different redshifts. At the present time, lighter halos exhibit higher concentrations. This is because lighter halos collapsed at earlier times, when the mean matter density of the universe was higher, leading to a greater accumulation of mass toward the center (Navarro et al., 1997). The substantial variation in the shape of the graph at different epochs is thought to arise from differences in the dominant growth processes over time, even for halos of the same mass. For example, the frequency of mergers with other halos and whether the merging counterparts are more massive could significantly influence the evolution (Klypin et al., 2016). The figure plots the median concentration c for halos of a given mass. However, there is a significant degree of scatter in c (~ -0.25 dex) for halos of the same mass. Adapted from Klypin et al. (2016).

form, the NFW profile remains widely used for modeling the density distribution of dark matter halos. Since the NFW profile includes only two parameters, assuming the halo density follows this profile allows the radial density distribution of a halo to be fully determined by providing the virial radius R_{vir} , the virial mass M_{vir} , and the scale radius r_s .

Therefore, the relationship between the concentration parameter c_{vir}

$$c_{\text{vir}} \equiv \frac{R_{\text{vir}}}{r_s} \quad (2.129)$$

and M_{vir} has been extensively studied. This relationship, known as the concentration-mass relation, indicates that larger values of c_{vir} correspond to higher central concentrations, with the halo's inner region ($r < r_s$) occupying a smaller fraction of the virial radius. Research on the concentration-mass relation has been actively conducted since the NFW profile was first proposed and continues to this day (e.g., Navarro et al. (1997); Ludlow et al. (2014); Klypin et al. (2016); Diemer & Joyce (2019)). Figure 2.4 provides an example of the concentration-mass relation. Currently, lighter halos tend to exhibit higher concentrations. This trend arises because lighter halos formed earlier, during periods when the universe's mean matter density was significantly higher, facilitating greater mass accumulation in their central regions (Navarro et al., 1997).

For relatively massive halos, it has been noted that the Einasto profile—a profile with three parameters—fits the radial density profile better than the NFW profile (Gao et al., 2008; Navarro et al., 2010; Dutton & Macciò, 2014; Wang et al., 2020). Fig. 2.3 shows the density profiles of halos in the mass range spanning from $10^{-6}M_{\odot}$ to $10^{14}M_{\odot}$, and the profiles can be fitted well by the Einasto profile (bottom panel) rather than the NFW profile (middle panel). On the other hand, the reason why the radial density distribution of dark matter halos can be described by either the NFW profile or the Einasto profile remains unclear. This represents an unresolved issue in the theoretical understanding of the structure formation of dark matter halos.

When halos with significantly different masses merge, the gravitational potential of the lighter halo does not substantially alter the shape of the more massive halo. Instead, the lighter halo is expected to undergo tidal disruption due to the gravitational force of the more massive halo. In this context, the more massive halo is referred to as the "host halo", while the lighter halo undergoing tidal disruption is called the "subhalo". However, when the subhalo has a mass comparable to that of the host halo, the merger is termed a major merger. In such cases, the merger can significantly impact the structure of the host halo (Wang et al., 2011).

2.4.4 Pseudo-phase-space density

Another notable universality of CDM halos is observed in their pseudo-phase-space density profiles, defined as $Q(r) \equiv \rho(r)/\sigma^3(r)$, where σ^2 represents twice the specific kinetic energy at radius r . The left panel of Fig. 2.5 illustrates the pseudo-phase-space density profiles for 21 N -body halos. As evident in the top panel, $Q(r)$ approximately follows a power law $r^{-1.875}$. While the exact slope index does not have to be -1.875 , the residuals from the best-fit $r^{-1.875}$ (middle panel) are comparable to those obtained when the slope is treated as a free parameter (bottom panel), strongly supporting the universal $r^{-1.875}$ behavior of these profiles.

The pseudo-phase-space density profiles were first analyzed by Taylor & Navarro (2001) as a proxy for the full phase-space density $f(\mathbf{x}, \mathbf{v}, t)$, where their universal nature

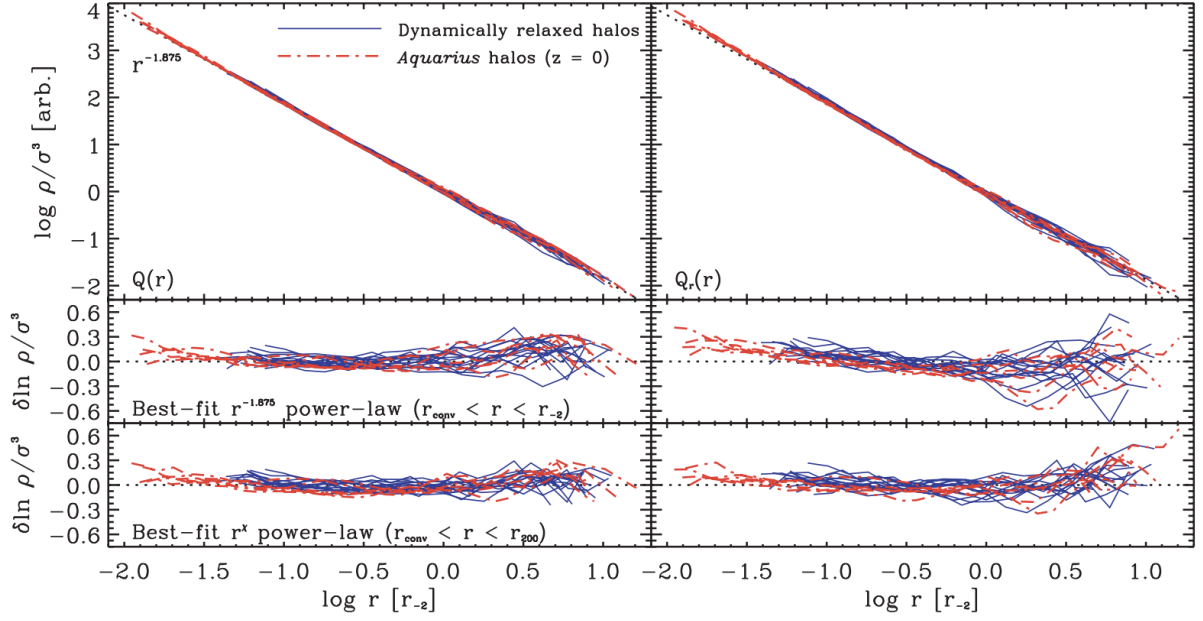


Figure 2.5: Spherically averaged pseudo-phase-space density profiles $Q(r)$ of 21 N -body halos. The left panel shows $Q(r) \equiv \rho/\sigma^3$, and the right panel shows $Q_r(r) \equiv \rho/\sigma_r^3$, where σ^2 is twice the specific kinetic energy at each radial bin and σ_r^2 is that of radial component. The middle and bottom panels show the residuals from the best $r^{-1.875}$ fit in different ranges of radii indicated in the panels. In general, all the pseudo-phase-space density profiles $Q(r)$ shown exhibit power-law behavior $Q(r) \propto r^{-1.875}$. This figure is taken from Ludlow et al. (2011).

was identified. They further explored the connection between the slope of $Q(r)$ and the density profile's slope, assuming hydrostatic equilibrium. From the diagonal component of equation (2.65), the Jeans equation can be written as

$$\frac{d(\rho\sigma^2)}{dr} = -\rho\frac{d\Phi}{dr} = -\rho\frac{GM(< r)}{r^2}. \quad (2.130)$$

Assuming the pseudo-phase-space density follows a power-law behavior,

$$\frac{\rho(r)}{\sigma^3(r)} = \frac{\rho_0}{\sigma_0} \left(\frac{r}{r_0} \right)^{-\alpha}, \quad (2.131)$$

where ρ_0 , σ_0 , and r_0 are arbitrary reference values for density, velocity, and radius, respectively, and α is the slope, we derive

$$\frac{d}{dx} \left\{ \frac{-x^2}{y} \frac{d}{dx} (y^{5/3} x^{2\alpha/3}) \right\} = \kappa y x^2, \quad (2.132)$$

where $y \equiv \rho(r)/\rho_0$, $x \equiv r/r_0$, and $\kappa \equiv 4\pi G\rho_0 r_0^2/\sigma_0^2$.

If we set $\alpha = 1.875$, the density profile slope becomes $\rho \propto r^{-9/4}$, identical to the slope derived from the secondary infall model of spherical collapse by Bertschinger (1985). This remarkable agreement suggests that N -body halos exhibit similarities with self-similar solutions. However, the physical interpretation of this correspondence remains an open question. The origin of the universal slope in pseudo-phase-space density profiles is occasionally dismissed as coincidental (Arora & Williams (2020); see also Lapi & Cavaliere (2011); Nadler et al. (2017); Colombi (2021)). Nevertheless, the power-law behavior has been observationally validated in both galaxies and galaxy clusters (e.g., Biviano et al. (2013); Chae (2014); Biviano & Mamon (2023)).

2.4.5 Prompt cusp

Thus far, our focus has primarily been on massive halos that emerge through the hierarchical clustering of smaller halos. However, as discussed in Section 2.2.3, the initial linear power spectrum includes a free-streaming damping scale determined by the properties of dark matter, such as its mass. This implies the existence of a characteristic free-streaming scale below which structures cannot form, leaving the density field smooth. Consequently, the first halos form around this scale, collapsing directly from the smooth density field. The free-streaming scale α can be approximated as

$$\alpha = 0.049 \left(\frac{m_{\text{DM}}}{1\text{keV}} \right)^{-1.11} \left(\frac{\Omega_{\text{DM}}}{0.25} \right)^{0.11} \left(\frac{h}{0.7} \right)^{1.22} h^{-1}\text{Mpc}, \quad (2.133)$$

as given by Viel et al. (2005). For the typical CDM case with $m_{\text{DM}} = 100$ GeV, this yields $\alpha \sim 10$ AU.

Ishiyama et al. (2010) conducted the first detailed investigation into the structure of these primordial halos. Snapshots from their simulations are displayed in Fig. 2.6. The upper panel illustrates the simulation with a free-streaming cutoff corresponding to $m_{\text{DM}} = 100$ GeV. In contrast to the lower panel, which shows simulations without a free-streaming cutoff, the upper panel lacks prominent structures apart from a few halos, representing the first-formed halos.

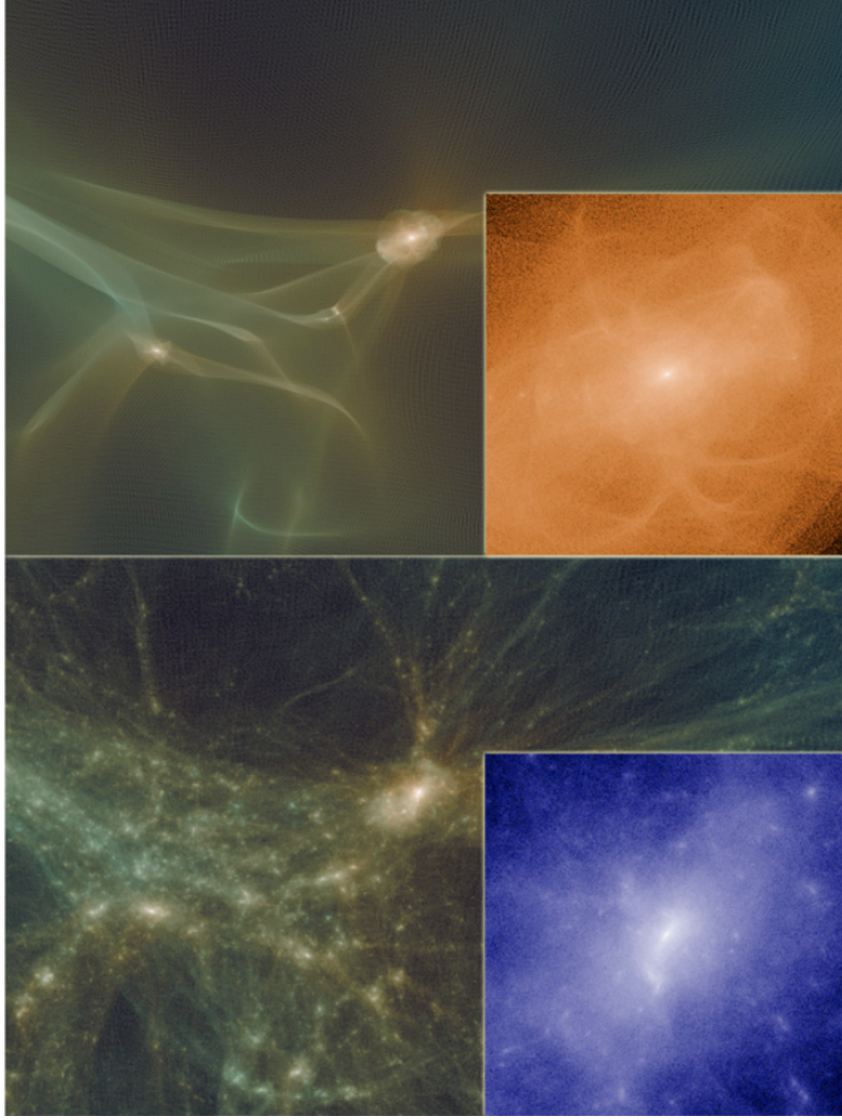


Figure 2.6: The snapshots of N -body simulations focusing on the emergence of the first halos at $z = 31$. The width of the panels is 12 comoving parsecs, while that of the enlargement of single halos in the square panel is 0.6 comoving parsecs (= 4900 AU in physical units). The simulation with free-streaming cut-off corresponding to 100 GeV WIMP (top), and that without cut-off (bottom) are shown. It is clearly shown that there is no smaller structure than the halos shown in the top panel while numerous halos exist on any small scale because there is no cut-off in the initial power spectrum in the bottom panel. This figure is taken from Ishiyama et al. (2010).

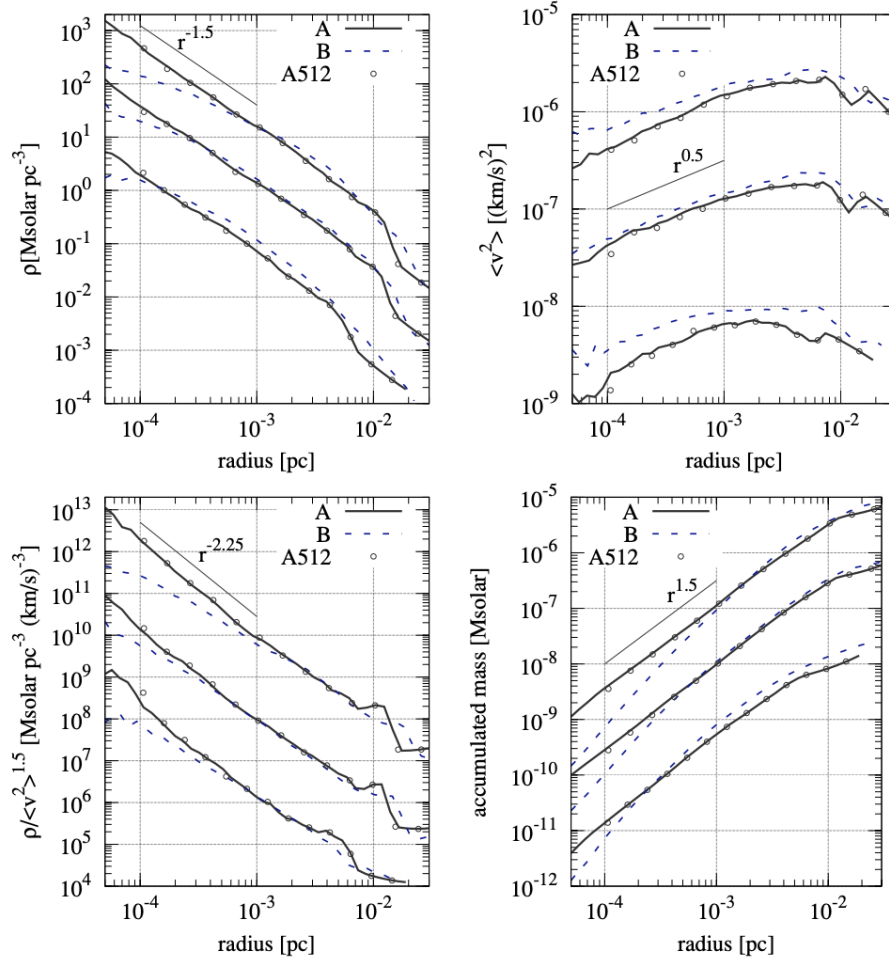


Figure 2.7: Radial profiles for three distinct microhalos from simulation A (upper panel of Fig. 2.6) and simulation B (lower panel of Fig. 2.6) are presented. The four panels illustrate the density, velocity dispersion, pseudo-phase-space density, and accumulated mass. To enhance clarity, two of the three profiles are shifted downward by 1 and 2 dex, respectively.

The internal structure of these first-formed halos differs substantially from those formed via hierarchical clustering. As shown in the top left panel of Fig. 2.7, the density profiles of the first-formed halos (solid lines) and halos formed through hierarchical clustering (dashed lines) are compared. Unlike the typical NFW-like profile seen in hierarchically formed halos, first-formed halos exhibit a steeper $r^{-1.5}$ power-law density profile. This sharp central cusp, referred to as the "prompt cusp," forms immediately following the initial collapse of density fluctuations (Colombi, 2021; Delos & White, 2023). Notably, these prompt cusps remain intact even through subsequent mergers.

Despite its clear presence, the physical mechanism behind the formation of prompt cusps remains unclear, making it one of the enduring mysteries of CDM halos alongside the universalities of NFW profiles and pseudo-phase-space density. Furthermore, prompt cusps display a distinct pseudo-phase-space density slope of $Q(r) \propto r^{-2.25}$, differing from that of massive halos. This unique characteristic may provide a key insight into the physical origin of the prompt cusp.

2.4.6 Multi-stream flows in CDM halos (Sugiura et al., 2020)

In this section, we review prior studies on the multi-stream regions of CDM halos. As discussed in Section 2.3.4, the evolution of phase-space density for collisionless dark matter follows the Vlasov-Poisson equation. This equation reflects the conservation of phase-space density along the trajectory of each phase-space element, as dictated by the Liouville theorem. Consequently, a phase-space element evolves without intersecting other elements or tearing apart from neighboring elements. In the case of the genuine cold dark matter model, where the initial thermal velocity is zero, the phase-space density initially forms a 3D sheet embedded within the 6D phase-space

$$f(\mathbf{x}, \mathbf{v}, t = t_i) = \rho_i(\mathbf{x}) \delta_D(\mathbf{v} - \mathbf{v}_i(\mathbf{x})). \quad (2.134)$$

Under this condition, the evolution of the 3D sheet is characterized by continuous foldings, as illustrated schematically over time.

Figure 2.8 provides examples of this folding process. The panels display the evolution of an initial cold 3D phase-space sheet projected onto the $x - v_x$ plane from top to bottom across three columns, each representing distinct initial sheet configurations. Details of these simulations are described in Section 5.2. It is evident that the initial phase-space sheet undergoes folding, creating multi-stream regions (as introduced in Section 2.3.4), consistent with the schematic depiction in the bottom row of Fig. 2.1. The defining features of CDM halos discussed earlier—such as the universal NFW density profiles, the power-law behavior of pseudo-phase-space density profiles, and the prompt cusps—are all products of this multi-folded (multi-stream) 3D sheet within 6D phase-space. In essence, the multi-stream regions hold the key to understanding these universal features, presenting an avenue for quantitative investigations aimed at uncovering their origins.

Building on this, Sugiura et al. (2020) examined the phase-space structure within and outside the multi-stream regions of CDM halos, comparing them to the self-similar solutions of the spherical collapse model (refer to Section 2.3.4). As demonstrated in Fig. 3.25 in Section 3.6, the location of each stream in the self-similar solutions is governed by the number of apocenter passages. Focusing on this aspect, they compared the phase-space positions of streams—characterized by the number of apocenter passages—in N -body simulations with those predicted by the self-similar solutions. An example of this comparison is shown in Fig. 2.9. Each column corresponds to a different number of

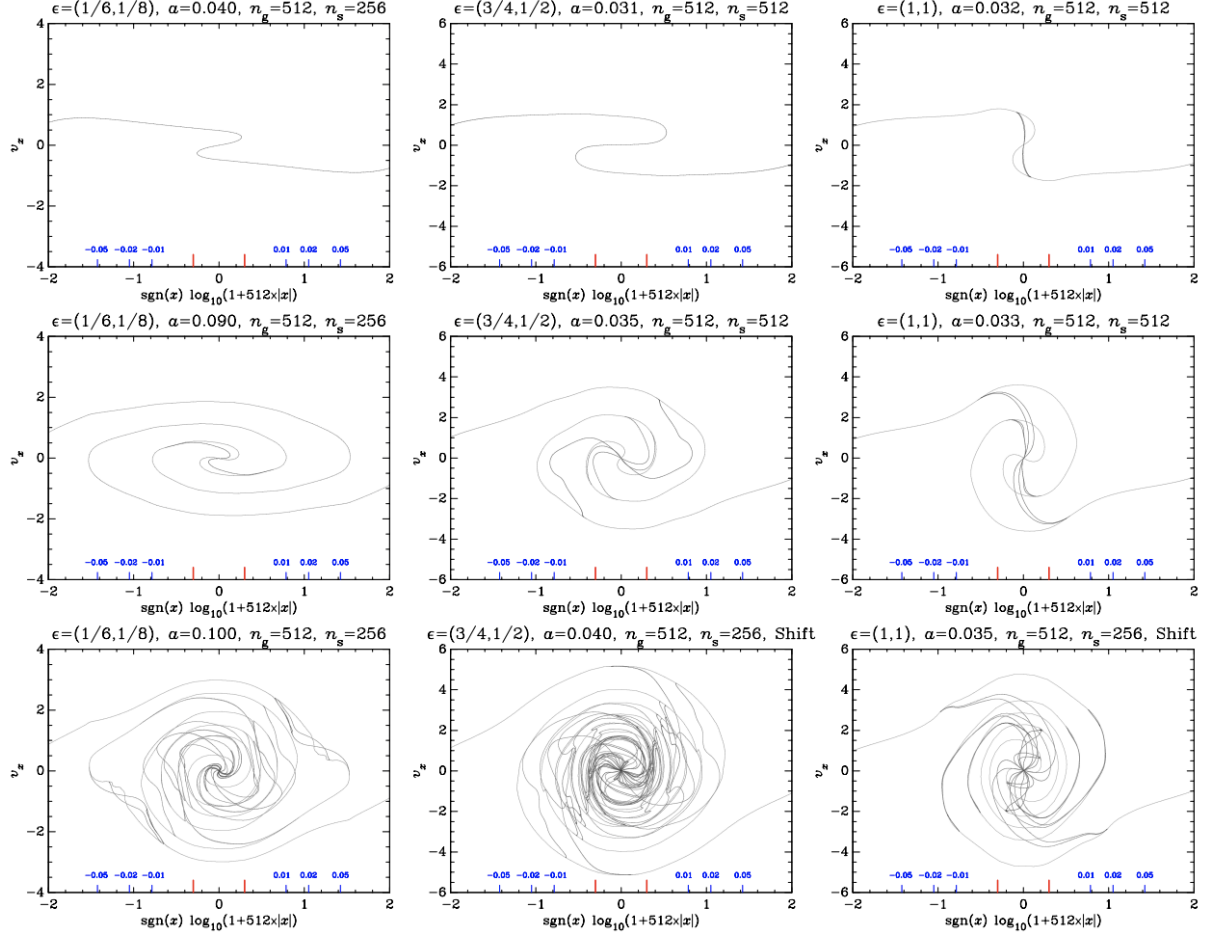


Figure 2.8: Phase-space slice ($x - v_x$ plane) of three-sine-wave simulations with various amplitudes. Each column represents a different scenario: quasi-1D (left), anisotropic (middle), and axisymmetric (right). Time progresses from the top row to the bottom. In accordance with the Liouville theorem, which dictates the conservation of the phase-space element, the evolution of the phase-space sheet occurs through intricate foldings, without tearing or intersection. However, intersections are apparent in these plots because they display the projected distributions. This figure is taken from Colombi (2021).

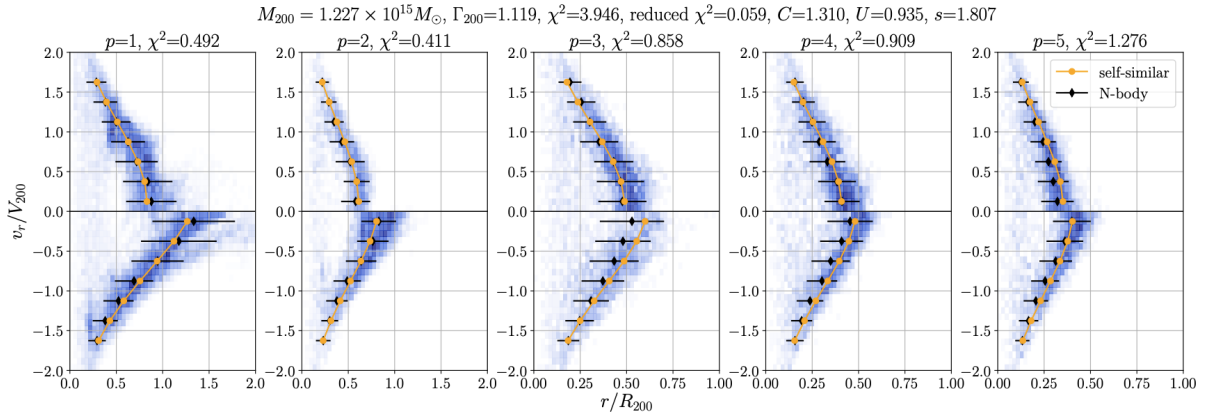


Figure 2.9: Radial phase-space distribution of N -body particles within the halo, which aligns closely with the self-similar solution. Yellow lines represent the best-fit streams derived from the self-similar solution. This figure is adapted from Sugiura et al. (2020).

apocenter passages. The distributions of N -body particles are shown in blue, with the mean and scatter represented by black markers and error bars, respectively. The best-fit phase-space distributions of streams from the self-similar solution are depicted as yellow markers, which align closely with the N -body particle distributions.

According to Sugiura et al. (2020), approximately 30% of simulated halos meet their criteria and are classified as being well-matched to the self-similar solution. This result is particularly intriguing given the contrasting formation processes: N -body halos form through numerous clumpy mergers, while self-similar solutions arise from the smooth accretion of perfectly spherical shells. Moreover, Sugiura et al. (2020) explored the outermost regions of halos, which are significantly influenced by recent mergers—scenarios that are completely beyond the assumptions of the self-similar solutions.

This discrepancy serves as the primary motivation for our subsequent work, which will be presented in Chapter 3. We focus on the inner multi-stream regions to investigate their phase-space structures and compare them to self-similar solutions that incorporate angular momentum.

2.5 Non-linear evolution of density fluctuation III: Alternative dark matter models

The biggest motivation to consider alternative dark matter models except for CDM is the so-called small-scale challenges in the Λ CDM paradigm. As described below, these issues have been raised particularly through comparisons between satellite galaxies observed in the Milky Way and subhalos in N -body simulations (Bullock & Boylan-Kolchin (2017); Sales et al. (2022), see also Pawlowski (2021) and Sales & Navarro (2023) as studies on a different problem known as satellite plane problem).

The first issue, known as the missing satellites problem, highlights the significant discrepancy between the number of subhalos in N -body simulations and the number of dwarf galaxies observed in actual galaxies. According to cosmological N -body simulations based on the Λ CDM model, a galaxy halo with a size comparable to the Milky Way typically contains more than 1,000 subhalos with masses greater than $10^7 M_\odot$. In contrast, the number of satellite galaxies identified in the Milky Way is only about $40 \sim 50$, even for those with masses greater than $300 M_\odot$.

The second issue is referred to as the core-cusp problem. In N -body simulations, halo density profiles exhibit a cusp-like distribution, represented by the NFW profile, where the density scales as $\rho \propto r^{-1}$ in the central region. However, when the density profiles of observed satellite galaxies are inferred from their rotation curves, they are found to be shallower than cusp-like profiles, resembling a core-like profile with $\rho \sim \text{const}$.

The third issue is the Too Big to Fail (TBTf) problem. This refers to the discrepancy where the central densities of the brightest satellite galaxies currently observed in the Milky Way are lower than the central densities of the most massive subhalos within Milky Way-class host halos in N -body simulations. These massive subhalos are expected to have sufficiently high central densities to initiate star formation, resulting in a mismatch with observational data.

These discrepancies may arise because pure CDM N -body simulations do not properly account for the processes of star and galaxy formation and their subsequent evolution. Incorporating these effects would reconcile the simulations with observations (e.g., Pontzen & Governato (2012); Santos-Santos et al. (2020), see also Sales et al. (2022) as recent

review). Alternatively, attempts have been made to modify the nature of dark matter itself to explain the observational discrepancies. Two notable candidates in this regard are WDM and Self-Interacting Dark Matter (SIDM).

WDM differs from CDM by having nonzero velocity dispersion, which suppresses density fluctuations below a certain scale due to free-streaming (see Section 2.2.3). Consequently, the number of low-mass halos below the suppressed scale decreases, resolving the missing satellites problem Polisensky & Ricotti (2011, 2014). Additionally, the suppression of small-scale fluctuations delays halo formation relative to CDM, reducing the central densities of halos and thereby addressing the TBTF problem. It is also known that the density profiles of WDM halos can be well-described by the NFW profile, similar to the CDM case (e.g., Lovell et al. (2014); Ludlow et al. (2016), but see also Polisensky & Ricotti (2015) claiming that density profiles of WDM halos are not universal.).

Figure 2.10 presents the estimated number of subhalos in the Milky Way based on hydrodynamical simulations and various assumptions (see Kim et al. (2018) for details). From the approximately 50 observed satellite galaxies, the total number of subhalos in the Milky Way is estimated to be between 120 and 240, depending on the adopted subhalo distribution model. If the infall mass of Segue I is $10^8 M_\odot$, the CDM model predicts approximately 200 to 300 subhalos (corresponding to the x -axis value of the blue solid lines at $y = 10^8 M_\odot$ in Fig. 2.10). In contrast, the 2.0 keV WDM model predicts about 4.5×10^1 subhalos, which is fewer than the estimated total of 120 to 240 subhalos. These findings place the current constraint on WDM particle mass at $\gtrsim 5.0$ keV (e.g., Dekker et al. (2022); Villaseñor et al. (2023); Iršič et al. (2024); Keeley et al. (2024)), significantly heavier than earlier estimates but still several orders of magnitude lighter than the typical CDM particle mass (~ 100 GeV). Further constraints are needed to narrow down the parameter space for WDM models.

SIDM models introduce self-interactions between dark matter particles. In SIDM halos, these interactions thermalize the inner regions, resulting in core-like density profiles with $\rho \propto \text{const}$ (e.g., Spergel & Steinhardt (2000); Rocha et al. (2013); Banerjee et al. (2020), see also Balberg et al. (2002); Kamada et al. (2017) as studies on the gravothermal catastrophe of SIDM halos, which is totally different of the standard picture of core-forming SIDM halos.). This solves the core-cusp problem. Additionally, the core-like central density distribution in SIDM halos is smaller than the cusp-like distributions seen in NFW profiles, thereby offering a potential resolution to the TBTF problem.

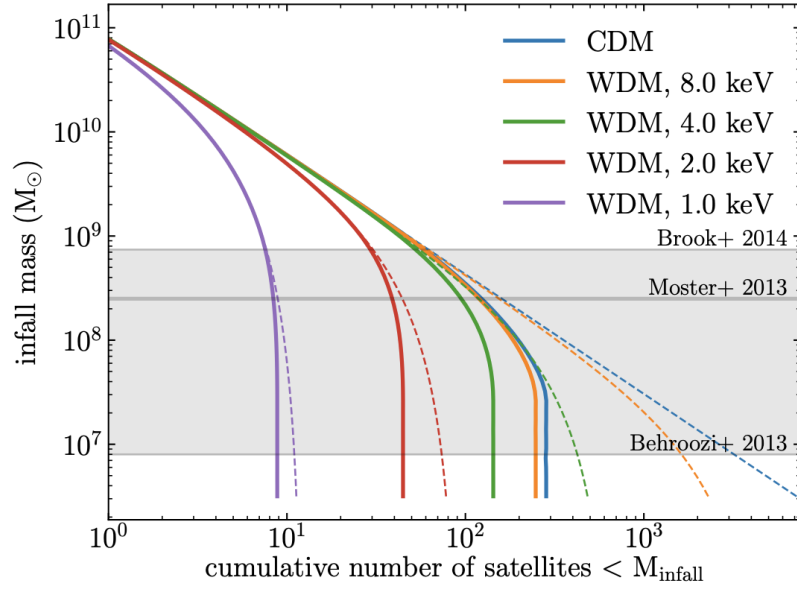


Figure 2.10: The figure illustrates the number of luminous (solid lines) and total (dashed lines) subhalos in the Milky Way as predicted by CDM and WDM models for various lower limits on subhalo mass. For instance, in the CDM model, the number of subhalos with infall masses exceeding $10^7 M_\odot$ is approximately 3.0×10^2 , whereas in the 2.0 keV WDM model, it reduces to about 4.5×10 . The shaded region highlights the estimated infall mass of Segue I, currently recognized as the smallest subhalo in the Milky Way. This figure is reproduced from Kim et al. (2018).

Chapter 3

Multistream structure of CDM halos

3.1 Introduction

As outlined in Section 2.4.6, Sugiura et al. (2020) (hereafter S20) proposed a method to explore the multi-stream nature of halos in radial phase space. Building upon the SPARTA algorithm developed by Diemer (2017), they leveraged cosmological N -body simulations to elucidate the outer regions of the multi-stream flows of CDM halos. Through comparison with the self-similar solution by Fillmore & Goldreich (1984), S20 quantified the radial phase-space structures and discovered that approximately 30% of the simulated halos could be effectively modeled by the self-similar solutions across a broad spectrum of mass accretion rates. As will be elaborated further, their approach necessitates numerous simulation snapshots across various redshifts to reconstruct the trajectories of DM particles currently associated with halos. By analyzing these trajectories, they counted the number of apocenter passages, denoted as p , for each DM particle orbiting the halo center. Particles sharing the same p value delineate specific streams within the multi-stream DM flow in radial phase space. Using this framework, S20 examined the multi-stream characteristics of DM particles up to $p = 5$.

It is important to note that the radial streams with $p \leq 5$ remain distant from the halo center. Consequently, their structures are likely influenced by the outer environment, which frequently displays irregular and extended features, particularly in the presence of merging halos or subhalos. This dependency on the outer environment could partially account for why only approximately 30% of halos align with the self-similar solutions. In contrast, uncovering the inner multi-stream structures could reveal universal features within each stream, potentially offering insights into the origin of the cuspy radial density profiles. From this perspective, characterizing the inner multi-stream structure may serve as a more fundamental property, providing a deeper understanding of the physical characteristics of CDM halos.

Building on this context, the primary objective of this chapter is to characterize the inner multi-stream structure using the methodology introduced by S20, incorporating significant enhancements in both the simulation dataset and numerical analysis techniques. This study serves as a follow-up to Enomoto et al. (2023) (hereafter E23), which identified a key result: the radial density profile of each stream, defined by the number of apocenter passages p , universally follows a simple double-power-law function regardless of p . In addition to reaffirming this discovery, this chapter delves deeper into the robustness

of these findings and provides a thorough examination of the radial phase-space distributions, comparing them with self-similar solutions. Furthermore, detailed descriptions of the method used to count the number of apocenter passages for N -body particles are presented.

This chapter is structured as follows. Section 3.2 introduces the simulations and halo catalog utilized in this study. In Section 3.3.1 and 3.3.2, we outline the methods for counting p for simulation particles, followed by the stacking and fitting procedures detailed in Section 3.3.3. The results for individual halos are presented in Section 3.4.1, while Section 3.4.2 focuses on stacked profiles and introduces the double-power-law density profile reported in E23. Section 3.4.2 provides the optimized χ^2 fits for the stacked profiles and examines the best-fit values for the inner and outer slopes of the double-power-law model. Next, Section 3.4.2 investigates the dependence of these two free parameters on p and halo mass. Section 3.5 explores the influence of halo concentration and recent mass accretion rate, while Section 3.6 discusses the physical implications of the inner slope values. Additionally, Section 3.7 examines the evolution of $\rho(r; p)$ to uncover the origins of the universal feature. Finally, Section 3.8 concludes with a summary of findings and prospects for future research.

3.2 Data and halo catalog

3.2.1 N -body simulations

Our study utilizes two cosmological N -body simulations (LR and HR) conducted in a periodic comoving box with a side length of $41, h^{-1}\text{Mpc}$. These simulations, each containing a different number of particles as specified in Table 3.1, assume a flat Λ CDM universe in line with recent Planck satellite findings: $\Omega_m = 0.3089, \Omega_\Lambda = 0.6911, h = 0.6774, n_s = 0.9667, \sigma_8 = 0.8259$ (Planck Collaboration et al., 2016a).

The initial conditions were established by displacing particles on a regular lattice using second-order Lagrangian perturbation theory (Scoccimarro, 1998; Crocce et al., 2006), seeded by a Gaussian random field derived from the linear matter power spectrum calculated with the CLASS Boltzmann solver (Blas et al., 2011). LR and HR simulations utilize identical random initializations, enabling direct comparisons between the two. Particle positions and velocities are evolved with the TreePM code GINKAKU (Nishimichi, Tanaka & Yoshikawa, in preparation), which is built upon the public Framework for Developing Particle Simulators (FDPS; Iwasawa et al., 2016; Namekata et al., 2018). This library, designed for high-efficiency particle simulations on modern supercomputers, employs a hybrid MPI-openmp parallel architecture to facilitate effective domain decomposition and processor communication. The short-range tree force benefits from SIMD instructions via the PHANTOM-GRAPE library (Nitadori et al., 2006; Tanikawa et al., 2012, 2013), while details on the long-range PM force are available in Yoshikawa & Fukushige (2005) (see also Ishiyama et al. 2009, 2012).

We analyzed particle trajectories using 1001 LR snapshots uniformly sampled in redshift from $z = 5$ to $z = 0$. With LR providing nearly five times the number of snapshots used in S20, we anticipate being able to examine particle orbits with enhanced detail, particularly those that pass their apocenters multiple times. To evaluate the convergence of the density profile given the limited softening length, we turn to the $z = 0$ snapshot of HR. However, due to disk storage constraints, we have not preserved higher-redshift snapshots for HR. Consequently, we study phase-space structure evolution using LR data

Table 3.1: The parameters of N -body simulations. N^3 denotes the number of simulation particles, m_p the mass of simulation particles, ϵ the softening length, N_{snaps} the number of snapshots.

Name	N	m_p ($h^{-1}M_{\odot}$)	ϵ ($h^{-1}\text{kpc}$)	N_{snaps}
LR	500^3	4.72716×10^7	4.10	1001
HR	2000^3	7.38619×10^5	1.025	1 ($z = 0$)

and validate the density profile accuracy at $z = 0$ through HR data.

3.2.2 Halo catalogue

We identified halo at $z = 0$ using a 6D phase-space temporal friends-of-friends halo finder, ROCKSTAR (Behroozi et al., 2013). As further detailed in Section 3.3.1, we traced both the centers and surrounding particles of halo identified at $z = 0$ backward in time to construct the most massive main progenitor branch of the halo.

While ROCKSTAR is capable of identifying halo involved in major mergers, the positions, velocities, and particle associations for these halo can often be highly uncertain. Additionally, as discussed in Section 3.3.1, robustly tracking the main progenitor branch within merger trees for these halo is typically challenging, particularly when they are near other massive halo. To address these issues, we focus on relaxed halo less impacted by recent mergers, selected based on two parameters: the spin parameter λ and the offset parameter X_{off} , both determined by the ROCKSTAR halo finder.

The parameter X_{off} denotes the distance between the location of the density peak and the center of mass of particles, normalized by the virial radius R_{vir} . Meanwhile, λ represents the amplitude of angular momentum divided by $\sqrt{2}R_{\text{vir}}V_{\text{vir}}M_{\text{vir}}$, commonly referred to as the Bullock spin parameter in Behroozi et al. (2013). Following Klypin et al. (2016), we classify halo that meet either of the following conditions as undergoing major mergers and exclude them from our halo catalogue:

$$\lambda > 0.07, \quad X_{\text{off}} > 0.07. \quad (3.1)$$

Fig. 3.1 illustrates the distribution of X_{off} and λ , along with the results after applying the criteria from equation (3.1). The distribution closely resembles that depicted in Fig. 4 of Klypin et al. (2016).

In addition to halo undergoing major mergers, we also exclude subhalo. Subhalo are generally surrounded by particles associated with a distinct nearby host halo, which complicates the determination of whether the DM particles are bound to the subhalo and challenges precise density profile calculations. Therefore, we exclude subhalo from our halo catalogue through the following process.

First, we compute the mass, $M_{\text{vir,all}}$, representing the total mass of particles within the virial radius R_{vir} as determined by the ROCKSTAR halo finder ($R_{\text{vir,ROCK}}$). The ROCKSTAR halo finder calculates $R_{\text{vir,ROCK}}$ and the virial mass $M_{\text{vir,ROCK}}$ after excluding unbound particles, ensuring that $M_{\text{vir,all}}$ consistently exceeds $M_{\text{vir,ROCK}}$. For host halo, most particles within $R_{\text{vir,ROCK}}$ are gravitationally bound to the host, making $M_{\text{vir,ROCK}} \approx M_{\text{vir,all}}$. However, for subhalo, many particles are bound to the host halo rather than to the subhalo itself, resulting in $M_{\text{vir,all}} \gg M_{\text{vir,ROCK}}$.

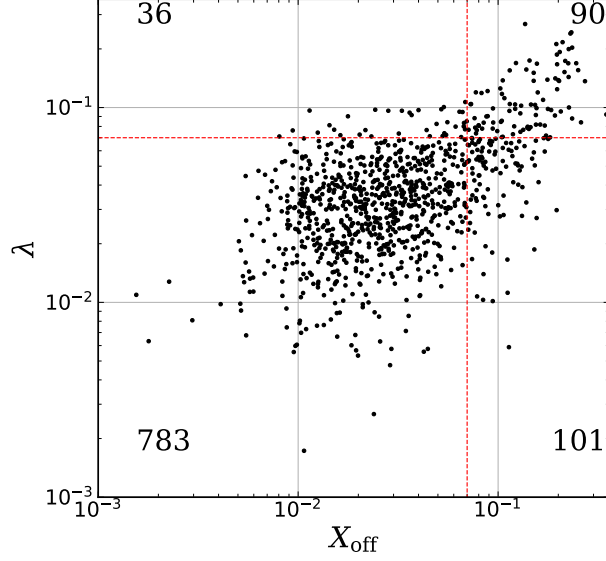


Figure 3.1: The distribution of the offset parameter X_{off} and the spin parameter λ for 1010 halos is depicted (black dots). Red dotted lines indicate the thresholds defined by equation 3.1. The numbers located near the four corners denote the counts of halos within each region outlined by the red dashed lines. This figure is adapted from Enomoto et al. (2024a).

While there is no strict theoretical threshold that separates host halo from subhalo, we identify halo that meet the condition:

$$M_{\text{vir,all}} > 1.3M_{\text{vir,ROCK}} \quad (3.2)$$

as subhalo, and we exclude these from our halo catalogue. Fig. 3.2 shows the distribution of $M_{\text{vir,all}}/M_{\text{vir,ROCK}}$ and the effect of applying equation (3.2). We retain the halo represented by black dots below the horizontal dashed line after applying both conditions, resulting in 685 halo remained. The two conditions appear largely independent, as the proportion of halo removed by the second condition does not vary based on the first condition.

In addition to the criteria discussed above, we apply two additional criteria to exclude two more halo identified as undergoing major mergers, as described in Section 3.3.1. Table 4.4.2 summarizes the halo catalog resulting from these selections. We also present the mass ranges from S to XL that will be used for stacking analysis after Section 3.4.

In summary, even after removing subhalo and halo undergoing major mergers, nearly 70% of all halo remain in our catalog. Equations (3.1) and (3.2) are the primary criteria used to construct this halo catalog.

3.3 Methods: tracking halos and particles

To analyze each stream within the multi-stream region, we focus on the number of apocenter passages, denoted as p , for particles and classify them according to their respective p values. In principle, the apocenter of a particle orbiting the halo center is defined as the point where its radial velocity changes from positive to negative along its path.

Once we have determined the position and bulk motion of the halo, represented by $\mathbf{x}_h$ and $\mathbf{v}_h$, at each redshift, as well as the starting time for counting p , denoted as t_s ,

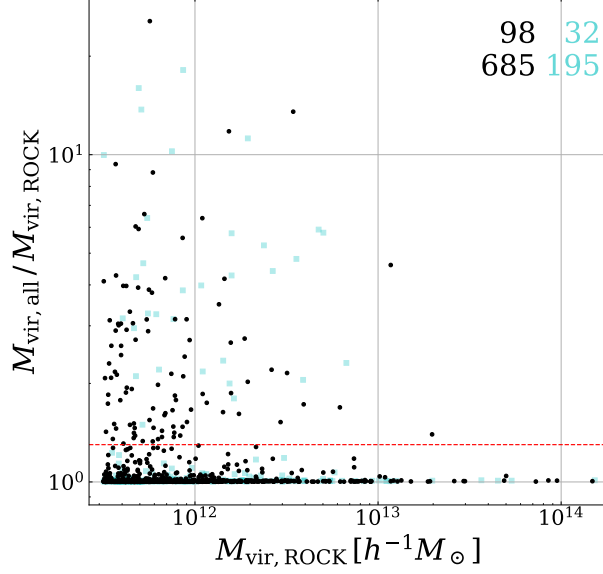


Figure 3.2: The distribution of $M_{\text{vir,all}}/M_{\text{vir,ROCK}}$ for the 1010 halo from the initial ROCKSTAR catalog. The 783 halos represented by black dots were kept after excluding those that underwent major mergers (equation 3.1), with cyan boxes marking the halos that were removed. The horizontal red dashed line represents the subhalo criterion (equation 3.2), and only halos below this line are investigated. The legend in the figure presents four values corresponding to the number of halos: for the criterion in equation (3.1) (left: retained, right: removed), and for equation (3.2) (upper: removed, lower: retained). This figure is adapted from Enomoto et al. (2024a).

Table 3.2: The number of halo meeting selection criteria in each mass range. The upper three rows describe the characteristics of our halo samples divided into four mass bins. The first and second rows respectively show the ranges of halo mass and radius, which are calculated by the ROCKSTAR halo finder. On the other hand, the third row shows the splashback radius of the stacked density profiles, defined by the radius at which the density slope takes a minimum value. The estimated values shown here are those normalized by the mean virial radius for each mass range. The rest of the lower rows summarize the number of total halo in each mass range, denoted by N_{all} , as well as the number of samples after setting the criteria shown in the left column. The number of halo shown at each row is the result when further adding the criterion to the upper rows.

Mass range	S	M	L	XL	Total
$M_{\text{vir}}[10^{11}h^{-1}M_{\odot}]$	[3.16, 5.71]	[5.71, 24.2]	[24.2, 134]	[134, 1530]	[3.16, 1530]
$R_{\text{vir}}[h^{-1}\text{Mpc}]$	[0.14, 0.17]	[0.17, 0.27]	[0.27, 0.48]	[0.48, 1.08]	[0.14, 1.08]
$R_{\text{sp}}[R_{\text{vir}}]$	0.64	0.70	0.65	0.96	
N_{all}	445	433	113	19	1010
+Eq. (3.1)	350	342	77	14	783
+Eq. (3.2)	301	301	70	13	685
+Eq. (3.4)	300	301	70	13	684
+Eq. (3.6)	300	300	70	13	683

the process of counting p for particles within the halo is straightforward: (1) compute the radial velocity v_r of each particle relative to the halo center at each snapshot after t_s , and (2) count the instances where the radial velocity transitions from positive to negative up to $z = 0$.

In the following sections, we describe the method for calculating $\mathbf{x}_h$, $\mathbf{v}_h$, and t_s of halo in Section 3.3.1. The procedure for steps (1) and (2) is presented in Section 3.3.2.

3.3.1 Halo center and merger trees

For a particle with position $\mathbf{r}$ and peculiar velocity $\mathbf{v}$, the radial velocity v_r is defined as:

$$v_r = \frac{(\mathbf{x} - \mathbf{x}_h) \cdot (\mathbf{v} - \mathbf{v}_h)}{|\mathbf{x} - \mathbf{x}_h|}. \quad (3.3)$$

We track the sign of this quantity across different snapshots to determine p . Therefore, to calculate p , we need to determine $\mathbf{x}_h$ and $\mathbf{v}_h$ for each halo in our catalog at each snapshot. While these values can be obtained by applying the halo finder to snapshots at higher redshifts, they tend to vary discontinuously over time due to the use of different simulation particles in each snapshot, resulting in abrupt changes in v_r between snapshots and making accurate counting of p challenging.

To address this, we calculate $\mathbf{x}_h$ and $\mathbf{v}_h$ as the average position and velocity of a fixed set of particles over time, containing 1000 particles considered to be the oldest progenitors. These progenitors are identified by tracking the most massive branch of the merger tree for each halo. By averaging the positions and velocities of these 1000 particles, we obtain smoothly varying values for $\mathbf{x}_h$ and $\mathbf{v}_h$ across snapshots, reducing the impact of individual particle noise.

The particles within the oldest progenitor are well bound within the halo at $z = 0$ and move coherently with it, making them appropriate for representing the position and bulk motion of the halo center. The specific number 1000 is chosen to minimize numerical noise and can be adjusted as long as it provides a sufficient particle count. In Section 3.3.4, we vary this number and confirm that the results for $p \leq 40$ remain quantitatively consistent. Consequently, for the following analysis, we focus only on particles with $p \leq 40$.

To construct merger trees, we begin by identifying the particles that compose each halo at $z = 0$. This is done in several steps. First, we determine the halo center using the shrinking sphere method, then expand this sphere around the fixed center until the overdensity within it drops to the virial overdensity Δ_{vir} , as defined in Bryan & Norman (1998). The particles within this final sphere are classified as members of the halo.

In the shrinking-sphere method, we start with the halo center identified by the ROCKSTAR finder as the initial center of the sphere. We systematically reduce the sphere's radius in logarithmic steps through 120 bins, from $R_{\text{vir,ROCK}}$ down to the radius closest to the initial center (typically around $\mathcal{O}(10^{-3}) \times R_{\text{vir,ROCK}}$). The shrinking process terminates when the particle count within the sphere falls below 100. We denote the center-of-mass position of these 100 particles as $\mathbf{x}_{h,\text{ss}}$. Note that the shrinking sphere method converges towards the main peak of the halo, so this approach may be unsuitable for subhalo or halo undergoing major mergers with additional peaks within R_{vir} . This is another reason we focus on relaxed, host halo.

The results are shown in Fig. 3.3. Most halo that satisfy the criteria in equations (3.1) and (3.2) display close alignment between $\mathbf{x}_{h,\text{ss}}$ and $\mathbf{x}_{h,\text{ROCK}}$, within 5% of $R_{\text{vir,ROCK}}$. However, there is one halo, despite meeting these criteria, with a mass of approximately

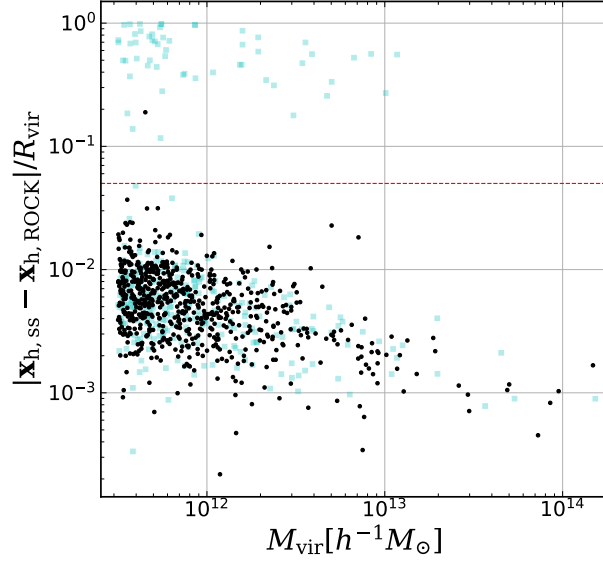


Figure 3.3: This figure compares the halo centers at $z = 0$ determined by ROCKSTAR ($\mathbf{x}_{h,ROCK}$) and the shrinking-sphere method ($\mathbf{x}_{h,ss}$). On the vertical axis, R_{vir} , and on the horizontal axis, M_{vir} , are derived from ROCKSTAR as $R_{vir,ROCK}$ and $M_{vir,ROCK}$, respectively. Cyan boxes indicate halos excluded based on the criteria in equations (3.1) or (3.2), while black dots represent the remaining halos. The distribution of halos in this plane is bimodal: most halos in the upper region are already excluded by the earlier criteria. Additionally, one halo with a significant discrepancy between the two center definitions is excluded, as indicated by the horizontal red dashed line marking a mismatch of $0.05 R_{vir,ROCK}$ (equation 3.4). Details of this exclusion are provided in the text. This figure is reproduced from Enomoto et al. (2024a).

$5 \times 10^{11} M_{\odot}$ and a relatively large positional discrepancy (around 20% of R_{vir}). In Section 3.3.6, we examine this halo in detail and find that it has a secondary peak, with both peaks fulfilling the previous criteria. Generally, halo with substantial deviations in $|\mathbf{x}_{\text{h,ss}} - \mathbf{x}_{\text{h,ROCK}}|$ can be classified as subhalo or halo undergoing major mergers. Thus, we introduce an additional criterion based on Fig. 3.3 to exclude halo that satisfy:

$$|\mathbf{x}_{\text{h,ss}} - \mathbf{x}_{\text{h,ROCK}}|/R_{\text{vir,ROCK}} > 0.05. \quad (3.4)$$

Next, we identify member particles of halo in each snapshot up to one step before $z = 0$, starting with those retained at $z = 0$ and using the same procedure. This involves determining the halo center via the shrinking-sphere method and expanding the sphere until its overdensity reduces to Δ_{vir} . This process is repeated back to $z = 5$, halting when the number of halo particles falls below 1000. If the particle count goes below 1000, we slightly expand the sphere to ensure it contains exactly 1000 particles, and define the starting time for counting p , t_s , as the cosmic time of this snapshot. Note that, as we move backward to each preceding snapshot, we only include particles retained from the previous snapshot in the shrinking-sphere process. This ensures that the list of member particles can only decrease as we trace back to higher redshifts. Consequently, we define particles within the oldest progenitor as consistently located near the halo center from t_s to the present.

As a result of this procedure, we can define the virial mass $M_{\text{vir}}(z)$ and virial radius $R_{\text{vir}}(z)$ at each redshift as the mass and radius of the sphere with an overdensity of Δ_{vir} . These values are used to calculate the accretion rate in Section 3.5. For clarity, we distinguish these from values computed by ROCKSTAR at $z = 0$, denoting the latter as R_{vir} and M_{vir} (i.e., without the argument z) in the following text.

Fig. 3.4 illustrates the distribution of t_s , which represents the cosmic time when the particle count in each halo’s progenitor drops below 1000. Most halo exhibit t_s values less than 4 Gyr, enabling particle tracking over more than ~ 10 Gyr. However, in 97 out of the 684 halo in our catalog, the tracking process reaches $z = 5$, our earliest snapshot, without dropping below 1000 particles. While we can still define the centers of these 97 halo using all progenitor particles at $z = 5$, the varying number of particles used to determine their centers may introduce systematic effects.

To address this, we select 1000 particles from the progenitors at $z = 5$, focusing on their phase space properties. First, we calculate the position and velocity dispersions, σ_x and σ_v , for the particles in the progenitor at $z = 5$. We then define the phase space distance, d_{ps} , of each particle relative to the average position $\mathbf{x}_{\text{ave}}$ and velocity $\mathbf{v}_{\text{ave}}$ of the particles as follows:

$$d_{\text{ps}}(\mathbf{x}, \mathbf{v}) = \left(\frac{|\mathbf{x} - \mathbf{x}_{\text{ave}}|^2}{\sigma_x^2} + \frac{|\mathbf{v} - \mathbf{v}_{\text{ave}}|^2}{\sigma_v^2} \right)^{\frac{1}{2}}, \quad (3.5)$$

where $\mathbf{x}$ and $\mathbf{v}$ are the positions and velocities of the particles. We then select the 1000 particles with the smallest d_{ps} values, ensuring a focus on particles that are well-bound within the halo progenitor.

Fig. 3.5 shows the positional differences between halo centers at $z = 0$, computed by averaging the positions of 1000 progenitor particles ($\mathbf{x}_{\text{h,pro}}$), and those determined by the ROCKSTAR halo finder ($\mathbf{x}_{\text{h,ROCK}}$). Aside from one halo with a mass of $\sim 10^{12} M_{\odot}$, which has a significant discrepancy ($\sim R_{\text{vir}}$), all 683 $\mathbf{x}_{\text{h,pro}}$ align with $\mathbf{x}_{\text{h,ROCK}}$ within an accuracy of less than 10% of R_{vir} , with 564 $\mathbf{x}_{\text{h,pro}}$ within 1%. This demonstrates that the motion

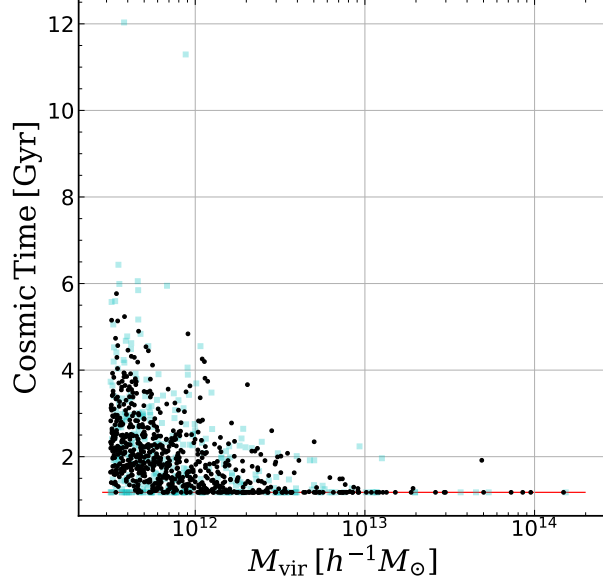


Figure 3.4: The starting time for counting p , denoted as t_s , is shown for each halo. Cyan and black markers correspond to the same halos as in Fig. 3.3. The horizontal red solid line represents $z = 5$. Among the 684 halos shown as black dots, 665 have t_s values below 4 Gyr. Notably, 97 black dots lie directly on the red line, indicating they are tracked up to $z = 5$. Since massive halos generally originate from similarly massive progenitors, the majority of their progenitors retain virial masses exceeding the equivalent of 1000 particles ($\sim 4.7 \times 10^{10} M_\odot$) even at $z = 5$. This figure is reproduced from Enomoto et al. (2024a).

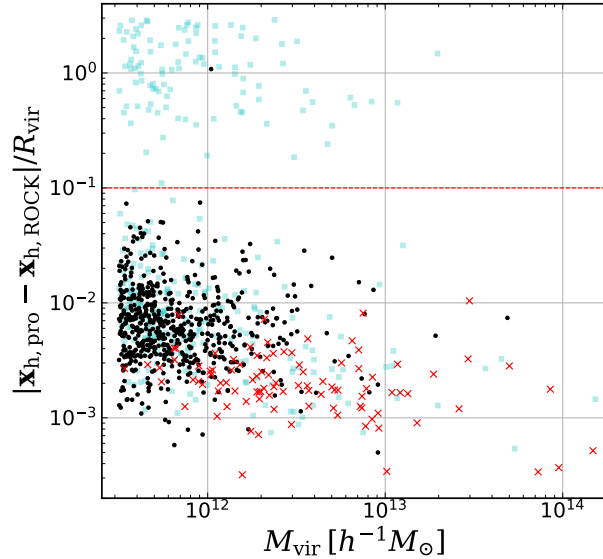


Figure 3.5: Difference between the center of the halo at $z = 0$ calculated by ROCKSTAR ($\mathbf{x}_{h,ROCK}$) and the average position over 1000 particles in the oldest progenitor ($\mathbf{x}_{h,pro}$). The cyan boxes and black dots are for the same halo as in Fig. 3.4. The red crosses represent 97 halo located on the red line in Fig. 3.4. Their $\mathbf{x}_{h,pro}$ is calculated by averaging over 1000 particles with the smallest phase-space distance, d_{ps} (defined in equation (3.5)) at $z = 5$. The red dashed line indicates the criterion from equation (3.6). This figure is adapted from Enomoto et al. (2024a).

of progenitor centers from t_s to the present can be reliably tracked using the fixed list of 1000 particles. These particles, the oldest population, occupy the central region of each halo throughout its evolution. We use them to define the positions and bulk motion of the centers, denoted as $\mathbf{x}_{\text{h,pro}}$ and $\mathbf{v}_{\text{h,pro}}$, at intermediate redshifts.

In Section 3.3.7, we demonstrate that the exceptional halo mentioned above contains a prominent secondary density peak, making it difficult to accurately trace the primary peak. This halo is likely undergoing a major merger. To address this, we introduce a final criterion and exclude halo that meet the following condition:

$$\frac{|\mathbf{x}_{\text{h,pro}} - \mathbf{x}_{\text{h,ROCK}}|}{R_{\text{vir,ROCK}}} > 0.1. \quad (3.6)$$

In summary, we constructed merger trees for each halo, selected 1000 particles within their progenitors, and defined $\mathbf{x}_{\text{h}}$ and $\mathbf{v}_{\text{h}}$ as the average position and velocity of these 1000 particles at each snapshot. As shown in Fig. 3.5, $\mathbf{x}_{\text{h}}$ aligns closely with those calculated by ROCKSTAR at $z = 0$.

3.3.2 Counting particles' apocenter passages

We are now positioned to calculate the radial velocity, v_r , of each particle across the 1001 snapshots, utilizing $\mathbf{x}_{\text{h}}$ and $\mathbf{v}_{\text{h}}$ as defined previously, and to quantify the number of apocenter passages by observing the changes in the sign of v_r . However, prior to conducting the actual counting of p , it is essential to consider particles that reside within subhalo. Our method for addressing this issue is based on the approach outlined in S20.

Typically, particles within a subhalo trace orbits around its center, and their radial velocities in relation to the host halo can sometimes shift from negative to positive (and vice versa) due to this internal orbital motion. While we define p as the number of apocenter passages concerning the host halo, this phenomenon can inadvertently inflate the count of p . To counteract this, we take into account the directional vector of each particle, denoted as $\hat{\mathbf{r}}(t)$, relative to the center of the host halo.

Consider a scenario where a particle experiences an apocenter passage at time t_1 , resulting in a change in the sign of its radial velocity from positive to negative. Subsequently, at time t_2 , a positive-to-negative transition in the sign occurs for the first time following t_1 . If the condition $\hat{\mathbf{r}}(t_1) \cdot \hat{\mathbf{r}}(t) > 0$ holds for all $t_1 < t \leq t_2$, it indicates that the particle has remained on the same side of the halo without completing a full orbit before the next apocenter passage. In such instances, the sign change at $t = t_2$ is likely attributable to internal motion within the subhalo. It can be demonstrated that a particle in a static spherical potential undergoes an apocenter passage only after moving at least 180° around the center of that potential since the last apocenter passage (refer to Sec. 3.1 of Binney & Tremaine (2008)). Thus, we only count the apocenter passages when $\hat{\mathbf{r}}(t_1) \cdot \hat{\mathbf{r}}(t) < 0$ occurs at least once for $t_1 < t \leq t_2$. This condition ensures that the particle has orbited a minimum of 90° from its previous apocenter passage. We have validated that this approach effectively eliminates spurious apocenter passages arising from internal motion within a subhalo.

To illustrate, Fig. 3.6 depicts the counting of p for an orbiting particle using the outlined method. In this illustration, the color of the data points varies as the particle traverses its orbital apocenter, highlighting the method's effectiveness. Around $t = 9$ Gyr, we can observe a sign change in v_r from positive to negative. However, the count of p remains unchanged because the particle has not moved significantly from the previous

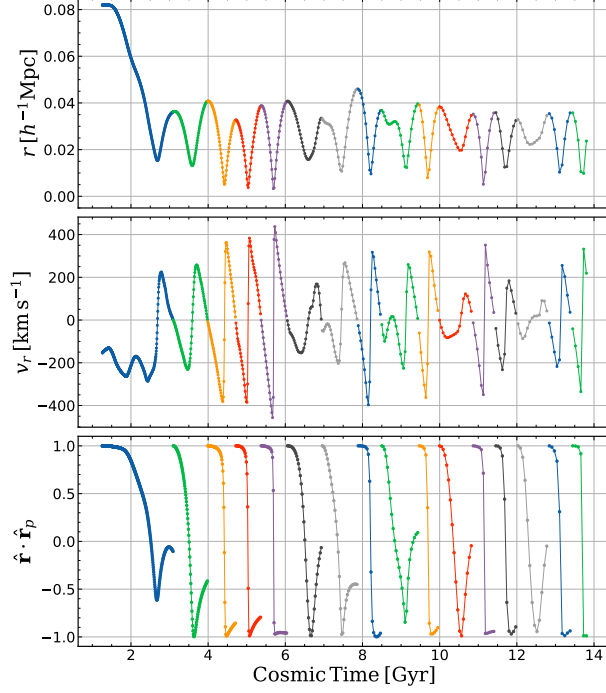


Figure 3.6: An example trajectory of a particle around the center of a progenitor halo is illustrated. From top to bottom, the rows depict the radial distance r , radial velocity v_r , and the inner product $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_p$, where $\hat{\mathbf{r}}$ is the particle’s direction from the center at a given time, and $\hat{\mathbf{r}}_p$ is the direction at the previous apocenter passage. Points are color-coded based on the value of p . To ensure accuracy, p is incremented only when the sign of $\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}_p$ becomes negative since the last apocenter passage. At the present time, this particle has $p = 15$. This figure is reproduced from Enomoto et al. (2024a).

apocenter passage (i.e., $\hat{\mathbf{r}}(t_1) \cdot \hat{\mathbf{r}}(t)$ stays positive since t_1). The duration since the last apocenter passage is relatively brief compared to typical intervals between other apocenter passages, suggesting that this sign change results from internal subhalo motion.

Moreover, the density of the data points in the figure is adequate to facilitate accurate counting of the apocenter passages. Insufficient snapshots could lead to missed detections of apocenter passages, possibly resulting in erroneous counts of p . To ensure precision, we perform a verification in Section 3.3.4, demonstrating that the 1001 snapshots employed offer sufficient time resolution to prevent miscounting of p for $p \leq 40$. Furthermore, in Section 3.3.5, we conduct a convergence study to assess whether our simulation setup possesses adequate mass resolution to effectively trace multi-stream flows over a large number of apocenter passages, again confirming that the current mass resolution yields a reliable estimate of p for $p \leq 40$.

3.3.3 Density profiles and fitting procedure

Before delving into our main findings, we summarize several technical aspects associated with the calculation of density profiles, the stacking procedure, and the fitting methodology. In Section 3.4.1, we present density profiles in physical coordinates. To create these density profiles, we utilize logarithmically spaced radial bins that range from 2 times the softening length up to $2.5 R_{\text{vir}}$.

Next, in Section 3.4.2, we employ stacked density profiles, which entail averaging

individual profiles within designated subgroups. These subgroups are defined based on factors such as the number of particles corresponding to a specific value of p and the mass. In our stacking process, we first normalize the radial distances of particles within each halo by their respective R_{vir} values to generate individual density profiles. Subsequently, we create stacked density profiles by averaging these individual profiles within the subgroup. For each mass bin's stacked profiles, we define 80 logarithmically spaced radial bins that extend from $0.0025 R_{\text{vir}}$ to $2.5 R_{\text{vir}}$. However, we exclude bins that fall below 1.2 times the maximum value of the softening length, scaled by R_{vir} , for halos in the subgroup to prevent numerical artifacts due to limited force resolution. A comparison of the LR and HR simulations confirms that the softening effect does not considerably affect the stacked profiles beyond this radius. These particular radii will be explicitly referenced in Section 3.4.2.

In Section 3.4, we will also present the fitted curves for the density profiles. Our fitting procedure is based on the standard minimization method for χ^2 , defined as follows:

$$\chi^2 \equiv \sum_i \left(\frac{\ln \rho_{i,\text{fit}} - \ln \rho_{i,\text{data}}}{\sigma_i / \rho_{i,\text{data}}} \right)^2. \quad (3.7)$$

In this equation, $\rho_{i,\text{fit}}$ denotes the fitted density in the i -th radial bin, while $\rho_{i,\text{data}}$ represents the individual or stacked density profile. The term σ_i signifies the uncertainty of $\rho_{i,\text{data}}$, which is determined by the Poisson scatter for individual profiles and the root mean square deviation of the profiles stacked within subgroups. For our analysis, we take into account all radial bins above the aforementioned lowest radial bins for each case.

3.3.4 Convergence tests I: the number of snapshots and halo center

In this subsection, we illustrate that the number of particles utilized to determine the centers of halo progenitors, specifically $\mathbf{x}_h$ and $\mathbf{v}_h$, along with the number of snapshots employed, do not influence our results. As outlined in Section 3.3.1, we define $\mathbf{x}_h$ and $\mathbf{v}_h$ as the average position and velocity of a fiducial set of 1000 particles in the progenitor. The number of particles used to define $\mathbf{x}_h$ and $\mathbf{v}_h$, denoted as N_{det} , can be selected arbitrarily. Consequently, we explore the implications of varying N_{det} by examining values of 500 and 2000, in addition to the fiducial value of 1000.

In Fig. 3.7, we display representative trajectories of $\mathbf{x}_h$ for each N_{det} . From our visual assessment, we observe that the trajectories corresponding to different values of N_{det} are closely aligned. However, the effect on the number of apocenter passages is non-trivial, particularly for particles with high values of p due to their smaller orbital sizes. Additionally, varying N_{det} can indirectly influence the value of p through its effect on t_s , which is the starting time for counting. Here, t_s is defined as the moment when the number of particles in the progenitors falls below N_{det} ; therefore, a decrease in N_{det} leads to a reduction in t_s (i.e., enabling tracking of the progenitor to higher redshifts) and initiates counting of p earlier. This earlier start may result in inflated p values for individual particles.

The two massive halos depicted in the upper left and upper middle panels of Fig. 3.7 possess substantial progenitors. We observe that the starting position of the counting, $\mathbf{x}_h$ at $t = t_s$ (indicated by star symbols), remains nearly the same across different N_{det} . Conversely, some of the less massive halos, especially the one shown in the lower right

panel, demonstrate significant variations in $\mathbf{x}_h(t_s)$. This variation indicates that progenitor trajectories can differ depending on the value of N_{det} .

Next, we illustrate the distribution of p for particles surrounding the same six halos in Fig. 3.8. Despite the differences noted in Fig. 3.7, the overall trend remains consistent. For low values of p , approximately below 50, the various symbols align closely within $\sim 10\%$. However, the dependence on N_{det} becomes more pronounced for higher p values. Hence, for particles with $p \lesssim 50$, both the value of t_s and the precise trajectories of progenitors have minimal impact on p counts. For the halo illustrated in the upper left panel, we consistently track the progenitor up to $z = 5$ across all three N_{det} values. Thus, the dependence of particle counts on N_{det} for $p \gtrsim 50$ arises solely from the slightly different center trajectories. It is noteworthy that for less massive halos, even at $p \lesssim 50$, the counts exhibit fluctuations due to Poisson noise, particularly when the number of particles for each p is below approximately 30.

Finally, we examine the influence of the number of snapshots utilized in our analysis. If the snapshots lack sufficient temporal resolution for tracking particles, some may complete multiple orbits between snapshots, potentially leading to inaccuracies in their p counts. Therefore, it is essential to verify that our set of 1001 snapshots, evenly sampled in redshift, maintains adequate time resolution to mitigate this issue. To evaluate this, we track the trajectories using every other snapshot from the 1001, with the results displayed as green star symbols in Fig. 3.8. Here, N_{det} is set to 1000, our fiducial number. We find that for most halos, the number of particles with high p values is less than in cases utilizing 1001 snapshots (as indicated by the triangles). However, we confirm that the distribution remains consistent for $p \lesssim 50$ even when utilizing half of the snapshots.

Based on the findings presented in this section, we decide to primarily focus on particles with $p \leq 40$ in the following text to adopt a conservative approach.

3.3.5 Convergence tests II: mass resolution

In this section, we conduct a convergence study to assess whether our simulation framework possesses sufficient mass resolution for accurately tracking particle trajectories and preventing miscounts in the apocenter passage counts.

To achieve this, we executed two additional simulations with identical initial conditions, but with distinct mass resolutions, while reducing the simulation box size to $10.25 h^{-1} \text{ Mpc}$. These simulations complement the LR and HR setups detailed in Table 3.1. One of these simulations, designated as N125, employs $N = 125^3$ dark matter particles, matching the mass resolution of the LR simulation. The other, termed N250, utilizes 250^3 dark matter particles. By maintaining the same box configuration as N125, N250 achieves an eightfold increase in resolution compared to both N125 and LR.

We then repeat the analysis outlined in Section 3.3, selecting six representative halo to compare their multi-stream characteristics across N250 and N125. Figure 3.9 illustrates the distribution of particle mass as a function of p for each halo. The results for $p \leq 40$ are nearly identical between the two simulations. An exception is noted in the middle-lower panel, where the rightmost bin for N125 exhibits significant scatter even at $p \leq 40$, markedly diverging from N250's results. This discrepancy can be attributed to the effects of Poisson noise due to the presence of only four particles in that bin.

Next, we turn to Fig. 3.10, which displays the stream profiles for various p values across the six selected halo, with color scales indicating p . Each panel features a lower sub-panel that depicts the fractional differences between the profiles obtained from N125 and N250.

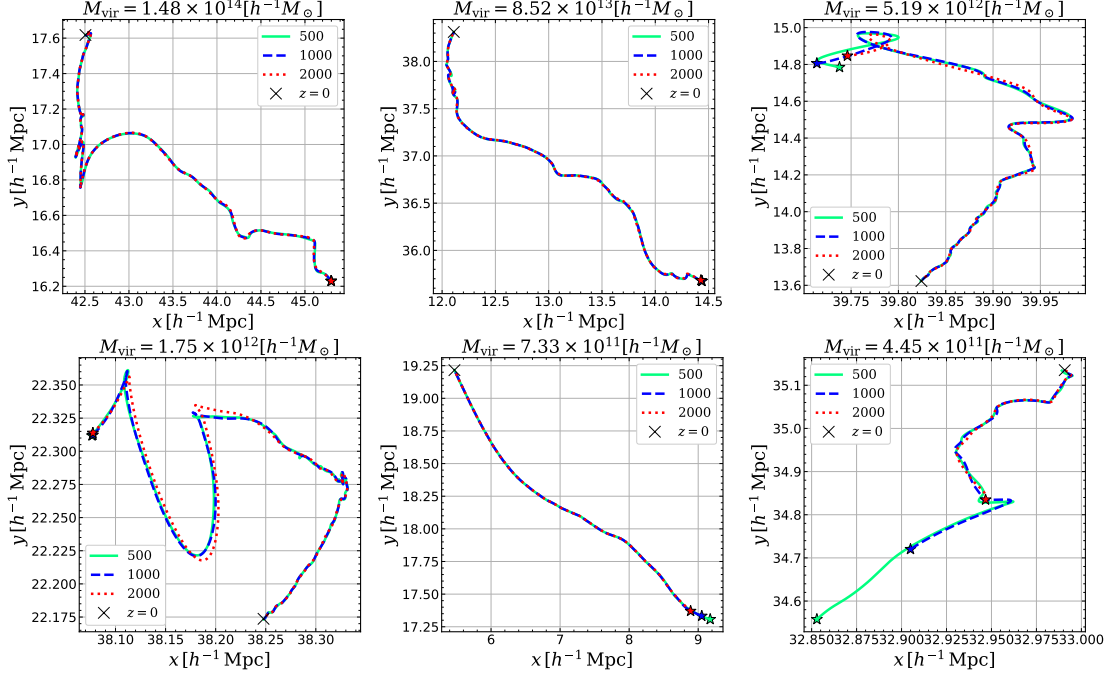


Figure 3.7: The representative trajectories of $\mathbf{x}_h$ from $t = t_s$ to $z = 0$. Line colors and styles correspond to each N_{det} as indicated in the legend. Stars mark the positions of $\mathbf{x}_h$ at $t = t_s$, while black crosses indicate $\mathbf{x}_h$ at $z = 0$. For cases where t_s matches the cosmic time at $z = 5$, N_{det} particles are selected using the phase-space metric described by Eq. 3.5 in Section 3.3.1. These figures are taken from Enomoto et al. (2024a).

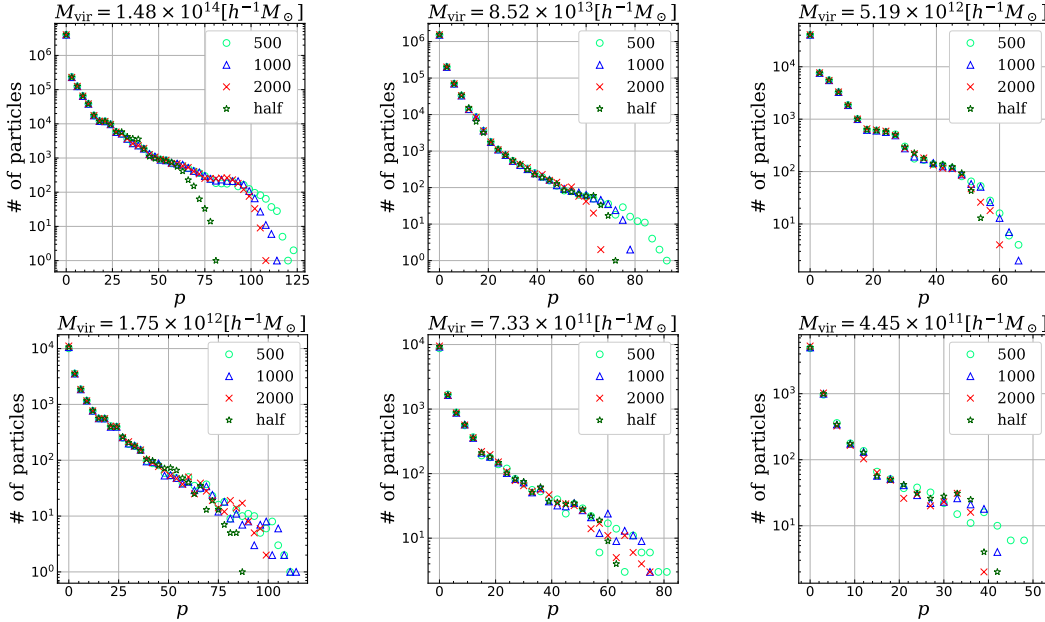


Figure 3.8: The particle distributions for each p of the halos presented in Fig. 3.7. Circle, triangle, and cross markers correspond to the fiducial value $N_{\text{det}} = 1000$, as indicated in the legend, with colors matching those in Fig. 3.7. The green star represents the case where trajectories are tracked using every alternate snapshot out of 1001. These figures are reproduced from Enomoto et al. (2024a).

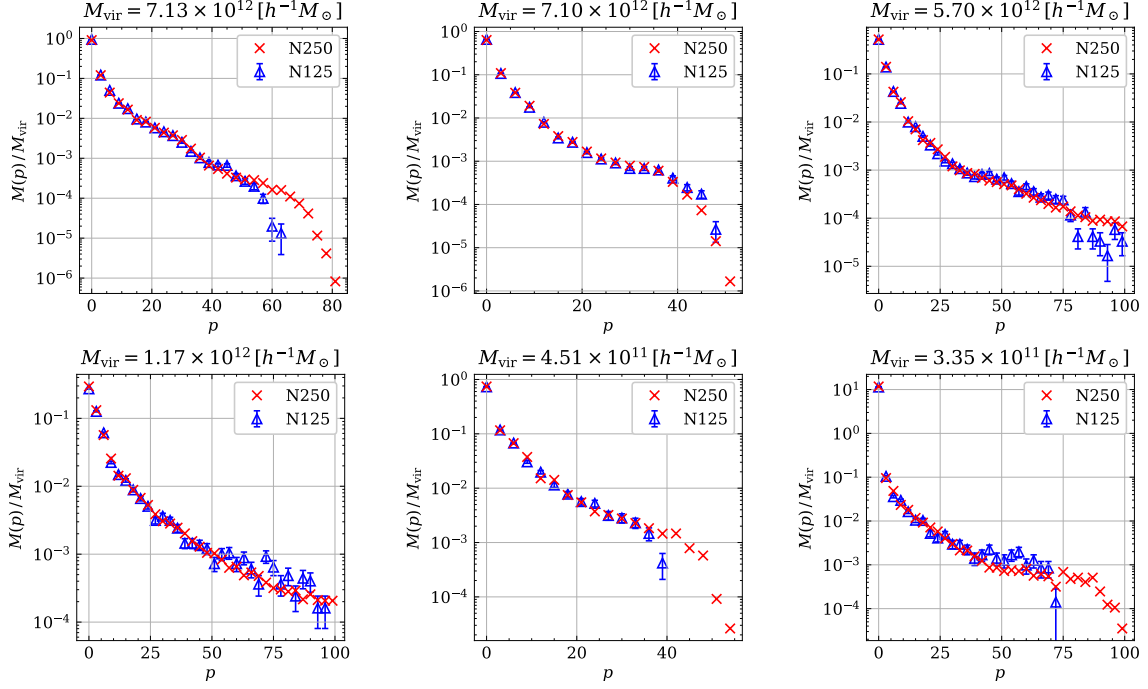


Figure 3.9: Mass distribution of the DM particles having the same number of apocenter passages p . Here, we compare the results between two additional simulations, N125 (open triangles) and N250 (crosses). While the former has the same mass resolution as in LR simulation, the latter simulation has a better mass resolution, eight times higher than that of the LR simulation. The errorbars indicate the Poisson noise estimated from the number of particles in each p bin. Note that we basically follow the methods to identify the halo centers and trajectories as well as to count the number of apocenter passages in Sections 3.3.1 and 3.3.2, but the number of particles used to determine the halo center in N250 is changed from 1000 to 8000 in order to be consistent with its mass resolution. These figures are adapted from Enomoto et al. (2024a).

The outcomes reveal no discernible systematic trend, with most profiles converging well within a 50% margin. We further examined additional halo in N125 and confirmed that for halo with masses exceeding $3.2 \times 10^{11} h^{-1} M_{\text{vir}}$ —the lower limit of the mass range S—the behaviors remain consistent with those illustrated in Fig. 3.10. Consequently, we conclude that the mass resolution in our LR simulation is sufficient for tracking particle trajectories and provides a statistically convergent result for counting apocenter passages.

3.3.6 The Exceptional halo I

In Fig. 3.3, we identify an unusual halo (marked by the black dot in the upper left corner) that remains in the catalogue despite the application of the criteria outlined in equations (3.1) and (3.2). These criteria are intended to eliminate subhalo and halo involved in major mergers. This particular halo shows a significant discrepancy between the two estimates of its center, $\mathbf{x}_{\text{h,ss}}$ and $\mathbf{x}_{\text{h,ROCK}}$.

To explore this halo further, we examine its counterpart in the HR run. We find another nearby halo recognized by ROCKSTAR (indicated by the green square symbol in the lower panel of Fig. 3.11), which does not have a corresponding entry in the ROCKSTAR catalog from the LR run. When we implement the shrinking-sphere method to locate the

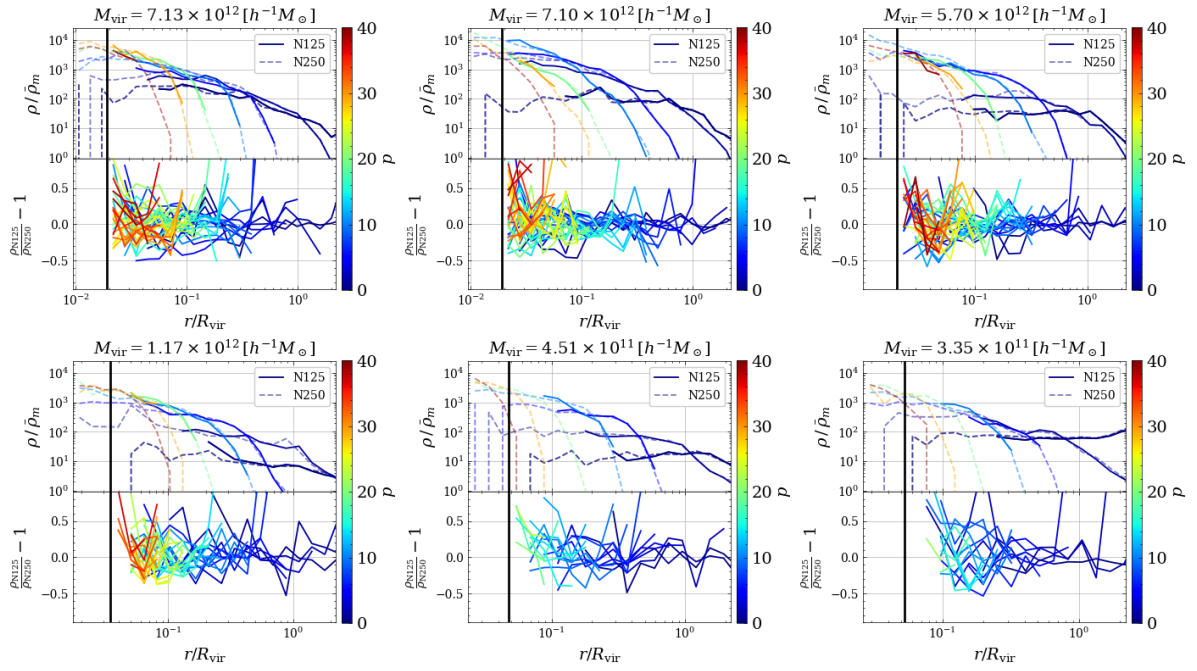


Figure 3.10: Density profiles of DM particles categorized by the number of apocenter passages, p . Similar to Fig. 3.9, results from two additional simulations, N125 (solid lines) and N250 (dashed lines), are compared for six representative halos. The upper panels present profiles for $p = 0, 1, 5, 10, 20, 30, 40$, excluding bins with fewer than 10 particles in N125. Vertical black lines denote three times the softening length of N125. The lower panels display the fractional differences between the results of N125 and N250. These figures are reproduced from Enomoto et al. (2024a).

density peak in this region of the HR run, we observe that the center aligns with this nearby halo, despite its virial mass being an order of magnitude less than that of the halo in question. This result is not unexpected, as the peak density around the less massive nearby halo—calculated from the 100 particles closest to its center in the final step of the shrinking-sphere method—aligns closely with that of the more massive halo. Consequently, we infer that even though the nearby halo is not detected by the ROCKSTAR finder in the LR run, the shrinking-sphere method opts for this halo instead of the more massive one, leading to a notable discrepancy between the two estimated centers.

Thus, while this halo is not ruled out by equations (3.1) or (3.2), it appears to be a candidate for undergoing a major merger. To mitigate potential technical complications arising from uncertainties in the center estimation, we apply a third criterion, specifically equation (3.4), to identify and exclude such cases from our catalogue.

3.3.7 The Exceptional halo II

In Section 3.3.1, we have highlighted another problematic halo that persists in the catalog despite the application of the first three conditions. This halo exhibits undesirable characteristics that complicate the accurate tracking of its trajectory for defining apocenter passages (the halo marked by the black dot near the top left corner of Fig. 3.5). In this section, we perform a thorough examination of this specific halo.

We provide particle snapshots of this unique halo at two redshifts, $z = 0$ and $z = 0.005$, in Fig. 3.12. These images illustrate the merger of two significant structures. At $z = 0$, both structures have similar masses, with the primary structure located near the origin having a mass of $1.79 \times 10^{12} h^{-1} M_{\odot}$, while the secondary structure below has a mass of $1.05 \times 10^{12} h^{-1} M_{\odot}$. The primary structure remains after applying the criterion defined by equation (3.1), whereas the secondary structure does not.

The bottom panel of Fig. 3.12 shows the inconsistent tracking of the secondary structure’s center (denoted by the star symbol) at $z = 0.005$ as determined by the shrinking-sphere method. In this case, a major merger event prevents consistent tracking of the center over time. To address this issue, we introduce a final criterion, specifically equation (3.6), to identify and exclude such halos from our catalog.

3.4 Main results

In this section, we report on the density profiles of halos based on particles with varying numbers of apocenter passages, p . We begin by selecting representative halos and displaying their individual profiles in Section 3.4.1. Following this, we analyze the stacked halo profiles in Section 3.4.2, where we thoroughly describe them using a double-power law model. To clarify the statistical properties, we specifically explore how these profiles vary with halo mass in this section.

3.4.1 Individual halo profiles

Applying the methods and algorithms outlined in Section 3.3 to 683 halos from our simulations (summarized in Table 4.4.2), we are now able to characterize the radial phase-space structure by analyzing particle distributions sorted by different numbers of apocenter passages. To illustrate this, we first select three representative halos with masses of 1.49×10^{14} , 1.36×10^{13} , and $1.45 \times 10^{12} h^{-1} M_{\odot}$, and present their radial phase-space and

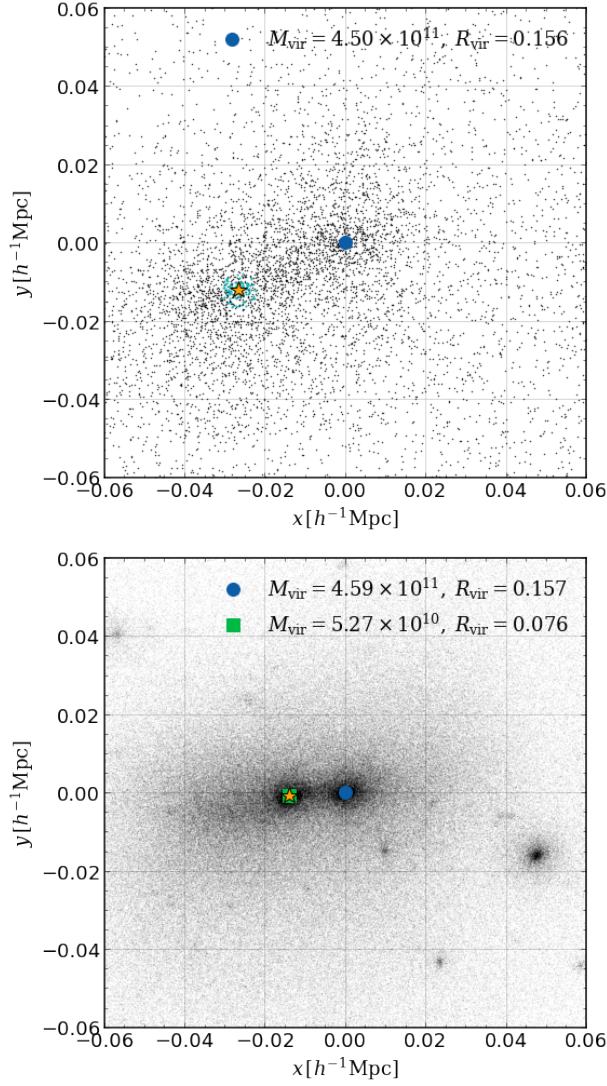


Figure 3.11: Projected snapshots of the exceptional halo, located near the top left corner of Fig. 3.3, are shown. The upper (lower) panel represents the halo in the LR (HR) simulation. In both panels, the orange star symbol marks the density peak where the shrinking-sphere method converges. The simulation particles, shown as cyan dots, correspond to the 100 particles within the sphere at the final iteration of the shrinking-sphere method. The blue-filled circle and green square represent the halo identified by the ROCKSTAR finder. The mass (M_{vir}) and radius (R_{vir}) of these halos are provided in the figure legend, with units of $[h^{-1}M_{\odot}]$ and $[h^{-1}\text{Mpc}]$, respectively. Note that the cyan dots in the lower panel are fully hidden behind the green square. In the LR run, ROCKSTAR fails to identify any halo corresponding to the density peak marked by the orange star, whereas it is successfully detected in the HR run. The overdensities relative to the background density $\bar{\rho}_m$ for the 100 particles nearest to the blue points and orange stars are as follows: LR – 1.0×10^5 and 1.2×10^5 , and HR – 2.2×10^6 and 1.6×10^6 , respectively. Notably, all three halos identified by ROCKSTAR in both runs satisfy the criteria defined by equations (3.1) and (3.2). However, the distance $|\mathbf{x}_{\text{h,ss}} - \mathbf{x}_{\text{h,ROCK}}|$ still reaches up to 8% of R_{vir} in the HR simulation. These figures are reproduced from Enomoto et al. (2024a).

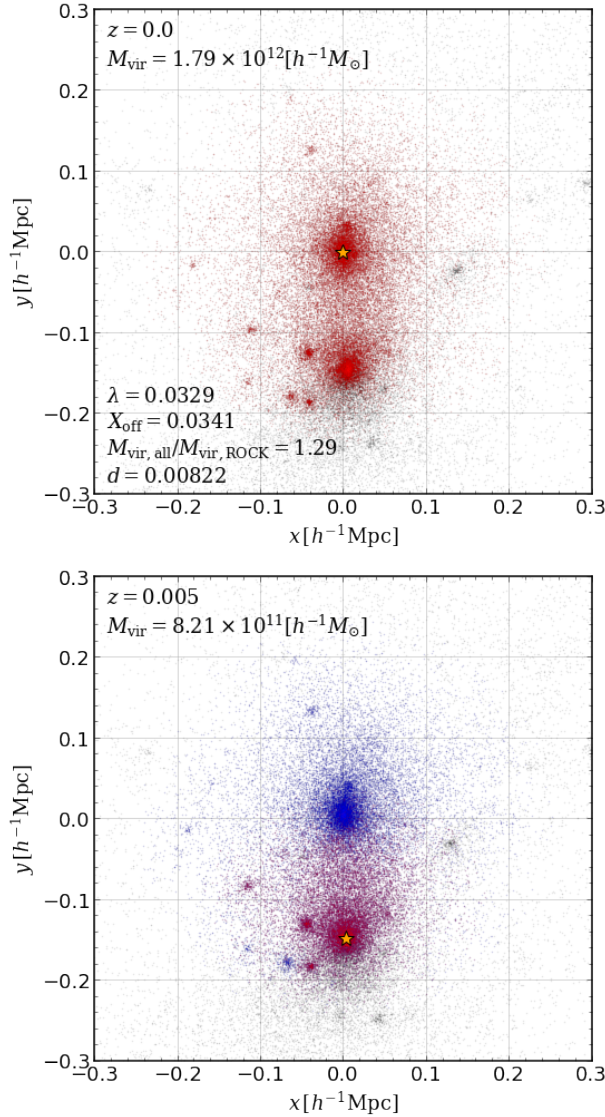


Figure 3.12: Projected snapshots of simulation particles surrounding an exceptional halo from Fig. 3.5 at $z = 0$ and $z = 0.005$, with the center of the halo located at $\mathbf{x}_{\text{h,ROCK}}$ at $z = 0$. Particles classified as members of the halo at each redshift (i.e., those within the virial overdensity sphere) are shown in red. The lower panel includes blue dots for particles that are members at $z = 0$ but not at $z = 0.005$. The orange stars mark the halo centers, $\mathbf{x}_{\text{h,ss}}$, determined using the shrinking-sphere method applied to the red particles at each redshift. At $z = 0$, this halo has $R_{\text{vir,ROCK}} = 0.206 h^{-1} \text{Mpc}$, and the star symbol shifts by $0.239 [h^{-1} \text{Mpc}]$ between the two snapshots. In the top panel, λ represents the spin parameter, X_{off} the offset parameter, and d the distance $|\mathbf{x}_{\text{h,ss}} - \mathbf{x}_{\text{h,ROCK}}|/R_{\text{vir}}$; these parameters comply with the criteria set forth in equations (3.1), (3.2), and (3.4). The mass M_{vir} in both panels refers to the mass of the red particles, not $M_{\text{vir,ROCK}}$. At $z = 0.005$, the secondary peak, which is observed at $z = 0$, is considered the primary peak by the shrinking-sphere method, causing a substantial shift in $\mathbf{x}_{\text{h,ss}}$ between the two snapshots. This secondary peak is also detected by ROCKSTAR, with a virial mass of $1.05 \times 10^{12} [h^{-1} M_{\odot}]$, but it has $\lambda = 0.0879$ and is excluded based on the criterion in equation (3.1). These figures are taken from Enomoto et al. (2024a).

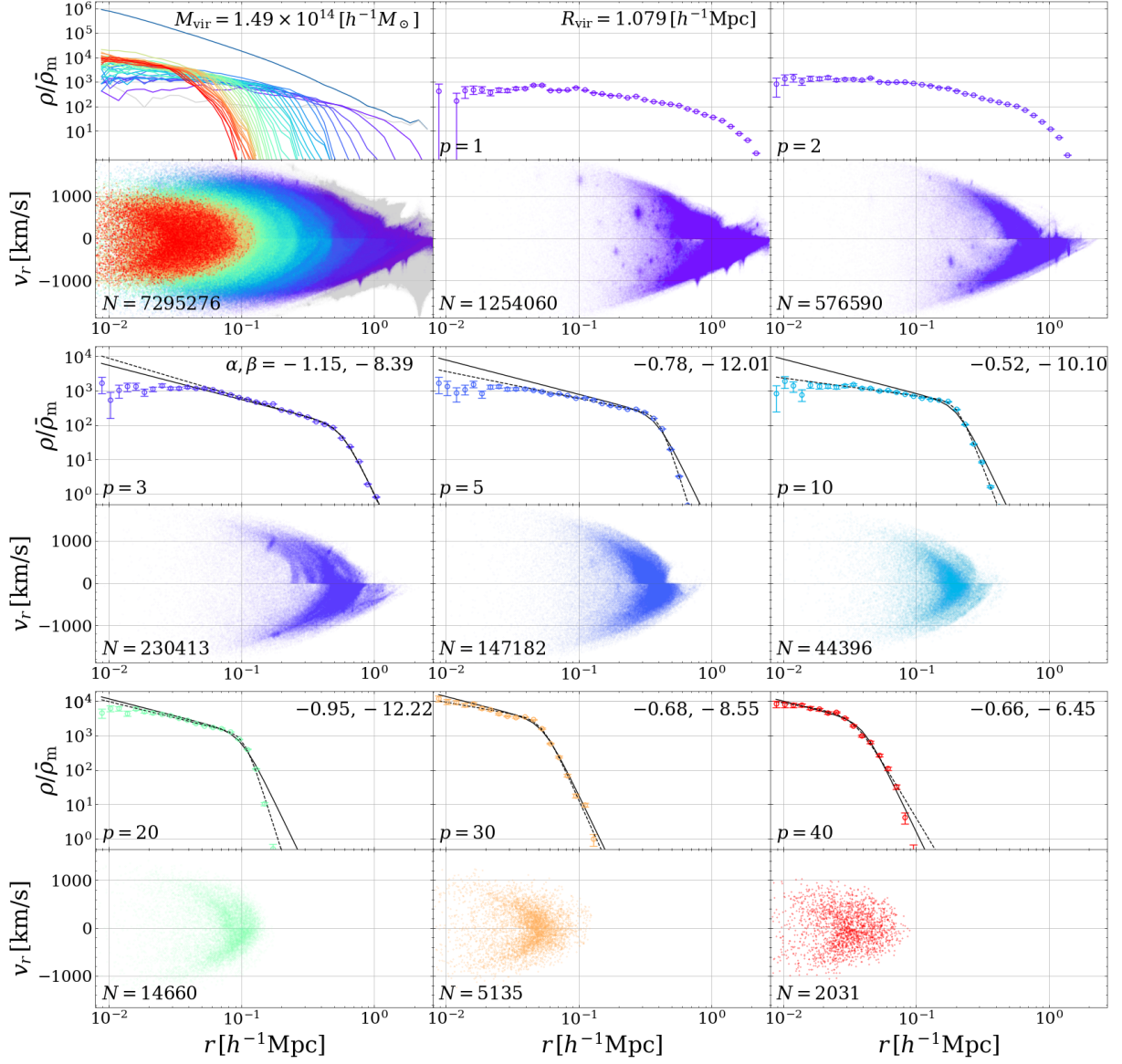


Figure 3.13: Radial density profiles and radial phase space distributions of particles for different values of $p = 1, 2, 3, 5, 10, 20, 30, 40$. The number of particles shown in each phase space is denoted by N . The black solid (dashed) lines represent the fits to the density profiles using equation (3.9) (equation (3.8)), with the inner and outer slopes α and β indicated for the dashed lines. In the upper left panel, the density profiles and phase-space distributions are shown for the total particles as well as for $1 \leq p \leq 40$ (with the same color coding as in other panels), and $p = 0$ (shown in light grey). As noted in Section 3.3.3, the minimum radial bin for the density profiles is set to be twice the softening length. This figure is taken from Enomoto et al. (2024a).

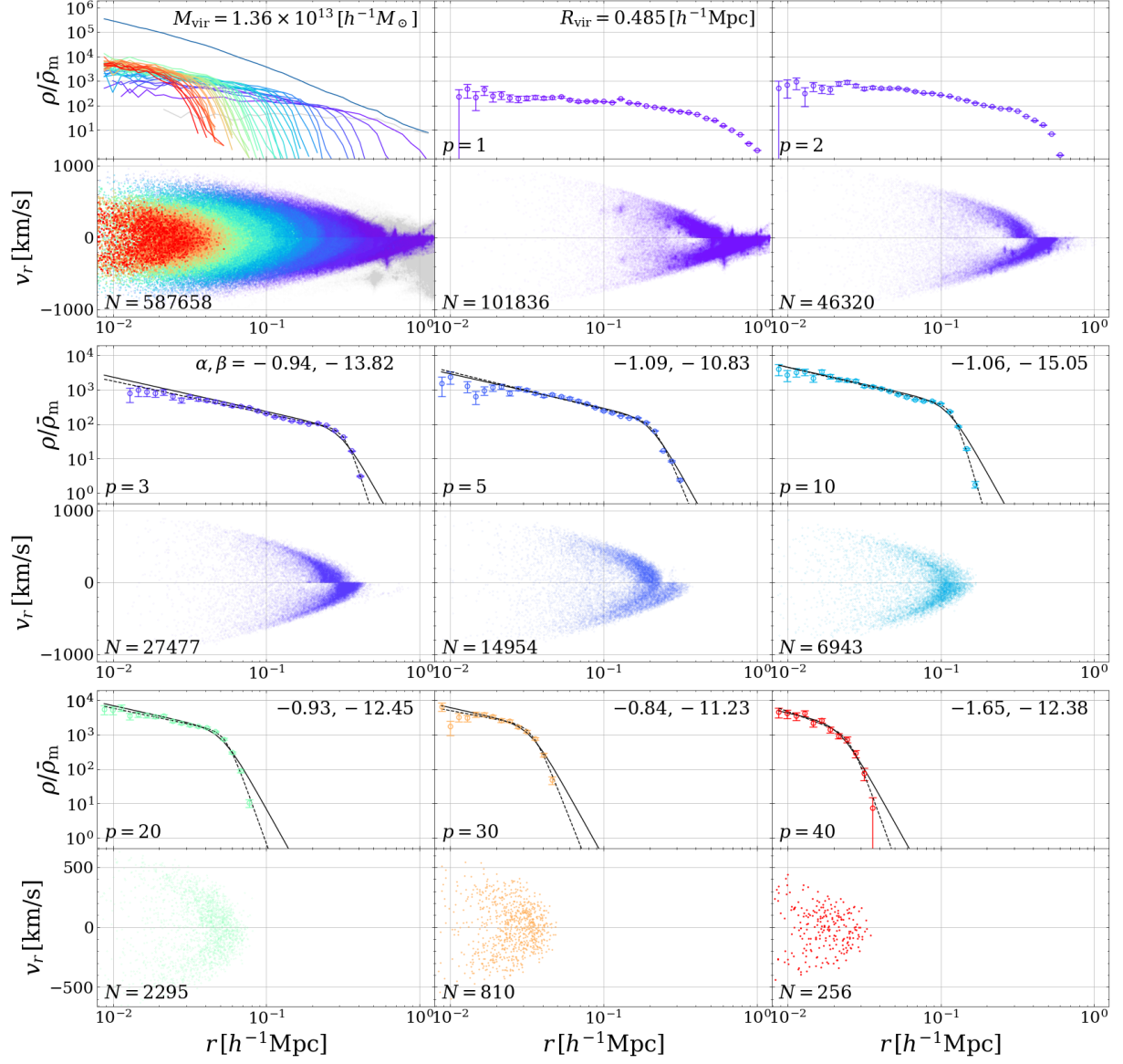


Figure 3.14: Same as Fig. 3.13 but halo mass is $1.36 \times 10^{13} [h^{-1} M_{\odot}]$. This figure is taken from Enomoto et al. (2024a).

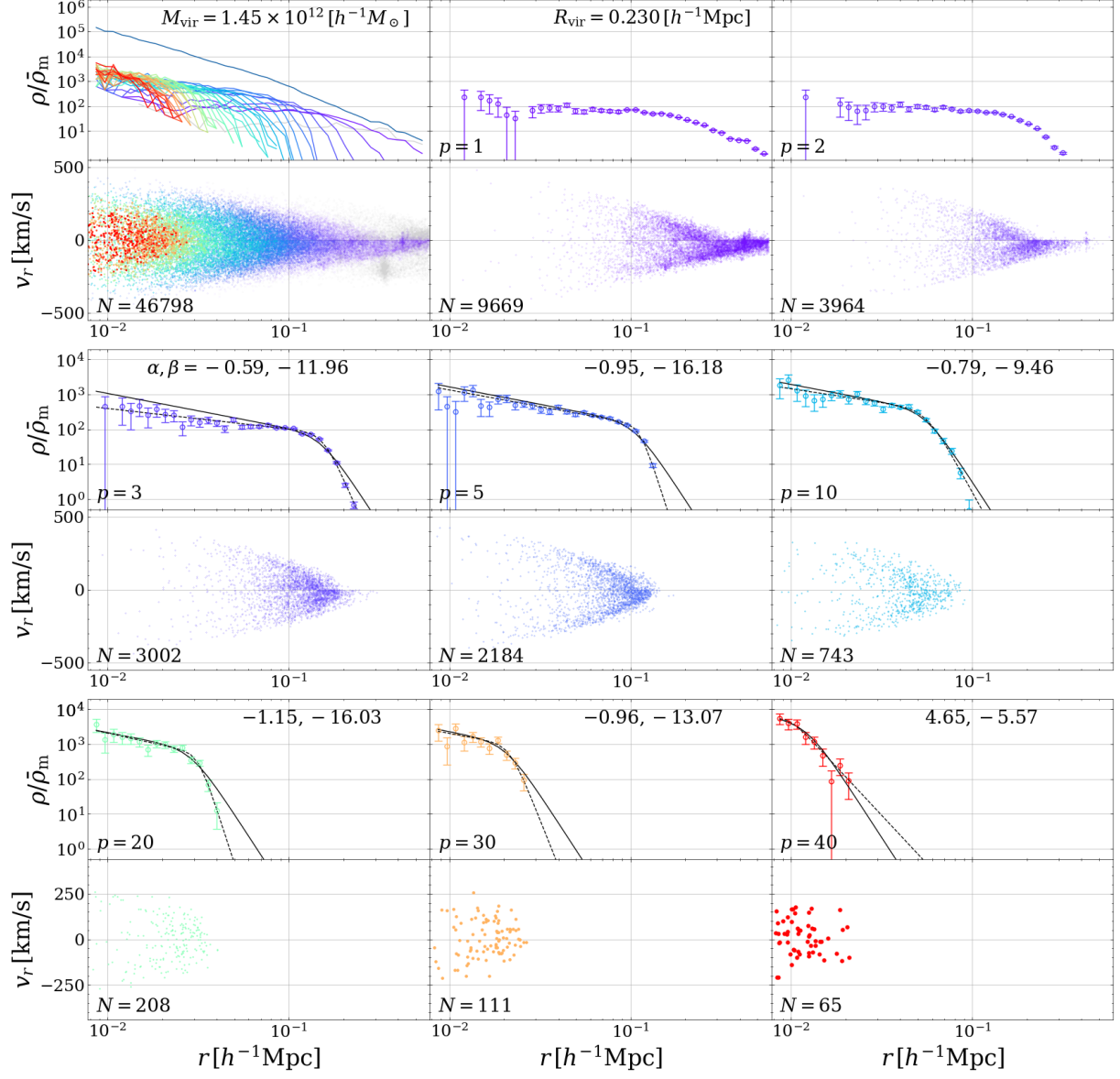


Figure 3.15: Same as Fig. 3.13 but halo mass is $1.45 \times 10^{12} [h^{-1} M_{\odot}]$. This figure is taken from Enomoto et al. (2024a).

density profiles in Figs. 3.13-3.15. In these figures, we display density profiles (top row) and phase-space distributions (bottom row) for particles with $p = 1$ to 40 (top left panel) and for $p = 1, 2, 3, 5, 10, 20, 30, 40$ (other panels).

Across this mass range, the density profiles and phase-space structures show largely consistent trends. In particular, the outer regions of the radial phase-space distributions reveal numerous clumps with a variety of spatial and velocity scales at $p = 1$ and 2. Although our sample excludes unrelaxed halos, this suggests that halo outskirts typically remain unrelaxed, displaying significant fluctuations in phase-space density due to residual substructures recently accreted or involved in mergers. This scatter corresponds to the halo-to-halo variability in infalling profiles, as also noted by Diemer (2022).

In contrast, at higher values of p , the inner structures of these halos show smoother phase-space distributions with fewer substructures. Furthermore, the density profiles for higher p values tend to follow a double-power law form, featuring a shallow inner slope transitioning to a steeper outer slope. This transition scale, where the slope sharply changes, marks the region where particles accumulate near zero velocity in phase space due to apocenter passages for particles with the same p . This transition roughly aligns with the radial caustic of each multi-stream flow. S20 noted that about $\sim 30\%$ of halos exhibit structural similarities to the self-similar solution from Fillmore & Goldreich (1984). Although their self-similar models typically predict spiked density at caustics, real halos show smoother transitions due to non-zero tangential velocities and deviations from sphericity, as seen in Figs. 3.13 to 3.15.

We now examine the density profile structure for each p in detail, aiming to model it using the following functional form:

$$\rho(r; p) = \frac{A(p)}{\{r/S(p)\}^{-\alpha(p)} \left[1 + \{r/S(p)\}^{\alpha(p)-\beta(p)} \right]}, \quad (3.8)$$

which behaves as $\rho \propto r^{\alpha(p)}$ at $r \ll S(p)$ and $\rho \propto r^{\beta(p)}$ at $r \gg S(p)$, assuming $\alpha(p) < 0$ and $\beta(p) < 0$. The four parameters, $A(p)$, $S(p)$, $\alpha(p)$, and $\beta(p)$, are obtained by fitting equation (3.8) to the measured density profile for each p as described in Section 3.3.3.

The fit results are represented by black dashed lines in the upper panels of Figs. 3.13–3.15. Here, we perform the fitting for $p \geq 3$, as the profiles for $p = 1$ and 2 deviate from a double-power law, partly due to substructure contributions in the radial phase-space distribution. The figures indicate that equation (3.8) fits the measured profile well, particularly near the transition scales, with improved fitting accuracy as p increases. For lower p values, some deviation from the double-power law form appears in the inner regions of massive halos (see Figs. 3.13 and 3.14), where a flatter core is evident. However, due to limited particle counts, it isn't easy to assess the significance of this deviation in individual halos. To address this, we will examine stacked halo profiles to assess the fit quality to the double-power law function statistically.

3.4.2 Stacked profiles for mass selected samples

In this subsection, we analyze the stacked density profiles of the selected 683 halos, grouped into four mass bins (S – XL) as summarized in Table 4.4.2. By fitting these profiles to a double-power law function, we examine the statistical properties of the radial profiles across different values of p , with particular attention to mass dependence.

Section 3.4.2 investigates the fit of the double-power law profile for the stacked profiles of each p . We demonstrate that setting the slope parameters to $\alpha = -1$ and $\beta = -8$ pro-

vides a good match to the measured profiles across all mass bins. With these slopes fixed, we proceed in Section 3.4.2 to quantitatively explore how the fitting parameters—namely, the characteristic density $A(p)$ and the radius $S(p)$ —vary with the number of apocenter passages p and halo mass.

Fitting to a general form of double-power law profile

We begin by presenting the stacked density profiles and their fitting results to the double-power law model given in equation (3.8). In Fig. 3.16, the radius and density are scaled by factors of p and $10^{p/4}$, respectively. Profiles for streams from $p = 4$ to 40 are shown for each mass bin, with the best-fit curves represented by dashed lines, which extend down to half the resolution limit of the LR run. These fits closely match the measured profiles.

In Fig. 3.17, we plot the best-fit parameters that characterize the double-power law profiles—namely, the characteristic density A (upper left), scale S (lower left), and inner and outer slopes, α (upper right) and β (lower right)—as a function of p . Error bars are derived from the curvature of the χ^2 function in equation (3.7), where σ_i represents the RMSD of the stacked profile for each mass bin. Distinct trends are evident with respect to p . While the characteristic density and scale vary monotonically (density increasing and scale decreasing), the outer slope β remains nearly constant around $\beta \sim -8$ to -10 , except at lower values of p . The inner slope α generally holds steady over p for the massive halo samples (L and XL), though it shows notable steepening for lighter samples, accompanied by increased error bars.

As Fig. 3.16 shows, profiles with higher p values tend to have fewer sampling points at inner radii. In the light mass bins (S and M), this limited sampling affects the resolved inner slope, which explains the systematic decrease in fitted values of α for these bins as p increases. To quantitatively assess the effect of reduced sampling on the determination of α , Fig. 3.18 focuses on the stacked halos for $p = 5$ in the M mass bin. Here, we manually reduce the number of inner sampling points and repeat the fitting procedure used in Fig. 3.16. The results reveal that removing inner points systematically shifts the best-fit values shown in the figure legend, leading to an apparent steepening of the inner slope α .

To mitigate bias in slope estimation, we explore fitting functions where α and β are set to fixed values. Fig. 3.19 displays the reduced χ^2 for different fit configurations across mass bins (S to XL) and values of p (even numbers from $p = 4$ to 40). In the left panel, only β is free while α is fixed, and the reduced χ^2 is plotted against values of α . For profiles with lower p ($\lesssim 10 - 15$), reduced χ^2 tends to minimize around $\alpha \sim 1$. For higher p , reduced χ^2 shows no minimum but rises slightly favoring smaller values of α , a trend consistent with Fig. 3.18 and the reduced bins at inner radii. Hence, setting $\alpha = 1$ appears optimal for representing the inner region of multi-stream profiles.

In the right panel of Fig. 3.19, we assess fits where both α and β are fixed, with α held at 1 and reduced χ^2 plotted against β . Results show that the reduced χ^2 generally reaches a minimum around $\beta \sim 8$ across most values of p , independent of halo mass bins.

Universal nature of multi-stream flows

Based on the findings illustrated in Fig. 3.19, we propose the following function to characterize the universal behavior of the radial density profile across streams:

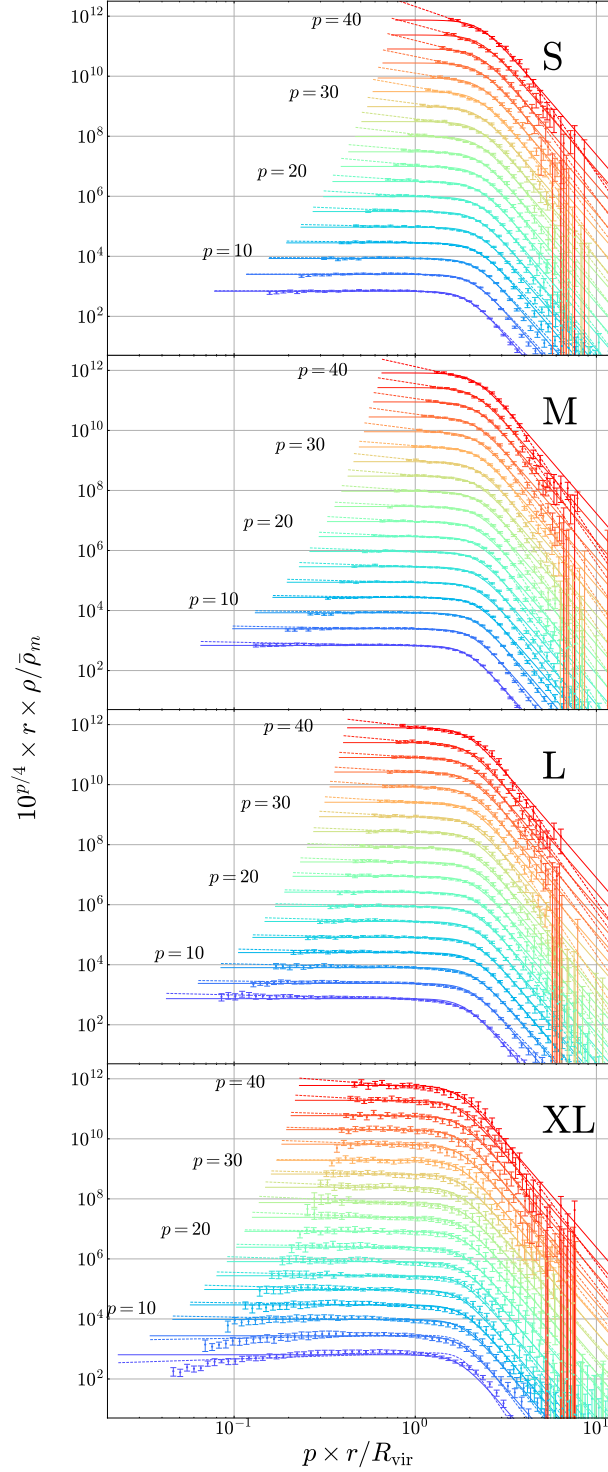


Figure 3.16: Stacked density profiles for $p = 4$ to 40, with markers spaced by every 2 values of p . The error bars represent the root-mean-square deviation of the profiles, normalized by the square root of the number of stacked halos. Dashed lines correspond to the best-fit models where all four parameters in equation (3.8) are varied, while solid lines represent fits with fixed slope parameters (i.e., $\alpha = -1$ and $\beta = -8$; equation 3.9). It is important to note that the vertical axis shows $r \times \rho$, and both axes are rescaled according to p for better visualization. The best-fit parameters for both cases are presented in Figs. 3.17 and 3.20, respectively. This figure is taken from Enomoto et al. (2024a).

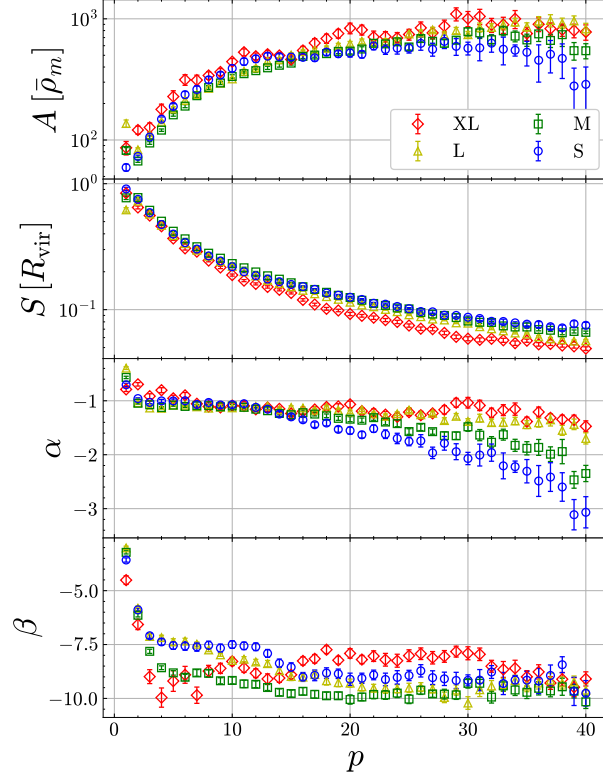


Figure 3.17: Best-fit parameters for the model (3.8) shown in Fig. 3.16 (markers). The error bars represent the square root of the diagonal components of the covariance matrix for the optimized parameters. This figure is adapted from Enomoto et al. (2024a).

$$\rho(r; p) = \frac{A(p)}{\{r/S(p)\} \left[1 + \{r/S(p)\}^7 \right]}. \quad (3.9)$$

Utilizing equation (3.9), the fitting results for individual and stacked profiles are displayed in Figs. 3.13–3.15 and Fig. 3.16, respectively, with each case depicted as solid curves. Overall, equation (3.9) successfully replicates the measured profiles in N -body simulations for large values of p . In particular, the stacked profiles in Fig. 3.16 show strong agreement with the model adopting $\alpha = -1$ and $\beta = -8$, as anticipated.

However, minor discrepancies are observed in the outer slope β for individual halos, which exhibit slightly steeper profiles with $\beta \lesssim -10$. Although the radius is scaled before stacking, the slope after stacking tends to be shallower due to residual halo-specific characteristics, such as the scale $S(p)$. Nevertheless, as demonstrated shortly, the total profile reconstructed by summing the profiles for individual p values is largely insensitive to the precise value of β , provided its absolute value remains high.

Focusing on the stacked profile, another noteworthy observation is the flatter inner profiles at $p \lesssim 10$ in the halo sample XL. This could be attributed to the influence of low- p orbits, which are sensitive to recent mass accretion or merger history Sugiura et al. (2020). Indeed, this effect diminishes after several orbits, with profiles converging to a universal slope of -1 for $p \gtrsim 10$. Thus, we find no compelling evidence against the form in equation (3.9), supporting that each stream’s density profile is universally described by the double-power law form in equation (3.9).

In equation (3.9), the free parameters, characteristic density A and scale S , are determined from fitting analysis on the stacked halo profiles. The following fitting formulas accurately represent the best-fit values of A and S for halos in different mass bins (Enomoto et al., 2023):

$$\log_{10} \left\{ A_{\text{fit}}(p)/\bar{\rho}_{\text{m}} \right\} = 4.89 - 0.119 \log_{10} (M_{\text{vir},10}) + \left\{ -3.89 + 0.243 \log_{10} (M_{\text{vir},10}) \right\} p^{-9/40}, \quad (3.10)$$

$$\log_{10} \left\{ S_{\text{fit}}(p)/R_{\text{vir}} \right\} = 2.46 - 0.0474 \log_{10} (M_{\text{vir},10}) + \left\{ -2.29 - 0.0639 \log_{10} (M_{\text{vir},10}) \right\} p^{1/8}, \quad (3.11)$$

where $M_{\text{vir},10} = M_{\text{vir}}/\{10^{10} h^{-1} \text{M}_{\odot}\}$. Fig. 3.20 compares the best-fit values of characteristic density and scale (symbols) with these fitting formulas (solid curves), showing a good agreement with an explicit yet weak mass dependence as specified by equations (3.10) and (3.11).

To quantitatively examine the double-power law nature of each stream via equation (3.9), we calculate the total density profile from the HR run for halos matched with the LR run. We then compare the stacked profiles for halos in each mass bin with predictions obtained by summing double-power law profiles (equation (3.9)) for $p \geq 1$ as defined by equations (3.9)–(3.11). The results are shown in Fig. 3.21, with upper and lower panels for each mass bin representing the fractional difference of the total density profile relative to the HR run, $(\rho - \rho_{\text{HR}})/\rho_{\text{HR}}$, and the logarithmic slope of the total profile, $d \log \rho / d \log r$, respectively. The solid curves represent predictions from the double-power law model, and the shaded regions indicate uncertainties due to variations in the coefficients in equations (3.10) and (3.11). We conservatively set an upper limit for the summation at $p = 3,000$, noting that stopping at $p = 300$ results in density changes of less than 0.2% over the range shown. For reference, we also plot the NFW profile (Navarro et al., 1997)

$$\rho_{\text{NFW}}(r) = \frac{\rho_s}{r/R_s (1 + r/R_s)^2}, \quad (3.12)$$

and the Einasto profile (Einasto, 1965)

$$\rho_{\text{Einasto}}(r) = \rho_s \exp \left[-\frac{2}{\alpha} \left\{ \left(\frac{r}{R_s} \right)^{\alpha} - 1 \right\} \right], \quad (3.13)$$

which we fit to the HR run profiles over the range $2 \text{Max}(\epsilon_{\text{HR}}/R_{\text{vir}}) \leq r/R_{\text{vir}} \leq 0.9$, where $\text{Max}(\epsilon_{\text{HR}}/R_{\text{vir}})$ represents the maximum ratio of the softening length ϵ_{HR} to R_{vir} , calculated for individual halos in each mass bin. Following Wang et al. (2020), we set $\alpha = 0.16$ in the Einasto profile. Our model agrees well with the HR run for all mass bins, accurately reproducing the profile below $1.2 \text{Max}(\epsilon_{\text{LR}}/R_{\text{vir}})$ (indicated by arrows), which marks the convergence radius where profiles from both runs agree within $\sim 3\%$. This supports the suitability of an inner slope of $\alpha = -1$ even below the LR run's softening scale.

In Fig. 3.21, we consider three variants: (i) a sum of equation (3.8) with fixed $\alpha = -1$ and $\beta = -30$, using characteristic density and scale from equations (3.10) and (3.11) (dotted); (ii) our main model with summation truncated at $p = 40$ (triangles); and

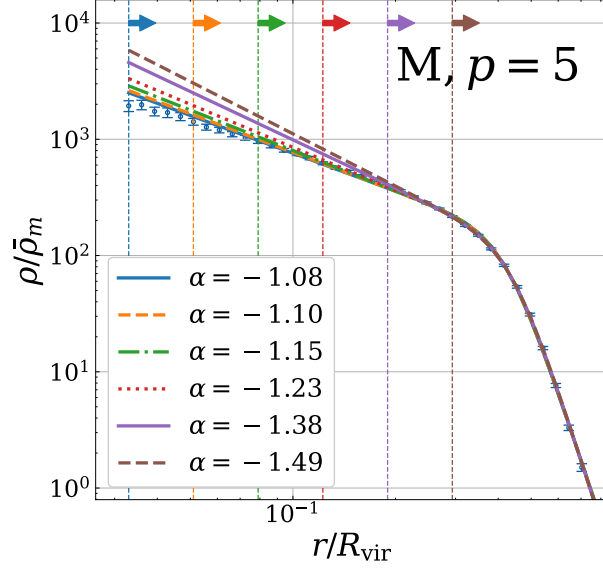


Figure 3.18: The impact of varying the lower limit in the fitting process is illustrated. Vertical dotted lines represent the lower limits, and the corresponding best-fit curves are displayed using matching colors. The legend indicates the best-fit values for the inner slope. This figure is reproduced from Enomoto et al. (2024a).

(iii) a sum of equation (3.8) with parameters obtained from fitting individual profiles in Fig. 3.17 (crosses). Notably, case (i) aligns closely with the solid curves, except at outer radii $r/R_{\text{vir}} \gtrsim 0.2$, implying that the inner profile is robust to variations in β provided it decays rapidly. Case (ii) predicts a declining profile at small radii due to the limited p range, while outer profiles align well with equations (3.9)–(3.11). Lastly, case (iii) agrees with case (ii), further supporting that fixed inner and outer slopes of -1 and -8 are optimal, as seen in Fig. 3.19. Consequently, the fitting forms in equations (3.10) and (3.11) accurately model halo structures, establishing the double-power law form in equation (3.9) as the most effective description among model variants.

As a result, the logarithmic slope displayed in the lower panels for the HR run (dot-dot-dashed) is effectively matched by our model (solid). Notably, the Einasto profile (dot-dashed) also aligns well with the results of the simulation. We anticipate that the two models will diverge asymptotically at the small scale limit (i.e., -1 vs 0), a distinction that cannot be explored with the current simulation due to its limited dynamic range. Addressing the slope at even smaller scales is reserved for future work.

3.5 Dependence on halo samples

So far, we have analyzed individual stream profiles for mass-selected halo samples, characterizing their properties using the double-power law profile. To test the robustness of these results, we now focus on a subset of 460 halos within a specific mass range of $[4.10 \times 10^{11}, 2.39 \times 10^{12}] h^{-1} M_{\odot}$. This sample is divided into two subgroups based on distinct criteria. The first criterion is the concentration parameter, c_{vir} , defined as the ratio R_{vir}/R_s . Here, R_s is the scale radius of the NFW profile (equation (3.12)), estimated using ROCKSTAR from the maximum circular velocity (Klypin et al., 2011). The second

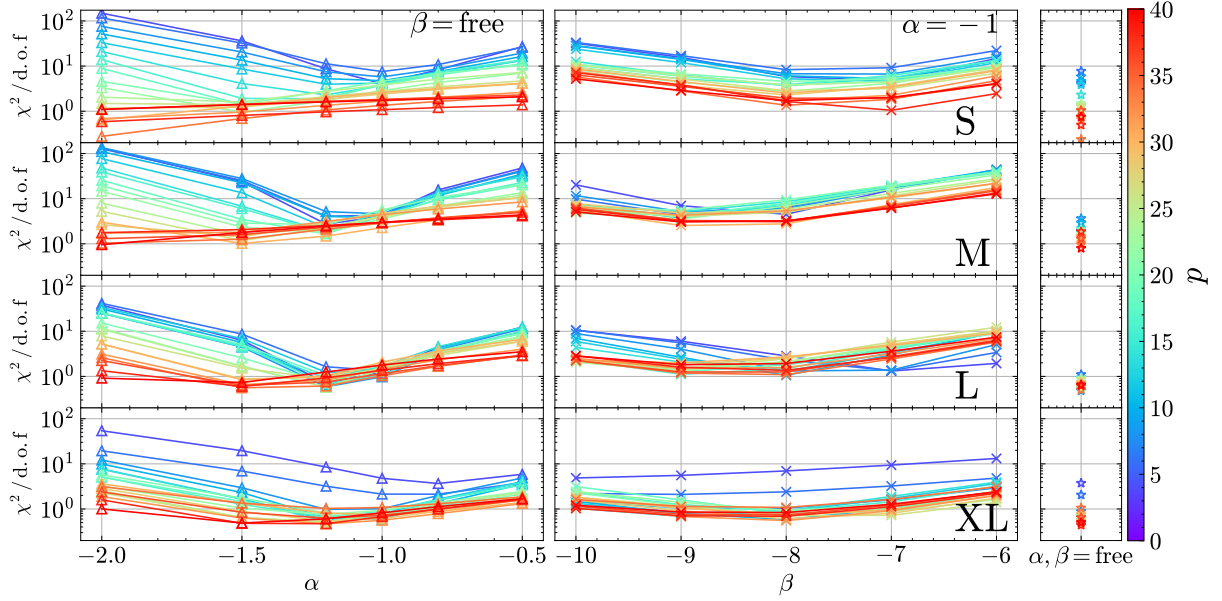


Figure 3.19: Dependence of χ^2 on the slope parameters for each density profile, with values every 2 between $p = 4$ and 40 as indicated by the color bar. In the left column, the inner slope α is fixed at the value on the horizontal axis, and the other three parameters in equation (3.8) are optimized. In the middle column, we set the inner slope α to -1 and examine how χ^2 depends on the outer slope β , optimizing the two remaining parameters. The right column shows the minimum χ^2 when all four parameters are varied. This figure is reproduced from Enomoto et al. (2024a).

criterion involves the mass accretion rate, which is defined as:

$$\Gamma_{\text{dyn}}(t) \equiv \frac{\log \left[M_{\text{vir}}(t) - M_{\text{vir}}\{t - t_{\text{dyn}}(z)\} \right]}{\log \left[a(t) - a\{t - t_{\text{dyn}}(z)\} \right]}, \quad (3.14)$$

where t_{dyn} denotes the dynamical time, computed from halo masses as in (Diemer, 2017)¹:

$$t_{\text{dyn}}(z) \equiv \frac{2R_{\text{vir}}}{V_{\text{vir}}} = H(z)^{-1} \left\{ \frac{8}{\Delta_{\text{vir}}(z)\Omega_m(z)} \right\}^{\frac{1}{2}}, \quad (3.15)$$

yielding a value of 4.04 Gyr at $z = 0$ in our case.

In both cases, the halos are divided into two groups: those with high values and those with low values of c_{vir} or Γ_{dyn} . The parameter distributions of c_{vir} , Γ_{dyn} , and M_{vir} are shown in Fig. 3.22, where the darker points indicate halos selected for this analysis. The sample boundaries are marked by red lines in the upper and lower left panels. A clear trend emerges: c_{vir} and Γ_{dyn} are strongly correlated, but inversely. Higher concentration halos tend to exhibit smaller Γ_{dyn} values. Consequently, we anticipate that the two selection methods will influence the results similarly.

The density profiles for these subgroups are shown in Fig. 3.23, with panels representing c_{vir} (upper) and Γ_{dyn} (lower). Profiles corresponding to low and high c_{vir} or Γ_{dyn} are displayed in red and black, respectively. For each p , the stacked profiles are fitted to the double-power law form (equation (3.9)), and the best-fit results are represented by dotted

¹In this study, we use the virial mass M_{vir} to calculate Γ_{dyn} , whereas Diemer (2017) relies on $M_{200\text{m}}$.

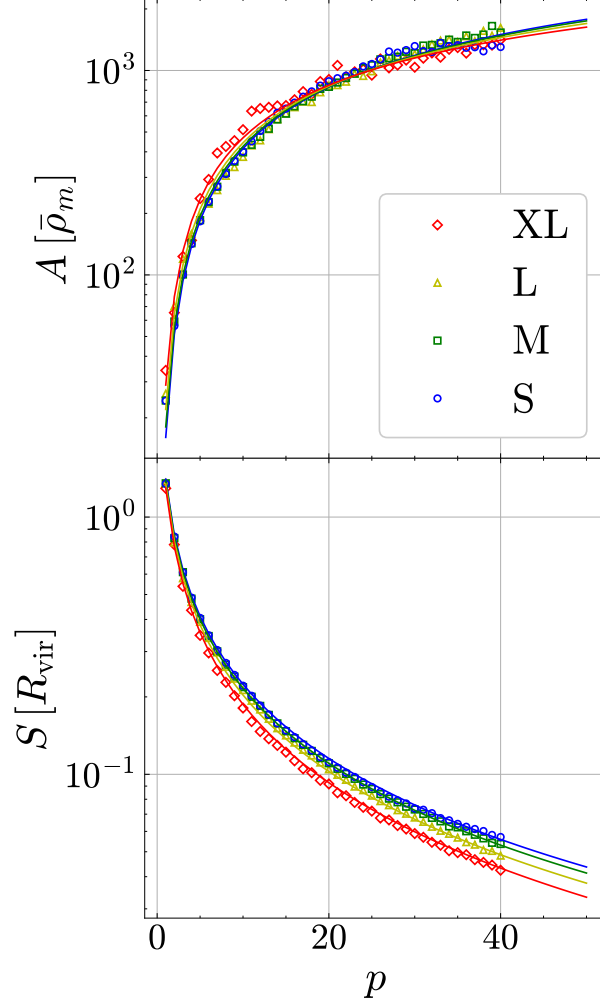


Figure 3.20: Best-fit values of the characteristic density A and scale radius S for the double-power law model with fixed slopes, as given in equation (3.9). The solid lines represent the fitting formulas outlined in equations (3.10) and (3.11), with the halo mass M_{vir} estimated from the average mass of each halo sample. Error bars are omitted for clarity, but they are generally similar to those in Fig. 3.17. This figure is reproduced from Enomoto et al. (2024a).

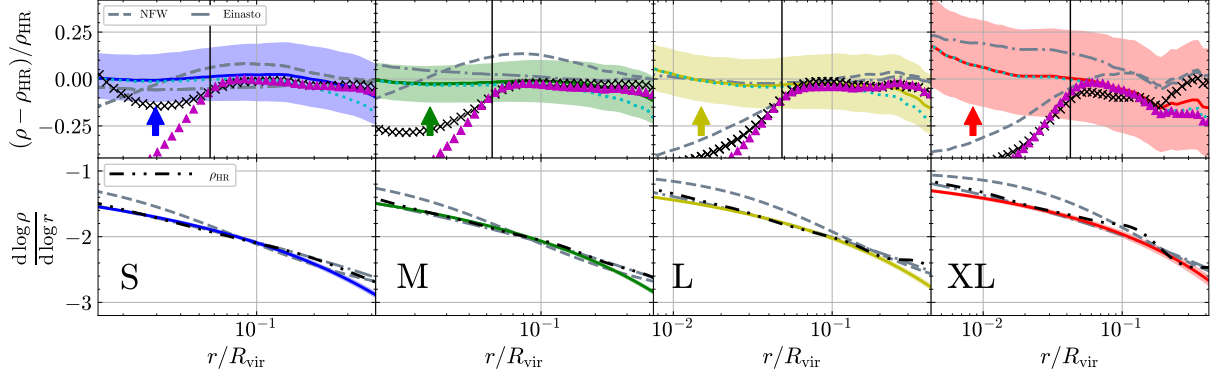


Figure 3.21: Comparison between the sum of fitting functions based on the LR run and the total profile in the HR run. The upper panels display fractional differences between the two. Solid lines represent the results when applying the model (3.10) and (3.11) for the p dependence of the parameters $A(p)$ and $S(p)$, assuming fixed slopes ($\alpha = -1$ and $\beta = -8$). Cyan dotted lines correspond to the same model but with $\beta = -30$. Magenta triangles represent the same model as the solid lines, but the summation is truncated at $p = 40$. The double-power law model, fitted with four free parameters to the measured profiles, is shown by cross symbols, also truncated at $p = 40$, as fitting the data at higher p values is not feasible. Additionally, the NFW profile (dashed) and the Einasto profile (dot-dashed) are shown. The lower panel shows the logarithmic slope of the total profile (dot-dot-dashed lines for the HR run). The solid, dashed, and dot-dashed lines correspond to those in the upper panels. This figure is taken from Enomoto et al. (2024a).

and solid curves for the two subgroups. Once again, the double-power law function closely matches the measured profiles across a broad range of p and radii.

A detailed inspection of each stream profile in Fig. 3.23 reveals that halos with high concentrations or low accretion rates tend to exhibit larger values of both the amplitude $A(p)$ and the characteristic scale $S(p)$, particularly for $p \gtrsim 14$. The contrasting trends in the subsamples divided by c_{vir} and Γ_{dyn} can be attributed to their inverse correlation, as illustrated in the lower right panel of Fig. 3.22. Interestingly, the finding that high-concentration halos exhibit larger $S(p)$ might initially seem counter-intuitive, given that a high c_{vir} inherently implies a smaller scale radius R_s . However, the enclosed mass within $r \leq R_s$ actually increases with c_{vir} . Since the inner halo mass is predominantly determined by the sum of stream profiles at larger p values, and each stream mass is proportional to $A(p)S(p)^3$, the high mass concentration naturally results in larger $A(p)S(p)^3$, as seen in Fig. 3.23².

The results presented in Figs. 3.23 suggest that the double-power law behavior of radial stream profiles is a generic feature, regardless of halo sample selection. While the specific

²Assuming the double-power law form of equation (3.9), the stream profile mass $M(p)$ can be expressed analytically as:

$$\begin{aligned} M(p) &= \int_{r=0}^{r=+\infty} \frac{4\pi r^2 A(p)}{\{r/S(p)\} \left[1 + \{r/S(p)\}^7\right]} dr \\ &= \frac{4\pi}{49} \left(2 \sin \frac{\pi}{7} - \cos \frac{\pi}{14} + 3 \cos \frac{3\pi}{14}\right) A(p) S(p)^3, \end{aligned}$$

yielding $M(p) \approx 0.574 A(p) S(p)^3$.

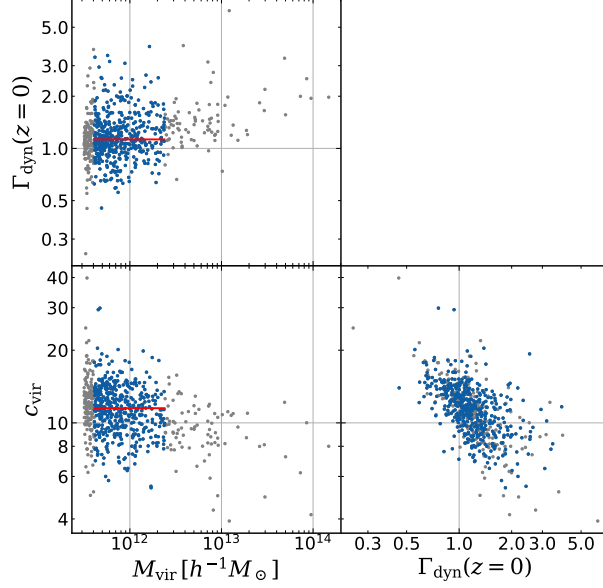


Figure 3.22: Distribution of concentration c_{vir} , halo mass M_{vir} , and mass accretion rate Γ_{dyn} at $z = 0$. Thick markers represent 460 halos within the mass range $[4.10 \times 10^{11}, 2.30 \times 10^{12}] h^{-1} M_{\odot}$, which are analyzed in Section 3.5. The thin markers correspond to the other halos in our catalog. The horizontal red lines denote the median Γ_{dyn} and c_{vir} values for the thick markers, which are used as boundaries for the 460 sample halos. Here, c_{vir} is calculated from R_s , which is estimated by ROCKSTAR. This figure is taken from Enomoto et al. (2024a).

values of $A(p)$ and $S(p)$ vary depending on the selection criteria, their p -dependence can be mapped between samples once their relationships are understood. In this context, it may be beneficial to rederive the fitting functions for $A(p)$ and $S(p)$ in terms of c_{vir} rather than M_{vir} , as the concentration parameter closely correlates with the inner structure of halos.

To achieve this, we further subdivide the halos by c_{vir} into four groups corresponding to the 10–30, 30–50, 50–70, and 70–90 percentiles, each containing an equal number of halos. We then derive new fitting functions for $A(p)$ and $S(p)$ as functions of c_{vir} . The following formulas provide a good fit to the data:

$$\log_{10} \left\{ A_{\text{fit}}(p) / \bar{\rho}_{\text{m}} \right\} = 4.09 + 0.133 c_{\text{vir}} - \left\{ 1.99 + 0.202 c_{\text{vir}} \right\} p^{-4/25}, \quad (3.16)$$

$$\log_{10} \left\{ S_{\text{fit}}(p) / R_{\text{vir}} \right\} = 1.69 + 0.0132 c_{\text{vir}} - 1.74 p^{4/25}. \quad (3.17)$$

The performance of these new fitting formulas is evaluated in Fig. 3.24, where the total density profile and its logarithmic slope are plotted for the four subgroups. Similar to the mass-selected case in Fig. 3.21, the sum of the profiles, using parameters from equations (3.16) and (3.17), accurately reproduces the total profile measured from the HR run, even beyond the LR run’s resolution limit used for calibrating the fits.

3.6 Comparison with self-similar solutions

The findings presented in Section 3.4 and 3.5 reveal that the inner structure of halos shares consistent features, with each stream profile well described by a simple double-power law.

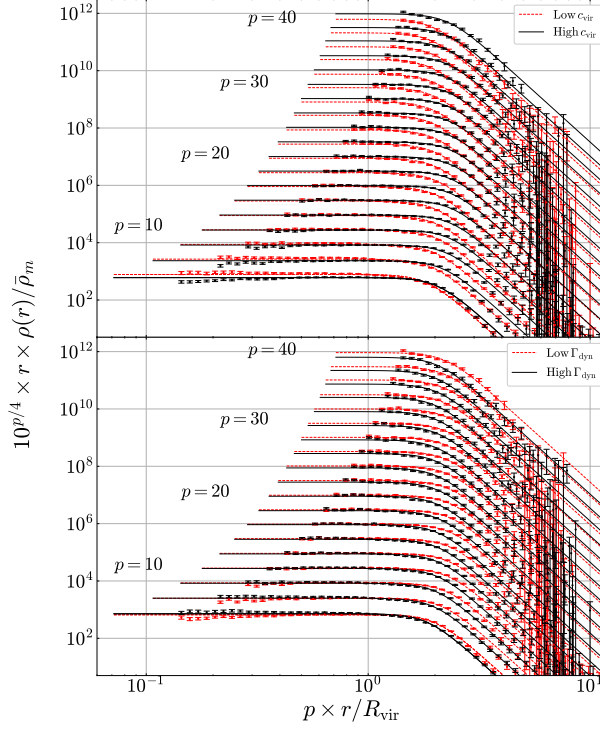


Figure 3.23: Stacked density profiles for $p = 4$ to 40 for every 2 (markers) of subgroups Low c_{vir} (dotted in upper panel), High c_{vir} (solid in upper panel), Low Γ_{dyn} (dotted in lower panel) and High Γ_{dyn} (solid in lower panel). Same as Fig. 3.16, the vertical axis denotes $r \times \rho$ and both axes are rescaled according to p for clarity. This figure is taken from Enomoto et al. (2024a).

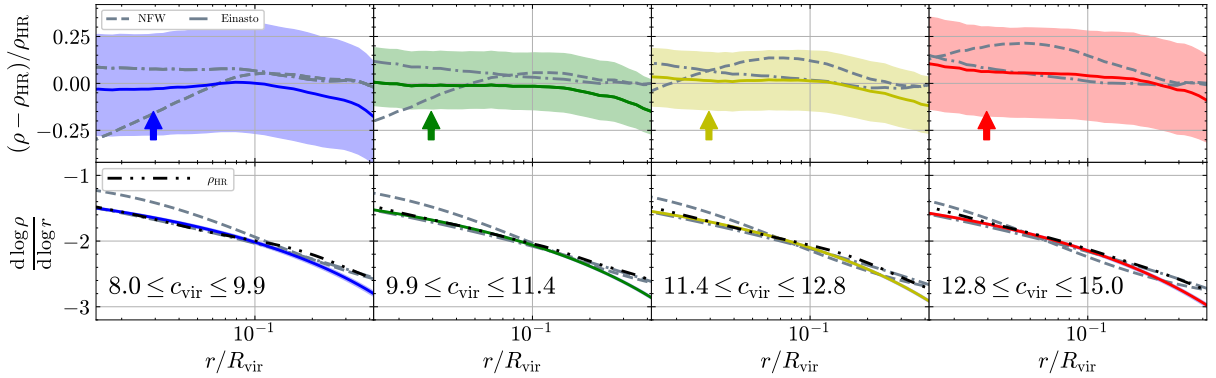


Figure 3.24: Same as in Fig. 3.21, but we use equations (3.16) and (3.17) for $A(p)$ and $S(p)$. In the upper panels, fractional differences are shown between the total profiles from the HR simulation and the sum derived using equation (3.9), with $A(p)$ and $S(p)$ given by equations (3.16) and (3.17) (solid line). Additionally, the best-fit NFW (dashed line) and Einasto (dot-dashed line) profiles are presented for the total HR profile. Vertical arrows mark the resolution limit of the LR simulation. The shaded regions represent the estimated uncertainties of the solid line, which are propagated from the statistical error in the stacked profile and the uncertainties in the fitting formulas of (3.16) and (3.17). This figure is taken from Enomoto et al. (2024a).

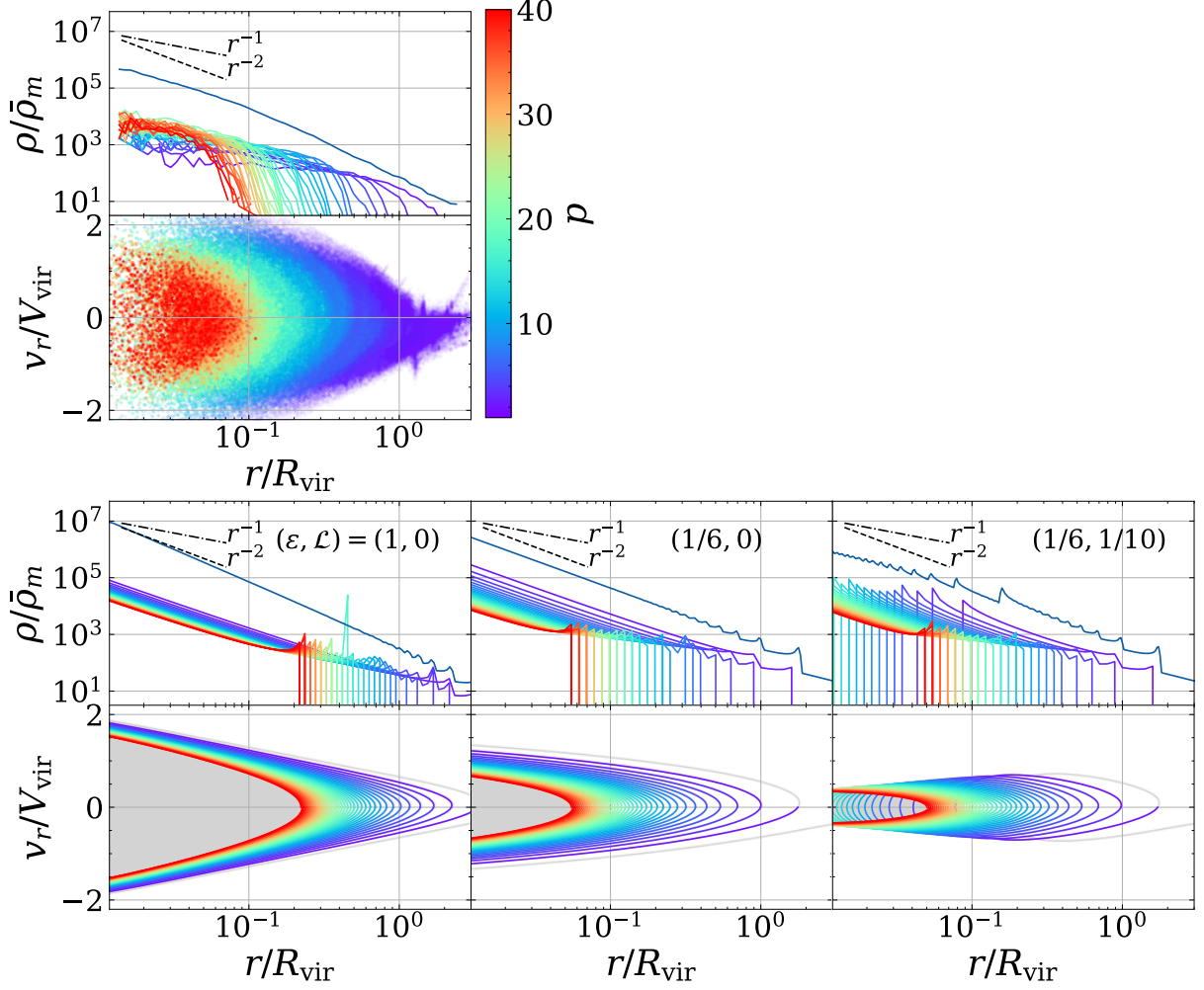


Figure 3.25: The density profiles and phase-space distributions of particles, grouped by p and color-coded as indicated by the color bar, are shown, along with the total density profile (blue solid line). As a representative example of an N -body halo, we display the distributions of the same halo shown in Fig. 3.14 in the top row. It is important to note that the virial overdensity Δ_{vir} is $18\pi^2$ in the EdS universe (the background metric for self-similar solutions), differing from the Λ CDM universe value of 313 at $z = 0$. In this analysis, we use $\Delta_{\text{vir}} = 313$ and normalize the coordinates in the self-similar solution, which does not affect the shape of the density profiles. These figures are taken from Enomoto et al. (2024a).

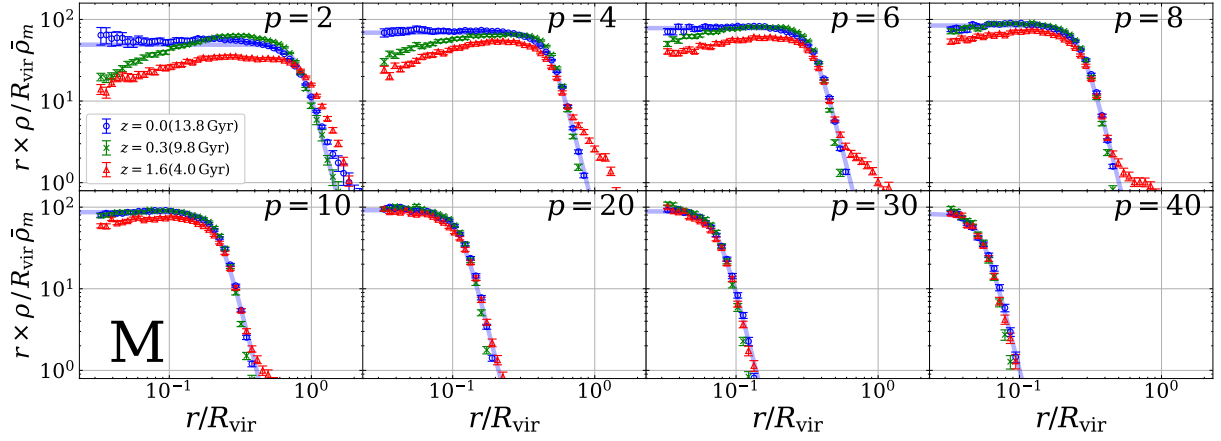


Figure 3.26: Stacked density profiles for the halo sample M at redshifts $z = 0$ (blue), $z = 0.3$ (green), and $z = 1.6$ (red) are shown. Each profile consists of identical particles categorized by p at $z = 0$. Additionally, the best-fit curve of equation (3.9) is displayed for the measured profiles at $z = 0$ as thin blue lines. The radial coordinates at each redshift are expressed in physical distance units, normalized by R_{vir} at $z = 0$. This figure is taken from Enomoto et al. (2024a).

This behavior, observed across different halos, suggests a self-similar evolution process. To explore this further, we compare the radial phase-space structure and density profiles of individual streams with predictions from self-similar solutions.

Strictly speaking, self-similar solutions are applicable only in an Einstein-de Sitter universe. Consequently, directly comparing these solutions to simulations within the Λ CDM framework lacks mathematical rigor. Nonetheless, the secondary infall model proposed by Bertschinger (1985), which corresponds to the self-similar solution of Fillmore & Goldreich (1984) for a particular parameter set (discussed below), has demonstrated success in replicating the $Q(r) \propto r^{-1.875}$ power-law slope of the pseudo-phase-space density observed in Λ CDM simulations.

In light of this, we aim to identify qualitative similarities between simulation results and self-similar predictions. For this comparison, we adopt the spherically symmetric solutions developed by Fillmore & Goldreich (1984) and Sikivie et al. (1997) (see also Nusser (2001)) as representative models for halos under stationary matter accretion. Unlike the former, the latter model incorporates non-zero angular momentum within the infalling matter. These self-similar solutions are characterized by two key parameters. The first is the slope of the initial linear overdensity, parameterized as $\delta_{\text{lin}}(r) \propto r^{-3\epsilon}$, with ϵ spanning $0 \leq \epsilon \leq 1$. This parameter links to the mass accretion rate via $M_{\text{ta}}(t) \propto \{a(t)\}^{1/\epsilon}$, where M_{ta} denotes the mass enclosed within the turn-around radius. The second parameter is the dimensionless angular momentum $\mathcal{L}$, defined as $\mathcal{L} \equiv L/(GM_*r_*)^{1/2}$. Here, r_* and M_* represent the turn-around radius and the total mass within it for each spherical shell accreting onto the halo.

The lower panels of Fig. 3.25 display the radial density profiles (upper row) and phase-space distributions (lower row) derived from the self-similar solutions for specific parameter choices: $(\epsilon, \mathcal{L}) = (1, 0)$ (left), $(1/6, 0)$ (middle), and $(1/6, 1/10)$ (right). Notably, the middle solution corresponds to the secondary infall model of Bertschinger (1985) mentioned earlier. These solutions were obtained numerically following the method outlined in Zukin & Bertschinger (2010). For comparison, the upper panels also include the density

profile of a representative halo from Fig. 3.14.

Due to spherical symmetry, the radial phase-space structure of the self-similar solution displays distinct streams, with all the examined cases exhibiting a qualitatively similar trend in their outer regions. Specifically, each stream features a spiky density peak around the caustics, followed by a sharp decline. Conversely, the inner regions of self-similar solutions typically follow a power-law behavior. For scenarios with zero angular momentum, both the total and individual stream density profiles are predicted to have slopes close to -2 . However, for the $(\epsilon, \mathcal{L}) = (1, 0)$ case, the asymptotic slopes deviate slightly from -2 : the total density scales as $\rho_{\text{tot}} \propto r^{-9/4}$, while each stream scales as $\rho \propto r^{-15/8}$ ³.

Although these predicted steep slopes align with theoretical expectations, they contrast starkly with the slopes observed in simulations. For $(\epsilon, \mathcal{L}) = (1/6, 1)$, introducing non-zero angular momentum creates a central potential barrier, yielding finite pericenter radii. This introduces a sharp inner cutoff for each stream and a total profile shallower than -2 , more consistent with simulation results. However, the individual stream profiles still maintain steep slopes near -2 , conflicting with the shallower inner slopes seen in simulations.

These discrepancies imply that the self-similar solutions considered here overlook critical elements or complexities intrinsic to cosmological N -body simulations. Addressing these limitations may require relaxing symmetry assumptions (Ryden, 1993; Lithwick & Dalal, 2011) or accounting for non-zero tidal torques (Zukin & Bertschinger, 2010). Additionally, the angular momentum distribution is likely a crucial factor. While the self-similar solutions examined incorporate angular momentum, they do so in a constrained manner. Allowing for a broader distribution could alter particle trajectories, potentially producing stream profiles more consistent with simulations. Breaking self-similarity, as explored by Lu et al. (2006), may also yield valuable insights. We defer these investigations to future work.

3.7 On the emergence of double-power law nature

For a deeper understanding of the origin of the universal double-power law nature, we focus on halo sample M from Table 4.4.2 and select particles with apocenter passage counts $p = 2, 4, 6, 8, 10, 20, 30, 40$ at $z = 0$. These particles are traced back to higher redshifts to analyze the evolution of their density profiles for each p , stacking the profiles across different halos. Importantly, the value of p is defined as the count up to $z = 0$ and

³The asymptotic slopes can be derived from the results of Fillmore & Goldreich (1984) and Nusser (2001), who provided analytical expressions for the total density profile ρ_{tot} in self-similar solutions:

$$\rho_{\text{tot}}(r) \propto \begin{cases} r^{-9\epsilon/(1+3\epsilon)} & (\frac{2}{3} \leq \epsilon \leq 1), \\ r^{-2} & (0 \leq \epsilon \leq \frac{2}{3}), \end{cases}$$

for $\mathcal{L} = 0$, and

$$\rho_{\text{tot}}(r) \propto r^{-9\epsilon/(1+3\epsilon)} \quad (0 \leq \epsilon \leq 1)$$

for $\mathcal{L} \neq 0$. To determine the slope of each stream, consider the interior mass $M_p(r)$ of particles with p apocenter passages within radius r . For stationary halos with equal-mass particles, $M_p(r)$ is proportional to the time $\delta t(r)$ that particles spend inside r , which scales as r/v_r , where $v_r \propto \sqrt{\Phi(r)}$. For solutions where $\epsilon > 2/3$, the potential $\Phi(r)$ diverges as $\Phi \propto r^{(2-3\epsilon)/(1+3\epsilon)}$, leading to $M_p(r) \propto r^{9\epsilon/2(1+3\epsilon)}$ and a stream profile $\propto r^{-3(2+3\epsilon)/2(1+3\epsilon)}$. For $(\epsilon, \mathcal{L}) = (1, 0)$, this results in a slope of $-15/8$.

remains unchanged for profiles plotted at earlier redshifts. Thus, the evolution reflects changes in the density profile for the same set of particles over time.

Figure 3.26 illustrates the profiles at $z = 0.3$ (green) and $z = 1.6$ (red), overlaid with the $z = 0$ profiles from Fig. 3.16 (blue). The amplitude of the profiles increases as redshift decreases. Notably, the evolution of the inner density profiles becomes progressively weaker as p increases, with the $p = 40$ profiles showing convergence, even at the outermost radii.

This observation suggests that the double-power law nature is established early in halo formation and remains stable, with subsequent matter accretion affecting only the outer regions of the density profile. This is evident for particles with $p \leq 6$, whose profiles evolve more significantly in the outer regions. This finding aligns with prior studies showing that accreting matter predominantly accumulates in the outskirts of halos (e.g., Fukushige & Makino, 2001; Zhao et al., 2003b; Wang et al., 2011). It also provides insight into why the characteristic scale $S(p)$ in equation (3.11) decreases with p : particles with larger p were accreted earlier and have since relaxed into the inner halo regions.

This analysis highlights the significance of early halo dynamics in shaping the universal double-power law nature of density profiles. Further investigation into the early stages of halo formation could elucidate the mechanisms underpinning this universal behavior.

3.8 Conclusions

Due to their cold nature, CDM halo inherently exhibit multi-stream regions, where the DM velocity at a given position becomes multi-valued in phase space. A key feature of these regions is the splashback radius, marking the outer boundary of multi-stream regions. This radius has garnered significant attention in both theoretical and observational contexts, serving as a test for the CDM paradigm. However, the phase-space structure beyond the splashback radius remains underexplored, even in N -body simulations.

In this chapter, we focused on the inner structure of CDM halo within cosmological N -body simulations, quantifying the phase-space distribution of DM particles in the multi-stream regions. Utilizing the methodology of Sugiura et al. (2020), an extension of the SPARTA algorithm (Diemer, 2017), we classified DM particles inside halo at $z = 0$ by their number of apocenter passages, denoted by p . Using 1,001 snapshots, we precisely tracked DM trajectories, reliably counting apocenter passages up to $p = 40$ over a halo mass range of $3 \times 10^{11} \leq M_{\text{vir}}/(h^{-1} M_{\odot}) \leq 1 \times 10^{14}$. This classification revealed the multi-stream structure within the splashback radius, allowing detailed analysis of individual stream density profiles. Additionally, by stacking halo across mass bins, we statistically characterized the stream profiles.

Our important findings are summarised as follows:

1. The density profiles of particles having the same number of apocenter passages, p , consistently display a double-power law nature. This consists of an inner region characterized by a shallow cusp with an asymptotic slope close to -1 , transitioning to a steep density decline with a slope of approximately -7 or steeper in the outer region. These characteristics are observed across a broad range of halo masses.
2. The analysis of stacked halo profiles demonstrates that the density profiles of individual streams can be effectively modeled by the fitting function provided in equation (3.9), which features fixed slopes of -1 in the inner region and -8 in the outer

region. The parameters characterizing this function—the characteristic density A and the scale S —are expressed as simple functions of the apocenter passage count p , with only a weak dependence on halo mass (see equations 3.10 and 3.11). Notably, summing the fitting function from equation (3.9) over all values of p yields total density profiles that align well with those measured from the high-resolution (HR) simulation run. This holds even beyond the resolution limit of the low-resolution (LR) run, which was employed to calibrate the A and S parameters. Consequently, our predicted total density profile, derived from equation (3.9), agrees closely with the best-fit Einasto profile, exhibiting an inner cusp slope in the range -2 to -1 , which is slightly steeper than that of the NFW profile.

3. The double-power law nature of each stream profile persists across various halo selection methods, extending beyond those selected solely by mass. Although the functional form of the profile universally adheres to equation (3.9), the parameters governing this form—the characteristic density A and the scale S , both functions of the apocenter passage count p —display a clear dependence on the selection criteria. For instance, recalibration of these parameters for halo samples categorized by the concentration parameter c_{vir} reveals distinct trends. The corresponding fitting formulas for A and S in this context are presented in equations (3.16) and (3.17), respectively.
4. A comparison between a class of self-similar solutions, which model the stationary accretion of dark matter (DM) under spherical symmetry, and our simulation results reveals significant discrepancies in the radial multi-stream structure. Notably, the self-similar solutions predict asymptotic slopes for the stream profiles that are steeper than those obtained from simulations, conflicting with the model in equation (3.9). This discrepancy persists even when non-zero angular momentum is incorporated, underscoring the importance of accounting for the dynamical complexities associated with halo accretion and merger histories or relaxing the strict symmetry assumptions.
5. However, by tracing the trajectories of DM particles with a fixed number of apocenter passages p (as determined at $z = 0$) back to higher redshifts, we observe that their density profiles exhibit early convergence, particularly for streams with larger p . This aligns with findings from earlier numerical studies, suggesting that the double-power law nature of the stream profiles is established during the initial accretion phases and remains robust over time.

The universal characteristics of halo identified in this chapter stem directly from the cold nature of cold dark matter (CDM) and provide crucial insights into the intrinsic properties of CDM halo. While our study leveraged N -body simulations to examine the inner multi-stream structure up to $p = 40$, recent advancements in simulating collisionless self-gravitating systems via the Vlasov-Poisson equations present a promising avenue for further exploration of phase-space structures (Yoshikawa et al., 2013; Hahn & Angulo, 2016; Sousbie & Colombi, 2016). These approaches could offer deeper insights into the underlying physics of the universal features.

Of particular interest is the so-called prompt cusp—an inner cusp of proto-halo with a density slope of -1.5 , which forms quasi-instantaneously. This feature, initially identified by Ishiyama et al. (2010) and recently examined in detail by Delos & White (2023), has been the subject of extensive study (see also Anderhalden & Diemand, 2013; Ishiyama,

2014; Ogiya et al., 2016a; Angulo et al., 2017; Ishiyama & Ando, 2020; Colombi, 2021; Ondaro-Mallea et al., 2023). Since the prompt cusp can survive until the present epoch, it serves as a potentially dominant site for dark matter annihilation radiation (Delos & White, 2022). Therefore, a comprehensive theoretical analysis of the prompt cusp could significantly impact indirect dark matter searches, particularly through observations of annihilation radiation such as gamma-ray excess (Delos et al., 2023). In Chapter 5, we will explore the phase-space structure through an analysis based on radial action J_r .

To explore observational evidence of these universal properties, understanding the role of baryonic feedback, especially in galactic halo, is critical. Feedback mechanisms from active galactic nuclei (AGN) or supernovae are known to reshape the inner density structure, potentially modifying the phase-space multi-stream configuration. Quantitative assessments using simulations that incorporate galaxy formation processes (e.g., Sawala et al., 2016; Springel et al., 2018) would be instrumental in evaluating the robustness of the universal features identified in dark-matter-only simulations. Notably, recent hydrodynamical simulations suggest that the phase-space structure of dark halo can be inferred from that of stellar halo (Genina et al., 2023). A detailed investigation of the relationship between dark and stellar components could lay the groundwork for observational tests of the universal features revealed here.

Lastly, it is worth emphasizing that our method for uncovering multi-stream structures is broadly applicable to other particle-based simulation data. This adaptability allows for the examination of radial phase-space structures in alternative dark matter models, such as warm dark matter (WDM) and self-interacting dark matter (SIDM) (e.g., Stücker et al., 2020; Banerjee et al., 2020; Stücker et al., 2022; Correa et al., 2022). Given that different dark matter models influence small-scale structure formation in distinct ways (e.g., Bullock & Boylan-Kolchin, 2017, for a review), variations in the inner multi-stream structures could serve as powerful observational probes to discern the nature of dark matter. Future work will focus on applying our method to these models to deepen our understanding of dark matter’s fundamental properties. In Chapter 4, we shall investigate the phase-space structure of WDM halos.

Chapter 4

Inner structure of CDM and WDM halos

4.1 Introduction

As discussed in previous chapters (e.g., Section 2.4), cosmological N -body simulations have played a pivotal role in providing robust predictions about halo structures, which serve as a foundation for observational verification of dark matter models. These simulations have successfully reproduced the observed large-scale structures in the universe, such as the galaxy distribution at the local scale (e.g., Springel et al. (2006)).

A key prediction of N -body simulations is the universal radial density profile of CDM halos, known as the Navarro-Frenk-White (NFW) profile (Navarro et al., 1996, 1997, 2010; Wang et al., 2020) (see also Section 2.4). Due to the cold and collisionless nature of CDM, the collapse of initial density fluctuations is represented as a folding three-dimensional sheet of dark matter distribution in a six-dimensional phase space, which is governed by the Vlasov-Poisson equation (e.g., Colombi (2021)).

Consequently, the internal structure of halos is composed of multi-folded three-dimensional sheets, with multi-valued velocity flows at each position, referred to as the multistream region (More et al. (2015), see also Diemer & Kravtsov (2014) and Sections 2.3.4 and 2.4.6). The universality of the NFW profile is a fundamental characteristic of this multistream region in CDM halos. Additionally, the multistream region exhibits another universal feature: the pseudo-phase space density follows a simple power law with respect to radius (Taylor & Navarro, 2001; Ludlow et al., 2011).

Despite numerous efforts to explain the origins of these universalities through various arguments, a comprehensive theoretical understanding based on fundamental principles is still lacking. Syer & White (1998) and Nusser & Sheth (1999) proposed that the central cusps of CDM halos result from the hierarchical clustering of substructures, with the slope index being related to the spectral index of the linear power spectrum. In contrast, Wang & White (2007) showed that the NFW profile also appears in halos that form without hierarchical clustering. More recent studies suggest that merger processes play a significant role in shaping the central density profile, driving it towards universality (Ogiya et al., 2016b; Angulo et al., 2017; Delos & White, 2023).

Another approach to deriving halo density profiles extends the maximum entropy theory initially proposed by Lynden-Bell (1967), aiming to explain the apparent insensitivity of collisionless self-gravitating systems to their initial conditions (Hjorth & Williams (2010); Pontzen & Governato (2013), see also Williams et al. (2010, 2014); Williams &

Hjorth (2022)). While the origin of the universality of the pseudo-phase space density slope remains unclear and is sometimes regarded as a fluke (Arora & Williams (2020), see also Lapi & Cavaliere (2011); Nadler et al. (2017); Colombi (2021)), the power law behavior has been observationally confirmed in galaxies and galaxy clusters (e.g., Biviano et al. (2013); Chae (2014); Biviano & Mamon (2023)).

Motivated by these studies, we have explored the phase-space structure of the multi-stream region using particle trajectories, as discussed in Chapter 3 and Enomoto et al. (2023, 2024a). Our approach extends the method introduced in Sugiura et al. (2020) (see also Diemer (2022, 2023, 2024) for similar, but distinct methods). In these works, we track particle trajectories starting from $z = 5$, count their number of apocenter passages p to characterize the multi-stream flows and investigate the radial density profile of particles classified by p .

Interestingly, we found that the radial density profiles of particles classified by the value of p universally follow a double power law, with inner and outer slope indices of -1 and -8 , respectively, independent of halo mass, concentration, or recent mass accretion rate. We also compared our findings with self-similar solutions derived from spherical collapse models (Fillmore & Goldreich, 1984; Sikivie et al., 1997), but none of these models could account for the r^{-1} slope, even when incorporating angular momentum.

Building on these studies, investigating the multistream region of halos in alternative dark matter models may offer valuable insights for distinguishing between different dark matter scenarios. In addition to cold dark matter (CDM), several other dark matter candidates have been proposed, motivated by observational data such as galaxy rotation curves and the number of satellite galaxies (see Bullock & Boylan-Kolchin (2017); Angulo & Hahn (2022) for recent reviews).

One promising candidate among several DM models proposed is warm dark matter (WDM), which is thermally produced in the early universe and has a non-negligible velocity dispersion corresponding to its mass (see Sections. 2.2.3 and 2.5). This results in a free-streaming cutoff in the linear matter power spectrum. Due to this cutoff, WDM halos typically contain fewer subhalos compared to CDM halos, which helps address some of the small-scale challenges faced by the Λ CDM model (e.g., Bode et al. (2001)).

As a result, many studies have been conducted on the structure of WDM halos (Viel et al., 2005; Villaescusa-Navarro & Dalal, 2011; Lovell et al., 2012; Macciò et al., 2012, 2013; Leo et al., 2017; Stücker et al., 2022; Shtanov & Zhdanov, 2024), and their density profiles are also well-fitted by the NFW profile (Lovell et al., 2014). Although the allowed mass range for WDM is becoming increasingly constrained, the current mass limit for WDM ($\gtrsim 5$ keV, Dekker et al. (2022); Villaseñor et al. (2023); Iršič et al. (2024); Keeley et al. (2024)) remains several orders of magnitude lower than that of typical CDM particles (~ 100 GeV), with further constraints expected in the future.

The classification of simulation particles based on the number of apocenter passages has revealed new universal features in phase space that could provide a foundation for testing dark matter models. However, calculating p for real astronomical objects, such as stars, presents significant challenges. To determine p , we require complete information about the trajectory of stars from the moment they accreted onto the halo. This entails a reliable estimate of the past history of the halo’s gravitational potential, in addition to the current positions of stars in the 6-dimensional phase space, which is very difficult to obtain. Therefore, from an observational standpoint, an alternative quantity is needed to characterize the phase-space structure of the multi-stream region.

As a promising candidate with both a strong theoretical foundation and observational

significance, the radial action J_r is a favorable quantity for characterizing the phase-space structure of halos. It is defined by (Binney & Tremaine, 2008)

$$\begin{aligned} J_r &\equiv \frac{1}{2\pi} \oint v_r(r) dr \\ &= \frac{1}{\pi} \int_{r_{\text{peri}}}^{r_{\text{apo}}} \sqrt{2E - 2\Phi(r; t) - j^2/r^2} dr \end{aligned} \quad (4.1)$$

where Φ is the gravitational potential at a given radius r , and $v_r(r)$, E , j , r_{peri} , and r_{apo} are the radial velocity as a function of r , the specific energy, the angular momentum, the pericenter, and the apocenter of the particle, respectively.

Radial action is an adiabatic invariant, meaning it remains conserved if the potential is static or changes slowly relative to the particle’s orbital period. This conservation property makes it useful for classifying stellar streams based on their clustering in action space (e.g., Naidu et al. (2020); Arora et al. (2022); Malhan et al. (2022); Brooks et al. (2024); Malhan & Rix (2024)). Furthermore, radial action plays a crucial role as a conserved quantity in linking the initial linear density field to the resulting radial density profile of objects that collapse from it (e.g., Fillmore & Goldreich (1984); Nusser (2001); Dalal et al. (2010)).

Additionally, Pontzen & Governato (2013) demonstrated that conserving radial action places a constraint on entropy maximization in self-gravitating systems, which enhances the prediction of the phase-space distribution function. However, their model still struggles to accurately describe the inner regions of halos. Burger et al. (2021) also studied the distribution of radial action in halos and developed a theoretical framework that explains the evolution of radial action in a quasi-equilibrium regime. This model accounts for the systematic differences observed in the distribution of radial action between dark matter-only and hydrodynamics simulations, as discussed in Callingham et al. (2020).

In this chapter, we aim to explore the multi-stream regions within both CDM and WDM halos, taking an initial step towards observational tests of dark matter models in phase space. To achieve this, we compute the number of apocenter passages, p , and the radial action, J_r , for particles in simulations. These two variables are then used to classify the particles within halos, and their radial density profiles are analyzed. We observe that the profiles follow a double power law, with an inner slope of -1 for both WDM and CDM halos. Additionally, we show that the inner profiles of these two models have different amplitudes, a difference that can be attributed to the varying distribution of the concentration parameter across halos in the two models.

The structure of this chapter is as follows: In Section 4.2.1, we introduce the simulation data, while the halo catalog is discussed in Section 4.2.2. The methods for counting apocenter passages and estimating radial action are outlined in Sections 4.3.1 and 4.3.2. Section 4.4.1 presents the profiles of individual halos, with stacking analyses following in Section 4.4.2. In Section 4.5, we discuss the similarities between classifications based on p and J_r . The chapter concludes with the summary of our results in Section 4.6.

4.2 Data and halo catalog

4.2.1 N -body simulations

In this chapter, we present results from three different cosmological N -body simulations. One simulation models cold dark matter (CDM), while the other two explore warm dark

Table 4.1: Parameters for N -body simulations. m_{DM} denotes the dark matter mass, k_{peak} the wavenumber at which the dimensionless linear power spectrum $k^3 P(k)/2\pi^2$ has its maximum value, L the size of simulation box per side, N the number of simulation particles, m_p the mass of simulation particles, ϵ the softening length, N_{snaps} the number of snapshots we stored.

Name	m_{DM} [keV]	k_{peak} [$h\text{Mpc}^{-1}$]	L [$h^{-1}\text{Mpc}$]	N	m_p [$10^6 h^{-1} M_{\odot}$]	ϵ [$h^{-1}\text{kpc}$]	N_{snaps}
CDMLR	-	-	20	500^3	5.61	1.2	1000
WDM1	1	4.1	20	500^3	5.61	1.2	1000
WDM05	0.5	2.1	20	500^3	5.61	1.2	1000
CDMHR	-	-	20	1000^3	0.70	0.6	$1(z=0)$

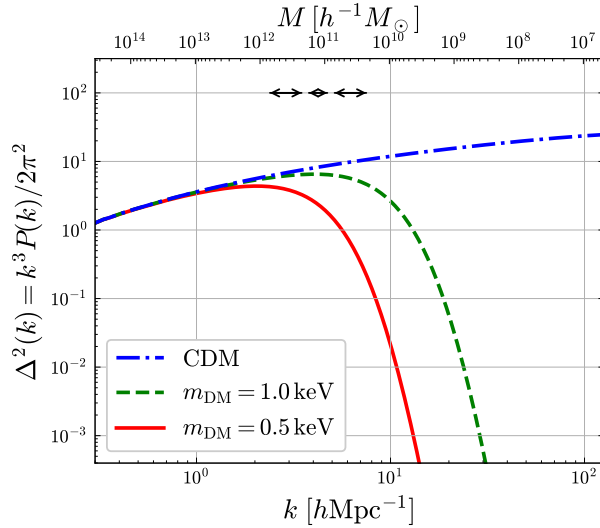


Figure 4.1: The dimensionless linear power spectrum $\Delta^2(k)$ for our simulations. For the two WDM cases, we compute $\Delta^2(k)$ by multiplying the transfer function from Viel et al. (2005) (see Section 4.2.1) with the CDM transfer function computed by the CLASS Boltzmann solver (Blas et al., 2011). The upper x-axis indicates the linear mass corresponding to the wavenumber k , defined as $M(k) = 4\pi/3(\pi/k)^3 \bar{\rho}_m$ (Bullock & Boylan-Kolchin, 2017). Horizontal arrows at the top of the panel mark the three mass ranges used in the stacking analysis. This figure is taken from Enomoto et al. (2024b).

matter (WDM) models with particle masses of 0.5 and 1 keV. All simulations use a periodic comoving box with a side length of $20 h^{-1}\text{Mpc}$ and contain $N = 500^3$ dark matter particles, adopting a flat ΛCDM universe with parameters consistent with Planck data: $\Omega_m = 0.3156$, $\Omega_\Lambda = 0.6844$, $h = 0.6724$, $n_s = 0.9645$, and $A_s = 2.1307 \times 10^{-9}$ (Planck Collaboration et al., 2016b). Additionally, we also conduct a high-resolution CDM simulation (CDMHR) with twice the resolution of the other simulations to assess the convergence radii of density profiles relative to the softening length. Table 4.1 provides the parameter specifications for these simulations.

Although current observational constraints place the WDM particle mass around $m_{\text{DM}} \simeq 5 \text{ keV}$ (Dekker et al., 2022; Villaseñor et al., 2023; Iršič et al., 2024; Keeley et al., 2024), which exceeds the masses considered here, our primary goal is to investigate how internal halo structures depend on initial conditions. This allows us to deepen our understanding of the universal features identified by Enomoto et al. (2023). Thus, even disfavored WDM models provide valuable insights for distinguishing between dark matter paradigms.

The initial conditions are generated using linear power spectra tailored for each dark matter model. Particle displacements and velocities are initialized via second-order Lagrangian perturbation theory (2LPT; Scoccimarro 1998; Crocce et al. 2006). The CLASS Boltzmann solver (Blas et al., 2011) is employed to compute the linear power spectrum for CDM. For WDM, we adopt the transfer function from Viel et al. (2005), based on the formulation by Bode et al. (2001), to modify the CDM power spectrum:

$$\begin{aligned} T(k) &\equiv \sqrt{\frac{P_{\text{WDM}}(k)}{P_{\text{CDM}}(k)}} \\ &= [1 + (\alpha k)^{2\nu}]^{-5/\nu}, \end{aligned} \quad (4.2)$$

where $\nu = 1.12$, and α is approximated by

$$\alpha = 0.049 \left(\frac{m_{\text{DM}}}{1\text{keV}}\right)^{-1.11} \left(\frac{\Omega_{\text{DM}}}{0.25}\right)^{0.11} \left(\frac{h}{0.7}\right)^{1.22} h^{-1}\text{Mpc}. \quad (4.3)$$

We utilize this fitting function since WDM power spectra derived from CLASS exhibit numerical oscillations at relevant scales. The same Gaussian random fluctuations, seeded with identical Fourier phases, are applied across all models to enable direct comparison. Fig. 4.1 illustrates the linear power spectra for each model. The GINKAKU TreePM code (Nishimichi, Tanaka, & Yoshikawa, in prep.) is used to evolve the density field.

As in Enomoto et al. (2024a), we store 1000 snapshots spanning $z = 5$ to $z = 0$ to precisely track particle trajectories and count apocenter passages (see Section 4.3.1). Unlike Enomoto et al. (2024a), who sampled snapshots uniformly in redshift, we adopt uniform sampling in physical time, improving trajectory tracking at low redshifts where orbital periods of particles within the virial radius significantly shorten.

4.2.2 Halo catalog

We identify halos at $z = 0$ using the 6D phase-space temporal friends-of-friends (FoF) halo finder ROCKSTAR (Behroozi et al., 2013). Following Enomoto et al. (2024a), we focus on relaxed halos that meet specific criteria, excluding those undergoing major mergers or classified as subhalos. To filter out halos likely undergoing major mergers, we impose the

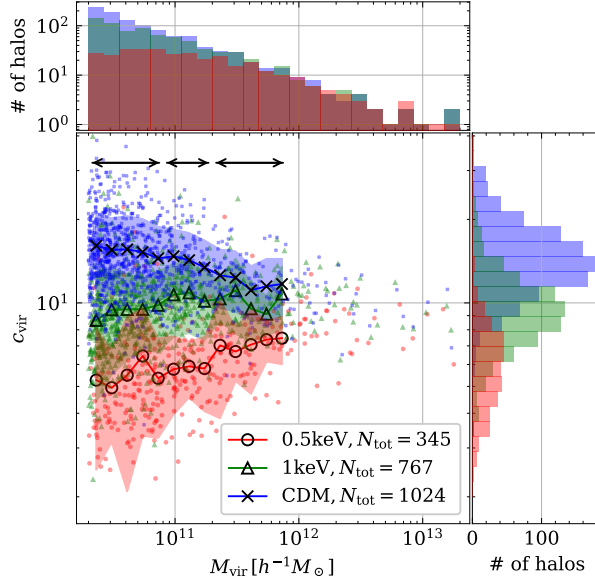


Figure 4.2: The concentration and mass distribution of our sample are shown in the bottom left, along with histograms for virial mass (top) and concentration (bottom right). The colors correspond to different dark matter models, as indicated in the legend. Solid lines and shaded areas represent the mean and standard deviation of concentration at each mass. Horizontal arrows denote the three mass ranges, S, M, and L, which are listed in Table 4.4.2. N_{tot} represents the total number of samples with a mass greater than $2 \times 10^{10} h^{-1} M_{\odot}$. This figure is adapted from Enomoto et al. (2024b).

following conditions from Klypin et al. (2016):

$$\lambda > 0.07, \quad X_{\text{off}} > 0.07, \quad (4.4)$$

where λ is the Bullock spin parameter (Bullock et al., 2001), and X_{off} is the offset parameter. Both are derived from ROCKSTAR for each halo. Halos satisfying equation (4.4) are excluded.

We also exclude subhalos using the criterion:

$$M_{\text{vir,all}} > 1.3 M_{\text{vir}}, \quad (4.5)$$

where $M_{\text{vir,all}}$ represents the total mass of simulation particles within R_{vir} , centered at the halo position determined by ROCKSTAR (Enomoto et al., 2024a). Here, R_{vir} and M_{vir} denote the virial radius and virial mass, respectively, as calculated by ROCKSTAR. For subhalos, the condition $M_{\text{vir,all}} \gg M_{\text{vir}}$ typically holds since particles inside subhalos are often only partially bound to them and remain bound to their host halos. Enomoto et al. (2024a) demonstrated that combining equations (4.4) and (4.5) effectively excludes subhalos.

However, to track the main progenitors of halos and ensure continuous trajectories across snapshots (see Section 4.3.1), the above conditions are insufficient to eliminate halos undergoing major mergers with significant secondary density peaks. Thus, we impose an additional condition, as discussed in Enomoto et al. (2024a):

$$\frac{|\mathbf{x}_{\text{h}} - \mathbf{x}_{\text{h,pro}}|}{R_{\text{vir}}} > 0.1, \quad (4.6)$$

where $\mathbf{x}_h$ denotes the halo center identified by ROCKSTAR, and $\mathbf{x}_{h,\text{pro}}$ represents the center of its main progenitor (defined in Section 4.3.1). This final criterion excludes a few irregular halos from each simulation after applying equations (4.4) and (4.5).

For the WDM cases, the suppression of small-scale power in the initial conditions amplifies the relative importance of N -body particle discreteness. This can significantly influence halo formation scenarios, leading to the formation of numerous spurious low-mass halos (Wang & White (2007), Melott (2007), Power et al. (2016); see also Stücker et al. (2022) and Liu et al. (2023) for recent developments). To reduce the impact of these spurious halos, we exclude halos with masses below the threshold M_{thre} , as proposed by Wang & White (2007):

$$M_{\text{thre}} = 10.1 \bar{\rho}_m d k_{\text{peak}}^{-2}, \quad (4.7)$$

where $\bar{\rho}_m$ is the mean matter density, d is the mean interparticle separation, and k_{peak} is the wavenumber corresponding to the maximum of the dimensionless linear power spectrum $k^3 P(k)/2\pi^2$. Table 4.1 lists the k_{peak} values for our WDM simulations. Based on equation (4.7), we obtain threshold masses of $M_{\text{thre}} = 2.1 \times 10^9 h^{-1} M_\odot$ for WDM1 and $8.0 \times 10^9 h^{-1} M_\odot$ for WDM05. Below these thresholds, the number of identified halos increases significantly. However, this criterion does not remove all spurious halos, as demonstrated by Lovell et al. (2014) (see their Figs. 9 and 10). To further mitigate contamination, we adopt a conservative mass threshold of $M_{\text{min}} = 2.0 \times 10^{10} h^{-1} M_\odot$. Our final halo catalog only includes halos with masses exceeding M_{min} .

To highlight the differences between CDM and WDM halos in our halo catalog, we plot the concentration-mass relation in Fig. 4.2. The top histogram shows the number of WDM halos as a function of M_{vir} , revealing a suppression of low-mass halos compared to CDM. This suppression becomes more pronounced as the dark matter particle mass decreases (e.g., Bode et al. (2001), Bose et al. (2016)). A similar trend appears in the concentration-mass relation, where c_{vir} values from WDM simulations are systematically lower as m_{DM} decreases. This suppression is especially evident for halo masses below the free-streaming scale, consistent with previous studies (e.g., Fig. 4 in Ludlow et al. (2016)). These effects are attributed to the reduced small-scale power in the WDM linear power spectrum, which delays halo formation and alters their accretion and merger histories (e.g., Zhao et al. (2009), Ludlow et al. (2016); see also Angulo & Hahn (2022) for a recent review).

4.3 Methods

In this chapter, we aim to characterize the phase-space structures of DM halos using the dynamical information of DM particles. Specifically, we adopt two different methods to quantify the phase-space properties of halos. The first method involves counting the number of apocenter passages p for each DM particle, as employed in our previous studies (Enomoto et al., 2023, 2024a). This approach, initially developed by Sugiura et al. (2020), extends the algorithm introduced by Diemer (2017) to identify the splashback radius, which corresponds to the radius of the first apocenter passage.

The second method involves evaluating the radial action J_r of DM particles (Binney & Tremaine, 2008). Radial action is a widely used quantity in galactic dynamics (see Helmi (2020), Deason & Belokurov (2024), and Bonaca & Price-Whelan (2024) for recent reviews). Its application in the cosmological context has been explored in previous studies, including Pontzen & Governato (2013) and Burger et al. (2021). In the following sections,

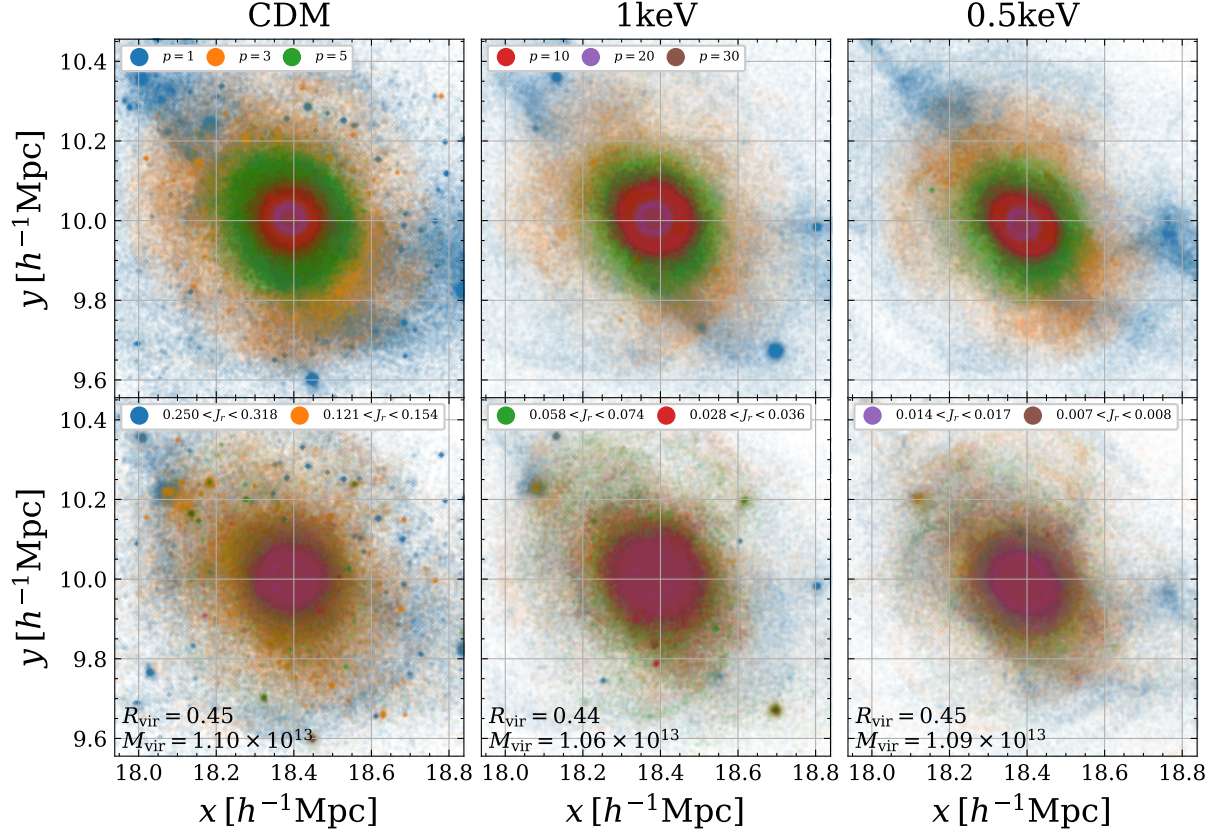


Figure 4.3: The distribution of particles classified by p and J_r is shown for a single halo with a virial mass of $M_{\text{vir}} \sim 1.1 \times 10^{13} h^{-1} M_{\odot}$, using a slice of $0.2 h^{-1}\text{Mpc}$ depth along the z -axis. Particles are categorized into four p values— $p = 1, 3, 5, 10$ (blue, orange, green, and red, respectively)—and are further divided into the four J_r bins as indicated in the legend. The virial radius R_{vir} and mass M_{vir} , shown in the bottom left corner, are given in units of [$h^{-1}\text{Mpc}$] and [$h^{-1}M_{\odot}$], respectively. The J_r values are normalized by the unit of [$R_{\text{vir}} V_{\text{vir}}$]. These figures are adapted from Enomoto et al. (2024b).

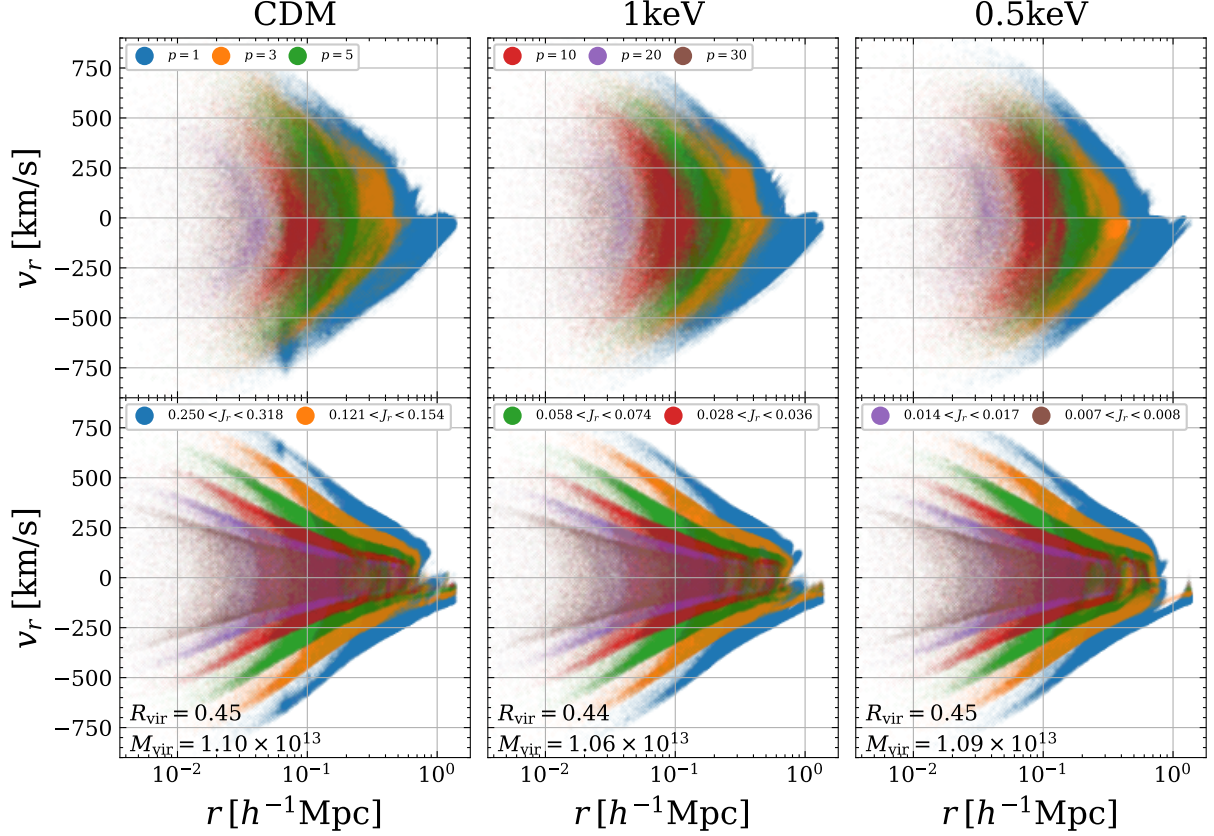


Figure 4.4: The phase-space distribution of particles classified by p (top row) and J_r (bottom row) is displayed for the same halos shown in Fig. 4.3. The plot includes all particles within a radius of $3R_{\text{vir}} = 1.35h^{-1}\text{Mpc}$, color-coded in the same manner as in Fig. 4.3. The leftmost radius in each panel is three times the softening radius. These figures are adapted from Enomoto et al. (2024b).

we provide detailed descriptions of these two methods.

4.3.1 Counting the number of apocenter passages p

The number of apocenter passages, denoted as p in the following, serves as a valuable phase-space indicator for investigating the multi-stream nature of DM flows within halos. This study employs the same approach outlined in Enomoto et al. (2024a) and Section 3.3, utilizing 1000 snapshots spanning $0 \leq z \leq 5$. In essence, we trace the halo centers, defined by their innermost particles, from $z = 0$ backward in time. For each DM particle, the number of apocenter passages p is determined by tracking its trajectory relative to the halo center. The final value of p is then recorded at $z = 0$. A detailed description of this methodology, based on Enomoto et al. (2024a) and Section 3.3, is provided below.

Starting with halos identified by ROCKSTAR at $z = 0$, we follow their main progenitors by tracing the particles within the virial radius back in time. The member particle list is updated using the shrinking sphere method, as explained in Enomoto et al. (2024a) (see their Section 3.1). This backward tracking continues until either the number of member particles falls below 500¹, or the first snapshot at $z = 5$ is reached.

At each snapshot, the bulk velocity and position of the halos, $\mathbf{x}_h$ and $\mathbf{v}_h$, are computed as the average over these 500 particles (i.e., the main progenitor’s position and velocity). Moving forward in time, we track the radial velocity of particles relative to the halo center within $2.5R_{\text{vir}}$ at $z = 0$, starting from the last snapshot where the main progenitor could be identified.

The number of apocenter passages for each particle is counted based on changes in the sign of its radial velocity, from positive to negative, with an additional requirement that its position relative to $\mathbf{x}_h$ changes by at least 90° compared to its last apocenter passage.

4.3.2 Calculating the radial action J_r

In addition to counting the number of apocenter passages, we employ the radial action, J_r , as a measure to characterize the phase-space structure of DM halos. Following the approach outlined in Pontzen & Governato (2013), we numerically compute equation (4.1) for each particle at $z = 0$.

To achieve this, we first calculate the spherically averaged gravitational potential for each halo, defined as

$$\Phi(r; t) = - \int_r^\infty \frac{GM(< r'; t)}{r'^2} dr', \quad (4.8)$$

where $M(< r)$ represents the mass enclosed within a sphere of radius r centered at the halo’s position, $\mathbf{x}_h$. The integration is performed by splitting the range into two regions: $[r, 2.5R_{\text{vir}}]$ and $[2.5R_{\text{vir}}, \infty]$.

For the first region, $[r, 2.5R_{\text{vir}}]$, the integral is computed directly using the N -body simulation data. In contrast, for the second region, $[2.5R_{\text{vir}}, \infty]$, we assume a Keplerian potential generated by a point mass equal to $M(< 2.5R_{\text{vir}})$ and integrate analytically. Although the boundary radius for dividing the regions is arbitrary, modifying it from

¹In Enomoto et al. (2024a), this threshold was set to 1000. Reducing it to 500 enables tracking halos further back in time. A comparison of results using both thresholds shows no significant impact on outcomes for $p < 50$ (see Fig. A2 of Enomoto et al. (2024a)). Even reducing the number of particles to 250 produces consistent results.

$2.5R_{\text{vir}}$ to $2R_{\text{vir}}$ affects the radial actions of only about 1% of particles within $1.5R_{\text{vir}}$, typically leading to deviations of no more than 5% from the fiducial values.

With the spherically averaged potential in hand, we calculate the specific energy per particle, E , defined as $E = v^2/2 + \Phi(r)$, where v is the velocity of the particle after subtracting the bulk velocity of the halo, v_h . Additionally, the specific angular momentum, j , is determined by $j = rv_t$, where v_t is the tangential velocity, expressed as $v_t = |\mathbf{v} - (\mathbf{v} \cdot \mathbf{r})/\mathbf{r}|$. Using these quantities as functions of r , we numerically evaluate the integral in equation (4.1) for each particle. The integration bounds, r_{apo} and r_{peri} , are identified as the zero-crossing points of the integrand. It is important to note that this approach does not apply to unbound particles at $z = 0$, which have positive specific energy. However, our analysis indicates that most particles within the virial radius are bound, with unbound particles comprising less than 1% of the total population.

The method outlined above relies solely on the particle data available at $z = 0$, under the assumption that the particle trajectories are governed by the potential measured at $z = 0$ and are not significantly influenced by other perturbations. As an alternative approach, we utilize the particle trajectories from the 1000 snapshot data to directly compute equation (4.1). Specifically, we extract the radial velocity, v_r , as a function of radius for each particle and integrate accordingly. In appendix 4.3.3, we compare the outcomes of these two methods and evaluate their influence on the inner structure of halos. Importantly, we confirm that the results obtained from the stacked halo analysis in Section 4.4.2 remain robust and are largely unaffected by the choice of method.

4.3.3 Convergence tests of the methods evaluating J_r

In this section, we demonstrate that our results are robust against the method used to compute J_r . As described in Section 4.3.2, we calculate the radial action for each particle by integrating equation 4.1, assuming that particles retain constant energy E and angular momentum j throughout their orbital motion.

To assess the validity of this assumption, we implement an alternative method that directly evaluates J_r using the radial velocities recorded at each simulation snapshot. This approach calculates J_r by tracking the radial velocities of particles between their most recent pericenter and apocenter passages, substituting these velocities into equation 4.1.

It is important to note that the direct evaluation method is limited to particles that have completed at least one apocenter passage (i.e., $p = 1$). As a result, it excludes particles currently on their first orbital traversal between apoapsis and pericenter, even if they are gravitationally bound. Conversely, the integration-based method encompasses all bound particles. This distinction primarily impacts the outer regions of the stacked density profiles, as illustrated in Fig. 4.8.

In Fig. 4.5, we compare the radial action of particles in a representative CDM halo, calculated using the two methods discussed earlier. Specifically, we denote the radial action computed via the method in Section 4.3.2 as J_r^{est} , and the value obtained by integrating the measured radial velocity as J_r^{dir} . Both panels display all particles within $2.5R_{\text{vir}}$ from the halo center as thin grey markers, with the mean and RMS scatter represented by grey solid lines and error bars. In the left panel, we highlight particles with $p \geq 3$ (cyan markers) and show their mean and scatter with cyan error bars. Similarly, the right panel focuses on particles within $1R_{\text{vir}}$ of the halo center (magenta markers), with their mean and scatter indicated by magenta error bars.

The results demonstrate a good agreement between J_r^{est} and J_r^{dir} , as reflected in the

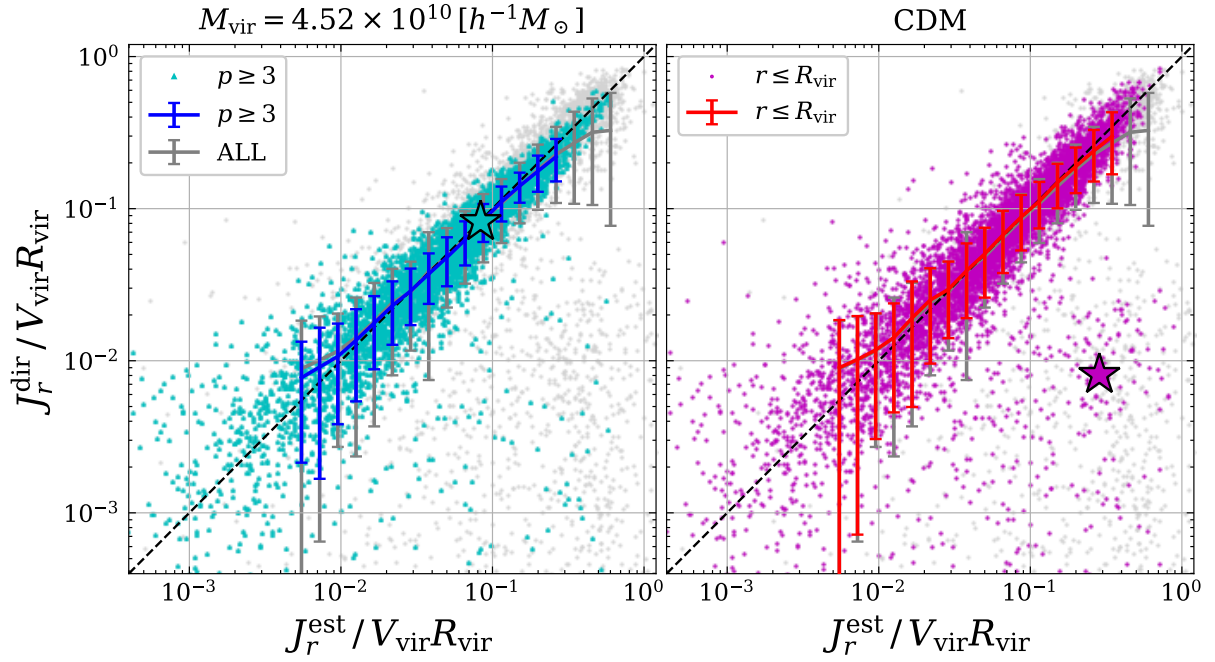


Figure 4.5: Comparison between J_r^{dir} and J_r^{est} of particles in a representative CDM halo. In both panels, we show all the particles within $2.5 R_{\text{vir}}$ as thin grey markers and their mean and RMS scatter as grey solid lines with error bars. Left: particles with $p \geq 3$ (cyan markers) and their RMS scatter (blue error bars) are shown. Right: particles located within $1 R_{\text{vir}}$ from the center of the halo (magenta markers) and their RMS scatter (magenta error bars) are also shown. In both cases, error bars are evaluated in bins with more than 100 particles. The stars correspond to the particles whose trajectories are shown in Fig. 4.6. These figures are taken from Enomoto et al. (2024b).

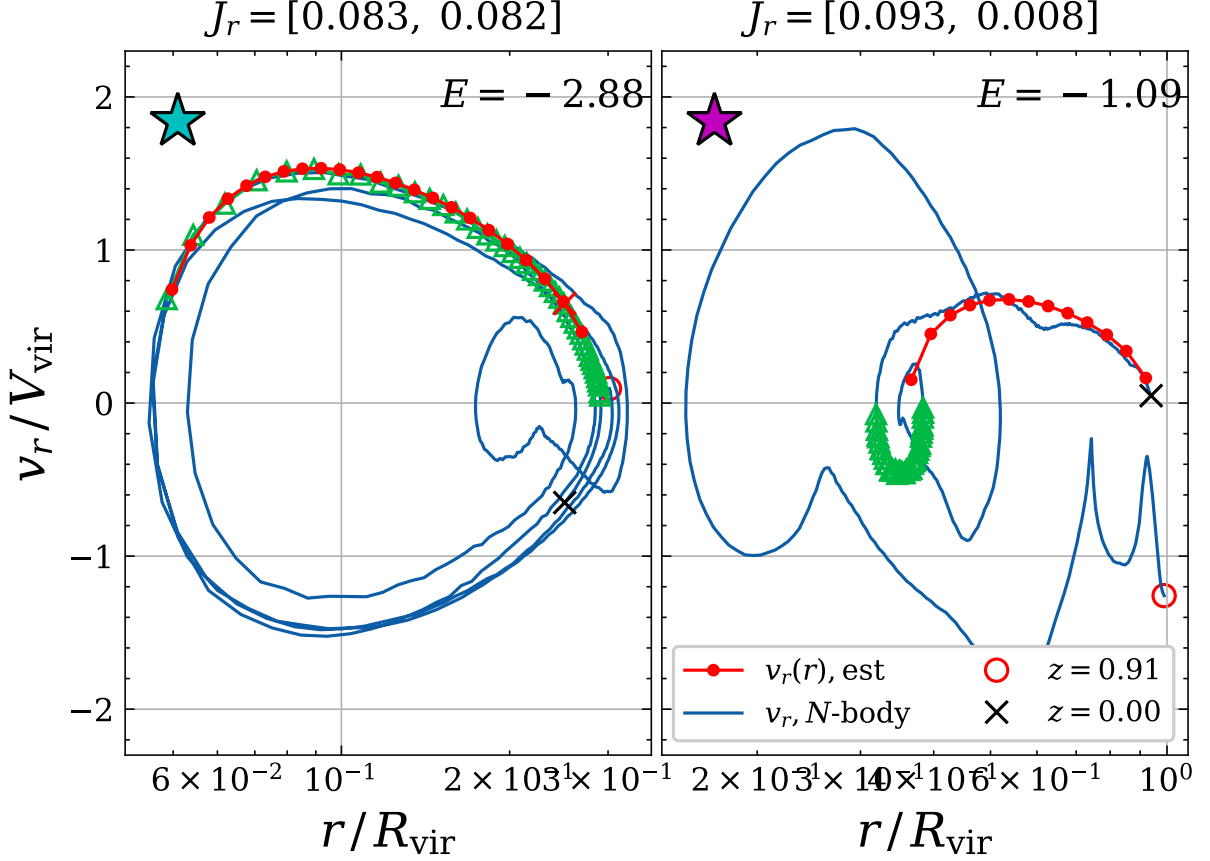


Figure 4.6: The phase-space trajectories of the particles marked as stars in Fig. 4.5. In both panels, the particles' trajectories are displayed as blue solid lines, spanning from $z = 0.91$ (red circle) to $z = 0$ (black cross). The trajectories are sampled at 600 redshift intervals. Red markers indicate the trajectory reconstructed using the energy and angular momentum at $z = 0$ (black cross, or red cross with the mirrored radial velocity of the black cross). J_r^{est} is calculated from the area enclosed by the line connecting the markers and the x-axis ($v_r = 0$). The J_r values shown in the titles of both panels represent each $[J_r^{\text{est}}, J_r^{\text{dir}}]$ in units of $R_{\text{vir}} V_{\text{vir}}$, and the energy is presented in units of $M_{\text{vir}} V_{\text{vir}}^2$ in the top right corner. Left: A representative particle (cyan star) from the left panel of Fig. 4.5, with $p = 10$, shows well-matching values of J_r^{est} and J_r^{dir} . Right: A representative particle (magenta star) from the right panel of Fig. 4.5, located inside $1R_{\text{vir}}$, has significantly differing values for J_r^{est} and J_r^{dir} . These figures are adapted from Enomoto et al. (2024b).

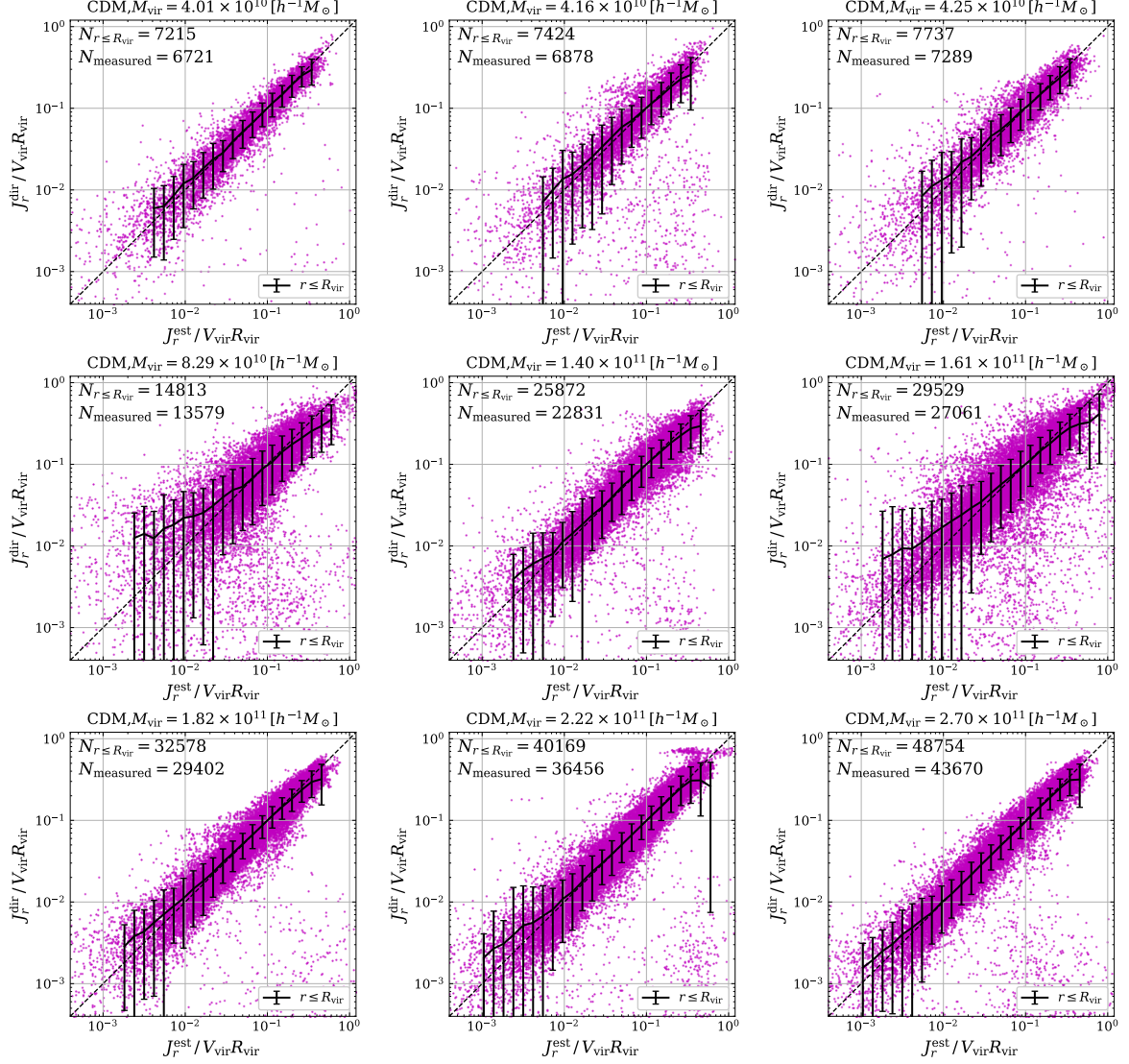


Figure 4.7: Comparison of J_r^{dir} and J_r^{est} for particles in selected CDM halos. Each panel displays particles within $1R_{\text{vir}}$ from the halo center as magenta markers, with their mean and RMS scatter represented by black solid lines, accompanied by error bars. These error bars are calculated for bins containing more than 100 particles. $N_{r \leq R_{\text{vir}}}$ denotes the total number of particles within the virial radius R_{vir} , and N_{measured} represents the count of particles for which both J_r^{dir} and J_r^{est} are measurable. About 90% of particles within the virial radius have measurable values for J_r^{dir} . These figures are adapted from Enomoto et al. (2024b).

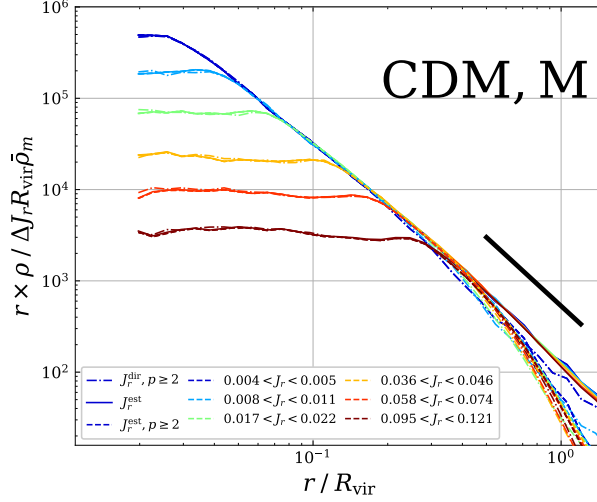


Figure 4.8: Stacked density profiles of particles classified by the radial actions J_r^{est} and J_r^{dir} in CDM halos within the mass range M . The profiles are shown for classification based on J_r^{est} (solid lines), as well as for particles with $p \geq 2$, categorized by J_r^{est} (dashed lines) and J_r^{dir} (dot-dashed lines). This figure is adapted from Enomoto et al. (2024b).

mean values. Moreover, the scatter is significantly smaller for particles with $p \geq 3$ or those located within $1R_{\text{vir}}$. This indicates that the radial actions are more consistent for particles with higher pericenter-apocenter counts or those confined to the virial radius. Despite the overall agreement, many particles exhibit lower values of J_r^{dir} compared to J_r^{est} , particularly noticeable in the bottom-right regions of both panels in Fig. 4.5. However, this discrepancy diminishes considerably when applying thresholds of $p \geq 3$ or $r \leq R_{\text{vir}}$, underscoring the improved reliability of J_r^{dir} for particles meeting these conditions.

To further examine the properties of these particles, we present the phase-space trajectories of two representative particles in Fig. 4.6. These particles are marked as cyan and magenta stars in Fig. 4.5. The left panel illustrates the trajectory of a particle with $p = 10$, where the computed radial actions, J_r^{est} and J_r^{dir} , are nearly identical. Conversely, the right panel depicts a particle located within $1R_{\text{vir}}$, which shows a notable discrepancy between the two methods of radial action calculation. In both panels, the red dots connected by lines represent trajectories reconstructed from the potential, energy, and angular momentum measured at $z = 0$ (denoted by a black cross). These reconstructed trajectories are used to calculate J_r^{est} by determining the area enclosed by the lines and the x-axis ($v_r = 0$). The green triangles correspond to the particle's measured trajectories during its most recent orbit, which is employed to compute J_r^{dir} . This value is obtained from the area enclosed by the line connecting the green triangles and the x-axis. The comparison highlights that J_r^{est} aligns closely with J_r^{dir} for particles with stable, well-sampled orbits (e.g., $p = 10$). However, for particles with fewer orbital crossings or those undergoing dynamic interactions within the virial radius, discrepancies between the two methods are more pronounced.

In the left panel, the particle follows a periodic orbit with a shorter orbital period, and its reconstructed trajectory (red dots) closely matches the measured trajectory (green triangles), leading to a good agreement between the radial actions. In contrast, the particle on the right exhibits a longer orbital period with high energy and fluctuates around $v_r = 0$. These irregular fluctuations, often caused by substructures, lead many particles with long orbital periods to show smaller J_r^{dir} values. On the other hand, the radial

actions of particles deep within the virial radius are less affected by such perturbations. In Fig. 4.7, we present J_r^{est} and J_r^{dir} for particles inside the virial radius from nine additional representative halos. As indicated by the mean values (black lines), most halos exhibit well-matched values of radial action, with some particles showing smaller J_r^{dir} values compared to their J_r^{est} , as previously discussed.

To demonstrate that the stacked profiles are not significantly affected by the choice of method for evaluating the radial action, we show the stacked density profiles of particles classified by J_r^{est} (solid lines) in Fig. 4.8. Additionally, we present the stacked profiles of particles with $p \geq 2$ classified by J_r^{est} (dashed lines) and J_r^{dir} (dot-dashed lines). For radii below $0.5R_{\text{vir}}$, all profiles corresponding to the same radial action value are in good agreement, but they diverge for $r \geq 0.5R_{\text{vir}}$. Since J_r^{dir} cannot be measured for particles that have not passed their apocenter by $z = 0$ (i.e., $p = 1$), the density profiles classified by J_r^{dir} are lowered. However, this difference only affects the outer regions of the density profiles and does not alter our results for $r < 0.5R_{\text{vir}}$, even when selecting particles with $p \geq 2$ (see dot-dashed and solid lines in Fig. 4.8).

It's important to note that the discrepancy between the profiles for $r \geq 0.5R_{\text{vir}}$ does not imply that our results in these radii lack robustness. The lowered profiles for J_r^{dir} arise because we cannot compute J_r^{dir} for particles that have not yet passed their apocenter, whereas J_r^{est} can still be defined for these particles. Furthermore, evaluating the radial action for particles falling into the halo for the first time is inherently challenging due to their typically long orbital periods and the significant oscillations caused by substructures (see Burger et al. (2021)).

4.4 Main results

In this section, we present the results of the phase-space structure of halos, utilizing the methods described in the previous section. In Section 4.4.1, we examine representative halos to highlight the unique characteristics of their inner structures. This analysis focuses on individual properties, providing detailed insights into the dynamics and density profiles at small radii. Subsequently, in Section 4.4.2, we conduct a stacking analysis to investigate the averaged radial density profiles across multiple halos. This approach allows us to quantitatively compare the inner structures of halos formed in cold dark matter (CDM) and warm dark matter (WDM) simulations, elucidating key differences between these dark matter models.

4.4.1 Individual halo properties

Spatial and radial phase-space structures

Here, we select representative halos from our halo samples to visually examine the similarities and differences among them, focusing particularly on their spatial and radial phase-space structures.

In Fig. 4.3, we display the spatial distribution of dark matter (DM) around a single halo with a mass of $M_{\text{vir}} \sim 1.1 \times 10^{13} h^{-1} M_{\odot}$. For this visualization, we collect particles within a radius of $3R_{\text{vir}}$ and project their distribution onto the x - y plane with a slice depth of $0.2 h^{-1} \text{ Mpc}$. The panels illustrate the results for CDM (left), WDM with $m_{\text{DM}} = 1 \text{ keV}$ (middle), and WDM with $m_{\text{DM}} = 0.5 \text{ keV}$ (right).

In Fig. 4.4, we show the radial phase-space distribution of DM particles for these three DM models. The upper and lower panels in both figures use colors to represent the number of apocenter passages p and the radial action J_r , respectively. Here, J_r is normalized by $V_{\text{vir}}R_{\text{vir}}$, where the virial velocity V_{vir} is defined as $V_{\text{vir}} = \sqrt{GM_{\text{vir}}/R_{\text{vir}}}$, and this normalization is used throughout the analysis.

As evident from Fig. 4.3, the degree of substructure development (primarily subhalos) varies significantly among the DM models. In the CDM model, numerous subhalos are visible in the outer regions of the halo. However, in the WDM models, the spatial DM distribution becomes smoother, and the number of subhalos decreases substantially as the DM particle mass decreases from $m_{\text{DM}} = 1 \text{ keV}$ to 0.5 keV . Moreover, the overall shapes of the halos differ due to variations in their mass accretion histories. These findings are consistent with previous studies (e.g., Bode et al. (2001); Allgood et al. (2006)).

Conversely, when focusing on the spatial patterns of particles classified by p and J_r , no significant differences are observed among the DM models considered. In general, particle distributions are stratified by p : particles with larger p values tend to reside in the inner regions of the halo, while those with smaller p values are found in the outer regions. This creates an onion-like structure in the configuration space, as illustrated in Fig. 4.3. Since both CDM and WDM halos exhibit multi-stream flows, this onion-like structure is also apparent in the radial phase-space distribution shown in Fig. 4.4. In contrast, for the radial action J_r , the projected distribution in Fig. 4.3 is more mixed, particularly near the central regions of the halo, where particles exhibit a wide range of J_r values. This is because particles with a given J_r can have various specific angular momentum values j (see equation (4.1)), allowing their motions to extend spatially, as demonstrated in Fig. 4.4.

Since the radial action J_r is considered an adiabatic invariant, the structure of the particle distribution classified by J_r tends to persist in a quasi-stationary state. While J_r gradually decreases over time due to the continuous growth of the halo mass (see, e.g., Burger et al. (2021)), particles with smaller J_r values are sufficiently relaxed, erasing the memory of initial conditions. In this context, it is reasonable to expect that common features will generally emerge in both CDM and WDM halos. This expectation holds particularly for particles with small J_r values, which typically correspond to those with large p values near the halo center. Therefore, a similar common feature is also anticipated in the particle distribution classified by p , as previously identified in the CDM model (Enomoto et al., 2023, 2024a). This point will be explored in greater detail in the following sections.

Density profiles

To investigate the inner structure of the halo in a more quantitative manner, we compute the radial density profiles for particles categorized by the number of apocenter passages p and the radial action J_r . For this analysis, we select five halos that are identical across the three DM models, including those shown in Figs. 4.3 and 4.4.

Figs. 4.9 and 4.10 show the radial density profiles as a function of radius, normalized by R_{vir} . The vertical axis is scaled by r and divided by $R_{\text{vir}}\bar{\rho}_{\text{m}}$. In Fig. 4.9, the results for particles with an odd number of p within the range $5 \leq p \leq 19$ are plotted. For the density profiles classified by J_r in Fig. 4.10, we divide particles into 20 logarithmic J_r bins, spanning the range $0.01/\pi \leq J_r/(R_{\text{vir}}V_{\text{vir}}) \leq 1/\pi$. From this, we compute the radial density profiles for six representative bins within the range $0.017 \leq J_r/(R_{\text{vir}}V_{\text{vir}}) \leq 0.25$,

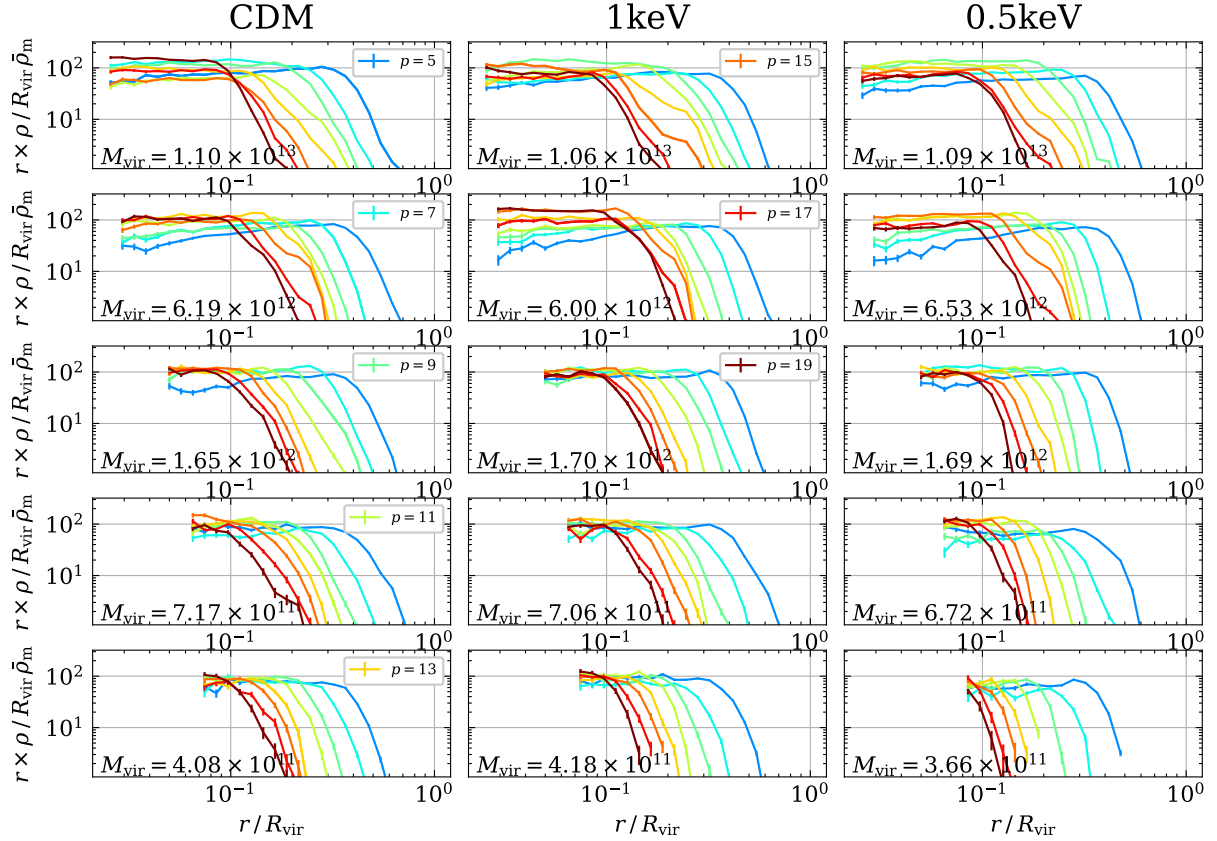


Figure 4.9: The density profiles of particles classified by the number of apocenter passages in different halos ($p = 5, 7, 9, 11, 13, 15, 17, 19$), with color coding for each value of p . The error bars are estimated based on Poisson scatter, and the profiles are only shown for radii larger than three times the softening radius. To highlight the relationship $\rho \propto r^{-1}$, the product of $r \times \rho$ is displayed. Most profiles exhibit a r^{-1} dependence in their inner regions, except for those with $p \lesssim 9$ in cluster-sized halos, which are likely undergoing a relatively rapid accretion phase. The three halos shown in the top row are the same as those in Fig. 4.3. These figures are taken from Enomoto et al. (2024b).

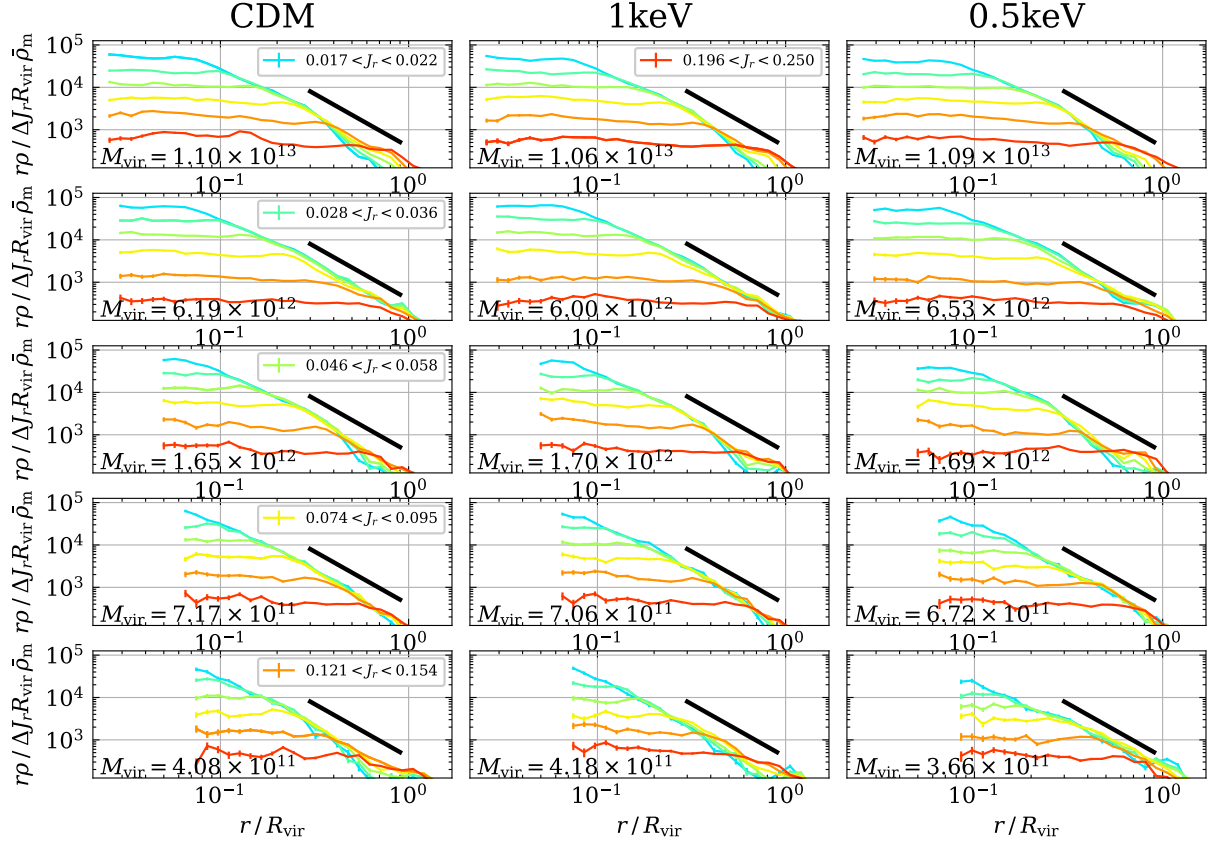


Figure 4.10: The density profiles of particles classified by radial action $J_r[R_{\text{vir}}V_{\text{vir}}]$ in different halos color-coded by the value of radial action. As in Fig. 4.9, the error bars are based on Poisson scatter, and the profiles are only displayed for radii larger than three times the softening radius. The particles are grouped into log-distributed bins of J_r , with the profiles divided by the width of each bin, ΔJ_r , as indicated on the y-axis. The halos shown are the same as those in Fig. 4.9, and the plot again shows $r \times \rho$. The black solid lines represent a proportionality to $r^{-2.5}$ in this plot. These figures are taken from Enomoto et al. (2024b).

The number of halos in each mass range, that are used for stacking analysis. The second column from the left shows the mass ranges, and the three columns on the right show the number of halos in the three simulations WDM05, WDM1, and

	Name	$M_{\text{vir}}[10^{10}h^{-1}M_{\odot}]$	$N_{0.5\text{keV}}$	$N_{1\text{keV}}$	N_{CDM}
CDMLR.	S	[2, 8]	153	476	718
	M	[8, 20]	76	149	165
	L	[20, 80]	77	104	104

where the values of J_r show good convergence between the two methods outlined in Section 4.3.2 (see Section 4.3.3).

Note that in both Figs. 4.9 and 4.10, the error bars in the radial density profiles represent the Poisson error, estimated from the number of particles in each radial bin.

The radial profiles presented in both Figs. 4.9 and 4.10 clearly demonstrate a double power-law behavior, characterized by a shallow inner cusp and a steeper outer slope. Notably, the inner profiles consistently show a slope of -1 , meaning $\rho \propto r^{-1}$. Upon closer inspection of Fig. 4.9, it becomes apparent that the inner slope slightly varies with p , being somewhat shallower than -1 for smaller p values. However, the slope converges to -1 for $p \geq 15$, a trend that is in agreement with our earlier results from CDM simulations (Enomoto et al., 2023, 2024a).

An important new finding is that these trends remain valid for WDM models, even in halos near the mass scale corresponding to the cutoff of the initial power spectrum, such as $M_{\text{vir}} \sim 10^{12}h^{-1}M_{\odot}$ for $m_{\text{DM}} = 0.5$ keV (shown in the lower right panel of Fig. 4.9). Regarding the profiles of particles categorized by J_r in Fig. 4.10, all the profiles exhibit an inner slope of -1 , with no clear dependence on J_r . Interestingly, in contrast to the p -classified profiles, the outer profiles display a common slope close to -3.5 , indicated by the thick black lines. This pattern is particularly noticeable for profiles with smaller J_r values, although minor fluctuations appear near the virial radius, likely due to the infall of non-stationary matter. These results align with the expectations discussed in the previous subsection (refer to the final paragraph).

4.4.2 Stacked halo profiles

In this subsection, we analyze the density profiles of halos averaged over a specific mass range, leveraging the large number of halo samples available. We limit our halo samples to those with $M_{\text{vir}} < 8 \times 10^{11} h^{-1} M_{\odot}$ and categorize them into three mass ranges: S, M, and L, as summarized in Table 4.4.2 (see also the arrows in Figs. 4.1 and 4.2). The profiles of individual halos are then rescaled by $R_{\text{vir}}\bar{\rho}_{\text{m}}$ for the amplitude and by R_{vir} for the radius. The results for each mass range are stacked to obtain the final profiles (Enomoto et al., 2023, 2024a).

To minimize any systematics arising from the softening scales, we restrict the plotted results to radii larger than the convergence radius, which is defined as follows. We first compute the total density profile for stacked halos in the CDM model using the high-resolution simulation (CDMHR). By comparing this with the profile obtained from the CDMLR simulations, we determine the minimum radius for each mass bin, where both simulations agree within 3%. This radius is referred to as the convergence radius, and we apply this to the stacked profiles in both the CDM and WDM models. Additionally, we exclude noisy data by discarding density profiles where the fractional error, estimated from the individual halo profiles, exceeds 80%.

In Section 4.4.2, we begin by presenting the stacked density profiles classified by p , focusing primarily on the differences observed between the DM models. In Section 4.4.2, we then compute the stacked density profiles for various values of J_r , highlighting the model-dependent trends that arise due to the halo mass concentration.

Classification by number of apocenter passages p

To explore the differences between dark matter models, we begin by examining the DM particles classified by the number of apocenter passages, p , and present the stacked density profiles in Fig. 4.11 for the mass ranges S (upper), M (middle), and L (lower). In each panel, we show the stacked density profiles for particles with $p = 10$ (blue), 20 (green), 30 (orange), and 40 (brown). The three distinct lines represent different DM models: WDM with $m_{\text{DM}} = 0.5 \text{ keV}$ (solid), WDM with $m_{\text{DM}} = 1 \text{ keV}$ (dashed), and CDM (dot-dashed).

Fig. 4.11 clearly illustrates that the profiles classified by p exhibit a double power-law feature with an inner slope of -1 , regardless of the DM model. While this is expected from the individual halo profiles in Section 4.4.1, other important features that are worth noting are summarized as follows:

- (i) The inner profiles asymptotically approach the form $\rho(r; p) \rightarrow A(p) r^{-1}$, independent of the DM model, where the amplitude $A(p)$ shows only a weak dependence on halo mass and is approximately $(50 - 100) R_{\text{vir}} \bar{\rho}_m$. This behavior is especially evident for larger values of p .
- (ii) The characteristic scale at which the slope of the density profiles transitions from -1 to a steeper value in the outer region systematically shifts towards smaller radii as the DM mass decreases from CDM to WDM with $m_{\text{DM}} = 0.5 \text{ keV}$.

For property (i), the weak dependence of the amplitude $A(p)$ on halo mass is indeed anticipated from Enomoto et al. (2023, 2024a) for the CDM model. They provided a fitting formula for the profiles $\rho(r; p)$, which is a function of p and M_{vir} (see equations 2 and 3 in Enomoto et al. (2023)). Although their formula was derived from halo samples with masses in the range $[3.2, 1500] \times 10^{11} h^{-1} M_{\odot}$, extrapolating it to the mass range of our halo samples yields a prediction of $A(p) \sim 100 - 110 R_{\text{vir}} \rho_m$ at $p = 40$, which is consistent with (i)². It is important to emphasize that the fitting formula predicts $A(p)$ to vary gradually with p , with a monotonic decrease. This suggests that the apparent convergence of $A(p)$ in Fig. 4.11 is misleading, and $A(p)$ likely changes with p , probably for both CDM and WDM halos. Nevertheless, an interesting point is the similarity between CDM and WDM halos, as both exhibit an inner slope of -1 .

Regarding feature (ii), a similar trend can be observed when the number of stacked halos is reduced (see figures 7-10 in Enomoto et al. (2024a)). Since the number of halos in WDM models decreases systematically with m_{DM} (see Table 4.4.2), this may account for the systematic shift in the characteristic scale discussed in (ii).

To clarify this point, we randomly select halos for each mass bin in the CDM and WDM model with $m_{\text{DM}} = 1 \text{ keV}$, matching the number of halo samples in the WDM model with $m_{\text{DM}} = 0.5 \text{ keV}$. We then compute the stacked density profiles using the reduced number of halo samples. Repeating this procedure 100 times, we obtain the

²The amplitude $A(p)$ here corresponds to the product $A^{\text{E23}}(p) S^{\text{E23}}(p)$, where $A^{\text{E23}}(p)$ and $S^{\text{E23}}(p)$ are the amplitude and scale radius, respectively, from the fitting formula $\rho_{\text{fit}}(r; p)$ obtained in Enomoto et al. (2023).

mean and RMS scatter for the stacked profiles, which are shown in Fig. 4.11 as crosses for CDM and filled triangles for WDM with $m_{\text{DM}} = 1 \text{ keV}$, accompanied by error bars.

As clearly shown, the resulting profiles almost coincide with those obtained before reducing the number of samples in each mass bin. Hence, the systematic shift in the characteristic radius is not artificial, arising from a limited number of halo samples, but inherent, possibly originating from the differences in the formation and mass accretion history between the DM models. Recalling that the concentration parameter of WDM halos is smaller than that of CDM halos (e.g., Ludlow et al. (2016)), the trends observed in Fig. 4.11 align with those reported by Enomoto et al. (2023, 2024a). In their study, halos in CDM simulations were divided into two halves based on the concentration parameter, with one group containing high-concentration halos and the other containing low-concentration halos. They found that the characteristic scales of the double power-law profiles were systematically smaller for low-concentration halos³. This highlights the difficulty in distinguishing between DM models based on the individual halo structure. Instead, the statistical properties of the halo structure, such as those shown in Fig. 4.11, are crucial, as they offer further insight when combined with halo concentration properties.

Classification by radial action J_r

Next, we examine the stacked radial profiles for particles classified by J_r . Since this is the first time we quantitatively characterize them using radial action, we begin by presenting the results for the CDM model in Fig. 4.12.

The upper panel illustrates how the stacked density profiles, classified by J_r , depend on halo mass. For each mass bin, we select six representative J_r bins from the 20 logarithmic J_r bins into which the samples are divided (see Section 4.4.1), and plot the results. As expected from the individual profiles, the double power-law feature is clearly visible. For the J_r range plotted, the profiles consistently show inner and outer slopes of -1 and -3.5 , respectively, with the latter indicated by the thick black lines. The characteristic scale where the slope transitions from -1 to -3.5 changes systematically with J_r . However, both the amplitude and the characteristic scale of the profiles exhibit little sensitivity to halo mass. In fact, we observe a very weak mass dependence. A more detailed inspection of the inner region at $r/R_{\text{vir}} \lesssim 0.1$ reveals that the dependence tends to manifest with decreasing J_r , where smaller mass halos appear to have slightly larger amplitudes and characteristic scales, although the differences remain minimal.

Next, we examine the WDM models and present their stacked density profiles in Fig. 4.13, alongside those from the CDM model. The upper panels display the density profiles selected from four representative J_r bins, while the lower panels show the fractional difference between the WDM and CDM models, $\rho_{\text{WDM}}/\rho_{\text{CDM}} - 1$. Overall, the double power-law feature is consistent across all models considered, with inner and outer slopes of -1 and -3.5 , respectively. Additionally, we observe differences between the models in terms of the amplitude and characteristic scales of the density profiles. Specifically, WDM halos exhibit systematically lower densities at inner radii and higher densities at outer radii compared to CDM halos. These differences become more pronounced as the DM mass decreases. As a result, the characteristic scale of the double power-law feature

³Enomoto et al. (2023, 2024a) also show that the amplitude of the inner density profile for low-concentration halos is systematically lower than that for high-concentration halos (see middle lower panel of Fig. 2 and upper panel of Fig. 17 in their papers, respectively). Indeed, in Fig. 4.11, we observe a systematic difference in amplitude between WDM and CDM halos, although it is subtle and quantitatively small, as described in property (i).

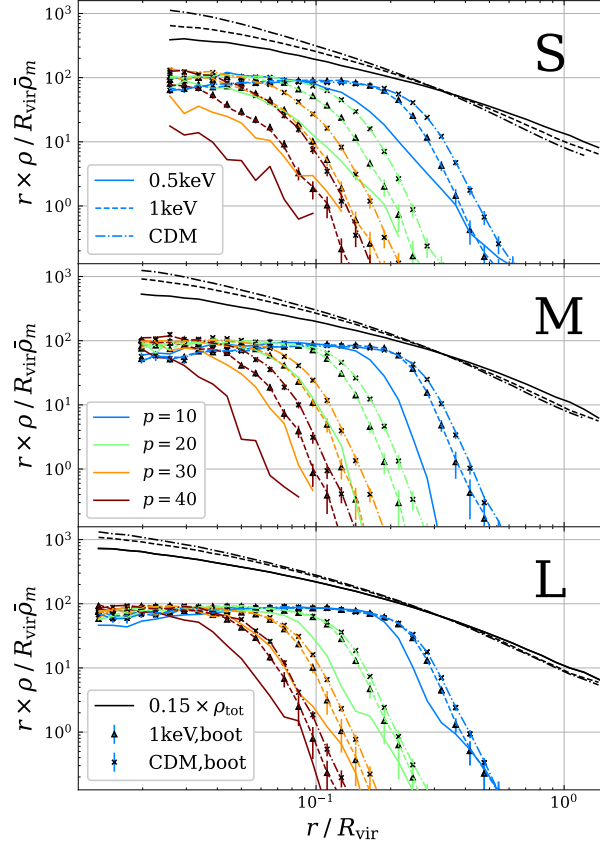


Figure 4.11: The stacked density profiles of particles classified by $p = 10, 15, 20, 25, 30$. From top to bottom, the results for the mass ranges S, M, and L are displayed, with solid, dashed, and dot-dashed lines corresponding to 0.5keV, 1keV, and CDM simulations, respectively. Similar to the individual halo profiles in Fig. 4.9, the stacked profiles exhibit a proportionality to r^{-1} , independent of the dark matter mass and models. As mentioned in Section 4.2.1, the stacked density profiles are shown at radii where their RMS error is less than 80% of their value. For clarity, the error bars for the stacked density profiles are omitted. This figure is taken from Enomoto et al. (2024b).

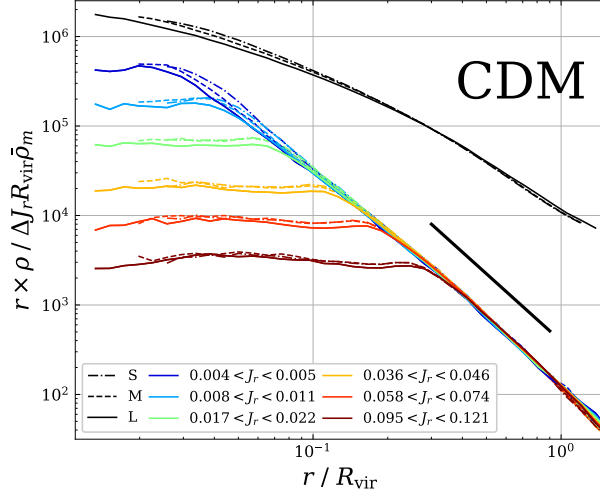


Figure 4.12: The stacked density profiles of particles in CDM halos that are classified by the radial action J_r . The top panel shows those for the mass ranges S, M, and L (dot-dashed, dashed, and solid respectively). Black solid lines are proportional to $r^{-2.5}$ in this plot. This figure is taken from Enomoto et al. (2024b).

shows model dependence, increasing as the DM mass decreases. Regarding halo mass dependence, the differences between models appear more significant for smaller halos.

While these model differences are striking and could provide insights for distinguishing between DM models based on the profiles, we have shown in Section 4.4.2 that such differences may originate from variations in halo concentration properties. Therefore, the discrepancies observed in Fig. 4.13 may be attributed to the concentration properties of the halos.

To explore this further, we focus on the halo mass bin S and select halos with concentration parameters within a specific range common to all models. We set the concentration parameter c_{vir} range to $[4.5, 6.5]$ for WDM models with DM mass $m_{\text{DM}} = 0.5$ and 1.0 keV, and $[9.5, 11.5]$ for both the WDM model with $m_{\text{DM}} = 1$ keV and the CDM model. The number of halos falling within these ranges is approximately 50 – 60 for the former and 80 – 90 for the latter. We then compute the stacked density profiles for each sample, as shown in Fig. 4.13.

The results are presented in Fig. 4.14, where we again select four representative J_r bins. The upper panels indicate that the model differences largely disappear, with the profiles for each J_r bin closely resembling each other. The lower panels display the fractional differences between the profiles, using one of the DM models in the upper panels as a reference, i.e., $\rho_{0.5 \text{ keV}} / \rho_{1 \text{ keV}} - 1$ for the lower-left panel, and $\rho_{1 \text{ keV}} / \rho_{\text{CDM}} - 1$ for the lower-right panel. Unlike the results in Fig. 4.13, no systematic trends or model differences are observed, and the fractional differences typically lie between -0.3 and 0.3 . However, there is some variation as the radius increases or decreases, due to the limited number of DM particles. Consequently, we conclude that even for the WDM models with DM masses of 0.5 and 1 keV, which are observationally disfavored, the phase-space structure of halos characterized by the radial action retains a common feature: the double power-law profile observed in the CDM model, with the concentration parameter c_{vir} being the key factor controlling the amplitude and characteristic scale of the profiles.

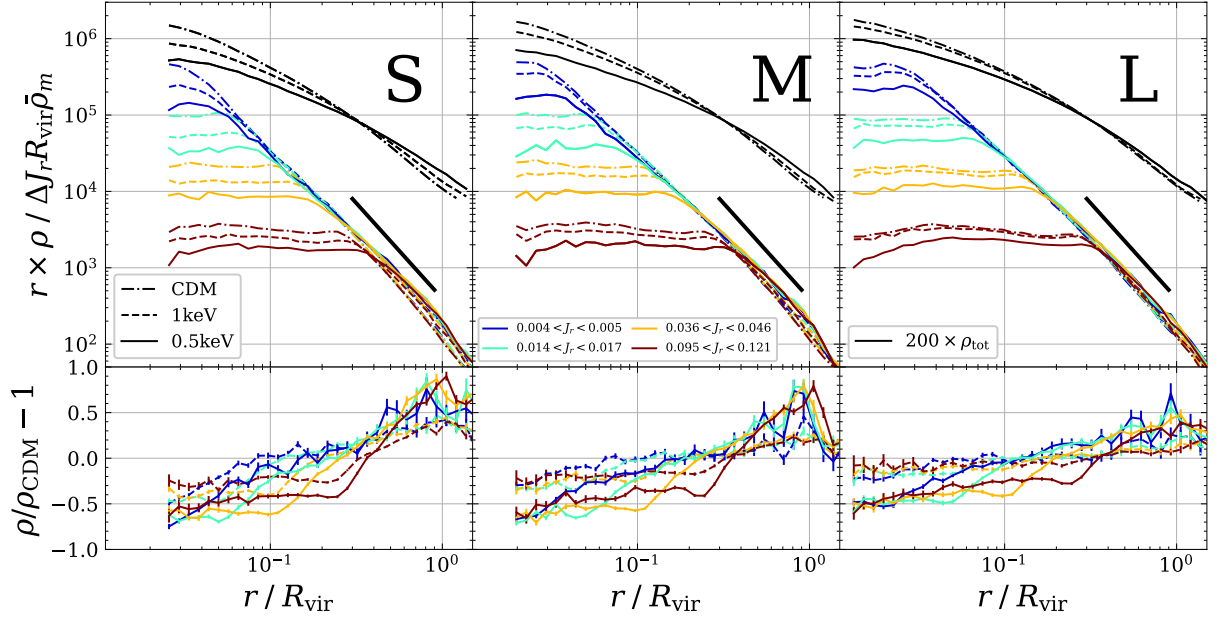


Figure 4.13: The stacked density profiles of particles in CDM (dot-dashed) and WDM halos (dashed and solid lines for 1keV and 0.5keV, respectively) are classified by the radial action J_r . The results for the mass ranges S, M, and L are shown from left to right. In the lower panels, the fractional differences of the profiles with respect to the CDM case are displayed. These figures are taken from Enomoto et al. (2024b).

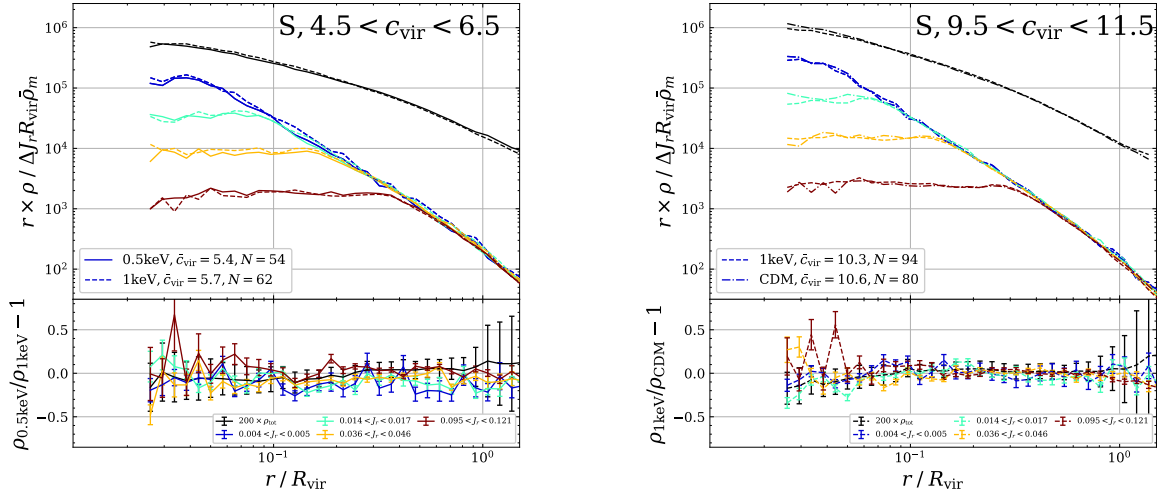


Figure 4.14: The comparison of the stacked density profiles of particles in halos within the mass range S and with concentrations between $9.5 < c_{\text{vir}} < 11.5$ (left) and $4.5 < c_{\text{vir}} < 6.5$ (right). In the left panel, we compare the 1keV (dashed lines) and CDM (dot-dashed lines) cases, while in the right panel, we compare the 0.5keV (solid lines) and 1keV (dot-dashed lines) cases. The values of $\bar{c}_{\text{vir}}$ and N in the legend represent the mean concentration and the number of halos in each subgroup, respectively. The bins for J_r are the same as those in Fig. 4.13. Bottom: The fractional difference between the profiles shown in the top panel. These figures are taken from Enomoto et al. (2024b).

4.5 Discussion

In the previous section, we demonstrated that the phase-space structure of halos consistently exhibits a double power-law feature for DM particles categorized by their number of apocenter passages p and radial action J_r . Specifically, the inner density profiles of the three DM models we analyzed all show a uniform slope of -1 . This behavior is persistent even when we consider halos based on their concentration parameters, suggesting a universal characteristic. This consistency points to a strong relationship between p and J_r , which seems to be independent of the specific DM model.

To examine this connection more closely, we select halos from the three simulations that share nearly identical mass and position. Among these, we focus on one halo with a mass of approximately $M_{\text{vir}} \simeq 1.7 \times 10^{12} h^{-1} M_{\odot}$ as a representative case. We then plot the distribution of DM particles in the j - J_r plane, categorized by different values of the number of apocenter passages: $p = 2$ (gray), 10 (dark blue), 20 (light blue), 30 (orange), and 40 (brown). These results are presented in Fig. 4.15, with both axes normalized by $V_{\text{vir}} R_{\text{vir}}$.

Notably, for each value of p , the particle distribution follows a distinctive rectangular pattern. This indicates preferential values for both j and J_r , with the DM particles predominantly aligning along either the preferred j or J_r value, up to a maximum threshold. Across all DM models, these preferential values decrease systematically as p increases.

From Fig. 4.15, an important observation is that for a fixed p , particles having a small value of j mostly lie around a certain value of J_r . Since the particles with small j are likely to reside near the halo center and can contribute to the inner density profiles, this means that there is a tight link between p and J_r for the inner DM particles, which explains why the radial density profiles commonly show an inner slope of -1 for particles classified both with p and J_r .

On the other hand, particles residing in the outer part of halos can generally take various values of j , and their orbital radius tends to become larger as increasing j . However, as seen in Fig. 4.15, there is an upper boundary in j for particles with a fixed p . This implies that the density profiles for these particles sharply drop toward outer regions, with typically the slope of $-7 \sim -9$. In contrast, particles with a fixed J_r have various values of p , and their radial density profile is thus described by a superposition of those with different p . Hence, the outer profiles are smeared and have a shallower slope than those for the particles classified with p . These are all consistent with Figs. 4.11 and 4.13. Nevertheless, the origin of the outer slope of -3.5 for the particles classified by J_r still remains puzzling, together with the common inner slope of -1 , and we need to look for other dynamical features imprinted in the halo formation and accretion history.

4.6 Conclusions and outlook

In this chapter, we have analyzed the inner structure of dark matter halos, with a particular focus on their phase-space properties. Specifically, by utilizing dynamical information about dark matter (DM) particles, we quantified the radial density structures in two complementary ways. One approach involved classifying the DM particles based on the number of apocenter passages p , as outlined in our previous works (Sugiura et al., 2020; Enomoto et al., 2023, 2024a). In this method, we tracked particle trajectories around the halo and counted p from 1,000 snapshots taken over the range $0 \leq z \leq 5$. The other approach involved calculating the radial action J_r for each DM particle and characterizing

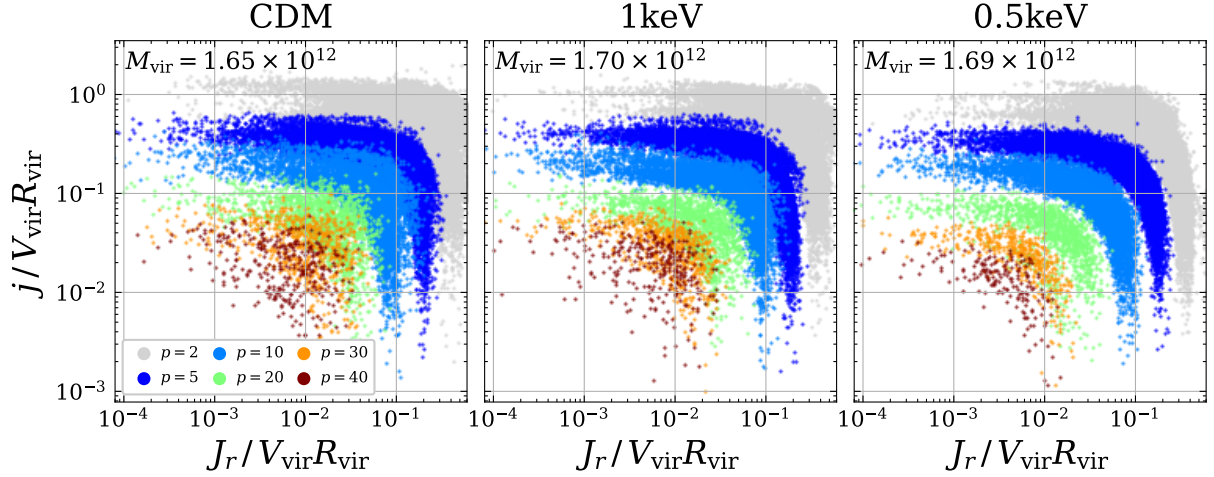


Figure 4.15: The radial action and angular momentum of particles in representative halos, which are the same as those shown in the third row of Figs. 4.9 and 4.10. The particles are color-coded according to the number of apocenter passages, as indicated in the legend. These figures are taken from Enomoto et al. (2024b).

the particle distribution by J_r , following the method described in Pontzen & Governato (2013). Using these characterizations, we examined the properties of the radial density profiles not only in the CDM model but also in WDM models with different DM masses, $m_{\text{DM}} = 0.5$ and 1 keV , both of which are observationally disfavored but are expected to display distinct features compared to the CDM model.

Our key findings are summarized as follows:

- (i) In all three DM models, the density profiles display double power-law features for particles classified by both p and J_r , with a shallow inner slope and a steep outer slope. These common behaviors are linked to the strong relationship between p and J_r , as shown in Fig. 4.15.
- (ii) In both classifications, the inner profiles consistently have a slope of -1 . However, the outer profiles differ between the two classifications: particles classified by p exhibit a steep slope of $-6 \sim -8$, while those classified by J_r have a common slope of -3.5 .
- (iii) Among the three DM models, there are systematic differences in the amplitude and characteristic scale of the radial profiles. As the mass of DM increases, the inner amplitude becomes larger, while the outer amplitude decreases, leading to a reduced characteristic scale. These trends are primarily attributed to differences in halo concentration. We found that for halos with the same concentration parameter, the density profiles for particles classified by J_r are nearly identical across the different DM models.

As noted in Enomoto et al. (2023) and Enomoto et al. (2024a), summing the individual contributions allows the double power-law structures found in particle distributions classified by p to accurately reproduce the total density profile. By calibrating the characteristic density and scale for the double power-law profile, the predicted total profile has been shown to be comparable to, or even outperform, the Einasto profile (Einasto, 1965). This result is expected to extend to the double power-law features of particles classified by J_r . In other words, the origin of the universal features in radial halo profiles, a

long-standing topic of debate in the literature, may be explained by the double power-law profiles discussed in this paper. In particular, the shallow cusp in the total halo profile likely corresponds to the inner slope of -1 seen in both classification schemes.

Although we do not yet have a clear physical explanation for the double power-law feature, the fact that this characteristic persists across different particle classifications offers an important clue. Specifically, the radial phase-space structure can be studied not only by classifying particles based on the number of apocenter passages p but also using the radial action J_r , which is more suitable for characterizing a relaxed halo system and has been widely utilized in the literature.

Building upon previous studies (e.g., Pontzen & Governato (2013); Muni et al. (2024)), we express the enclosed mass for particles classified by J_r at a radius r , denoted as $M(< r; J_r)$, using the following equation

$$M(< r; J_r) = m_p \int_0^\infty p(j, J_r) P(< r; j, J_r) dj, \quad (4.9)$$

where m_p represents the mass of a particle in the simulation. The function $P(< r; j, J_r)$ indicates the fraction of time a particle with orbital parameters (j, J_r) spends within a radius r , and is defined as

$$P(< r; j, J_r) = \frac{2}{t_{\text{orb}}(j, J_r)} \int_{r_{\text{peri}}}^r \frac{dr}{v_r(r)}, \quad (4.10)$$

where $t_{\text{orb}}(j, J_r)$ is the orbital period of the particle. Using this expression, the density profile for particles classified by a given J_r , denoted $\rho(r; J_r)$, can be written as

$$\rho(r; J_r) = \frac{1}{4\pi r^2} \frac{dM(< r; J_r)}{dr} \propto r^{-3} M(< r; J_r). \quad (4.11)$$

Equation (4.9) illustrates that the double power-law nature of $\rho(r; J_r)$ arises from the specific functional forms of $p(j, J_r)$ and $P(< r; j, J_r)$. Several studies have theoretically predicted these functions and discussed how the power-law behavior emerges⁴. Quantifying these functional forms using N -body simulations is also feasible, and such an analysis is currently underway. We report the results in Chapter 6.

From an observational standpoint, the phase-space information of stellar halos provides an opportunity to test the universal features we have identified. With the Gaia satellite's data (Gaia Collaboration et al., 2016), complemented by spectroscopic surveys like GALAH (De Silva et al., 2015), APOGEE (Majewski et al., 2017), and H3 (Conroy et al., 2019), six-dimensional phase-space information for over a billion stars is now available (Gaia Collaboration et al. (2021)). This vast dataset has been instrumental in probing the formation history of the Milky Way (e.g., Naidu et al. (2020); Belokurov et al. (2023); Donlon et al. (2024); see also recent reviews by Helmi (2020); Deason & Belokurov (2024); Bonaca & Price-Whelan (2024)). Recent work based on cosmological hydrodynamical simulations has found a correlation between the kinematics of dark matter and

⁴For instance, in a nearly flat potential, $P(< r; j, J_r)$ can be approximated as $P(< r; j, J_r) \propto r/v_r(r) \sim r$ (Dalal et al., 2010; Lithwick & Dalal, 2011). If $p(j, J_r)$ follows a power law, such as $p(j, J_r) \propto j^\alpha$, the slope of $\rho(r; J_r)$ behaves as $\rho(r; J_r) \propto r^{\alpha-1}$. The joint distribution function $p(j, J_r)$ is calculated by integrating the full distribution function of action variables $f(\mathbf{J} \equiv (J_r, j, j_z))$ over j_z (see equation (B2) in Pontzen & Governato (2013)). Models for $f(\mathbf{J})$, such as those proposed by Posti et al. (2015) and Williams & Evans (2015), are widely used to model our galaxy (e.g., Hattori et al. (2021); Binney & Vasiliev (2023, 2024)) or M31-like galaxies (Gherghinescu et al., 2024).

stellar halos (Herzog-Arbeitman et al., 2018a; Necib et al., 2019b; Deason et al., 2020; Genina et al., 2023). This correlation has been used to estimate the velocity distribution of dark matter around the Sun using stellar data (Herzog-Arbeitman et al., 2018b; Necib et al., 2019a). Notably, the radial density profile of the stellar halo in the Milky Way exhibits a broken power law (e.g., Watkins et al. (2009); Deason et al. (2011); Sesar et al. (2011); Pila-Díez et al. (2015); Han et al. (2022)). If we classify stars in the stellar halo by their radial action, we might reveal a similar universal feature in their density profiles, akin to the behavior observed in dark matter particles. If this is the case, the slope, amplitude, and scale radius of these profiles could potentially act as important indicators for distinguishing between different dark matter models.

Nevertheless, several challenges remain in verifying dark matter models observationally. First, it is essential to clarify the extent to which the distributions of dark matter and stars classified by radial action align in cosmological hydrodynamical simulations. Fortunately, radial action can be calculated from a single snapshot, unlike the number of apocenter passages, making it easier to extend our results from N -body simulations to hydrodynamical simulations. If the density profiles of stars classified by radial action are similar to those of dark matter, they could become powerful tools for distinguishing between dark matter models, or at least for confirming the CDM model.

Second, to make robust theoretical predictions, it is necessary to analyze simulations with a larger number of halos (e.g., Ishiyama et al. (2021)). As shown in Table 4.4.2, each mass range includes only a few hundred halos at most, which may not provide a sufficiently reliable prediction for the average density profile across each mass range. Larger and higher-resolution simulations are required to produce predictions that can be confidently compared with observational data.

It is also valuable to investigate simulations of warm dark matter (WDM) within the mass ranges allowed by observations, as well as alternative dark matter models such as self-interacting dark matter (SIDM) and fuzzy dark matter (FDM). By thoroughly examining how the amplitude or scale of $\rho(r; J_r)$ depends on cosmology or dark matter models, we could generate robust predictions for comparison with observational data.

Chapter 5

Inner structure of prompt cusp

5.1 Introduction

So far, our analysis has primarily concentrated on massive halos at $z = 0$, which are formed through successive mergers involving halos of various masses. In the WDM scenarios examined in Chapter 4, the merger frequency is generally lower than in CDM, owing to the lack of small-scale structures. However, the halos analyzed were still massive and experienced numerous merger events. We have discussed how the frequent accretion of dark matter onto halos and repeated mergers contribute to regularizing the phase-space structure, thereby establishing the universal characteristics of the density profiles $\rho(r; J_r)$.

In contrast, the prompt cusp, discussed in Section 2.4.5, emerges directly from the collapse of smooth density fluctuations in the early universe. Unlike NFW profiles, prompt cusps exhibit a density slope of approximately $\rho \propto r^{-1.5}$, reflecting a nearly power-law behavior rather than the broken power-law typically observed in massive DM halos.

In this chapter, before delving deeper into the origins of universal feature of $\rho(r; J_r)$ in Chapter 6, we turn our attention to the internal structure of prompt cusps, revisiting the methodology from Chapter 4. Using N -body simulations that track the evolution of prompt cusps, we calculate the radial action for simulation particles and reanalyze $\rho(r; J_r)$. Our findings reveal that, similar to massive DM halos, prompt cusps also exhibit a double-power law structure in their $\rho(r; J_r)$ profiles.

5.2 Data

We analyze one CDM simulation dataset and two datasets from 3 sine wave simulations, with the same initial conditions as in Colombi (2021). These simulations were just recently performed by S. Colombi and A. Parichha with the adaptive mesh refinement code RAMSES (Teyssier, 2002), using TSC interpolation as implemented by Li et al. (2012). These simulations were generated in the framework of an ongoing collaboration with S. Colombi, A. Parichha, S. Saga and A. Taruya and will be described in detail in a separate publication (Colombi et al., in preparation). Below, we offer a brief summary of the datasets.

5.2.1 CDM simulation

One of the simulations is a CDM simulation with a box size of $12.5 h^{-1}\text{pc}$ and the number of simulation particles is 512^3 with a maximum refinement level corresponding to a

mesh element size of $\lambda_{\min} = L/2^{13} = L/8192$, which defines, within some factor (in the figures below, we define a grey exclusion zone given by $r \leq 2\lambda_{\min}$), the maximum spatial resolution attainable. While the cosmological parameters are consistent with the Planck CMB results (Planck Collaboration et al., 2020), the initial power spectrum includes a free-streaming cutoff corresponding to neutralino dark matter with a mass of 100 GeV and a decoupling temperature of $T = 30$ MeV. This small box size is specifically chosen to simulate the formation of the first halos, though it does not account for tidal forces from larger-scale structures.

In Fig. 5.1, we present the latest snapshot from this simulation. Since the simulation focuses on the formation of the first halos, no smaller substructures are present, making the snapshots resemble those of the WDM simulation. We store three snapshots at $a = 0.084$, $a = 0.111$, and $a = 0.161$, focusing on two halos highlighted with triangle and hexagon in Fig. 5.1. Halo 1 is formed at $a = 0.084$, while Halo 2 is formed at $a = 0.111$. Consequently, we can analyze the structure of Halo 1 across three snapshots, whereas Halo 2 is examined using two snapshots.

5.2.2 Three-sine-wave simulations

In addition to the CDM simulation, we also investigate two different simulation sets of three-sine-wave simulations with the number of simulation particles 512^3 and the same maximum refinement level. The initial conditions of these simulations are generated from linear Lagrangian perturbation theory as follows. The initial comoving position and peculiar velocity of particles $\mathbf{x}_{i,j,k}, \mathbf{v}_{i,j,k}$ (i, j, k spans from 1 to 512) are given by

$$\mathbf{x}_{i,j,k}(t=0) \equiv \mathbf{q}_{i,j,k} = \frac{L}{512}(i, j, k) \quad (5.1)$$

$$\mathbf{v}_{i,j,k}(t=0) = (0, 0, 0), \quad (5.2)$$

and their displacements at the time when we start the simulation $t = t_i$ are given by

$$\mathbf{x}_{i,j,k}(t = t_i) = \mathbf{q}_{i,j,k} + \mathbf{D}_+(t_i)\mathbf{P}(\mathbf{q}_{i,j,k}) \quad (5.3)$$

$$\mathbf{v}_{i,j,k}(t = t_i) = a(t_i)\frac{d\mathbf{D}_+}{dt_i}\mathbf{P}(\mathbf{q}_{i,j,k}) \quad (5.4)$$

where the displacement field $\mathbf{P}(\mathbf{q}_{i,j,k})$ is given by

$$\mathbf{P}(\mathbf{q}_{i,j,k} \equiv (q_x, q_y, q_z)) = \frac{L}{2\pi} \left[A_x \sin\left(\frac{2\pi}{L}q_x\right), A_y \sin\left(\frac{2\pi}{L}q_y\right), A_z \sin\left(\frac{2\pi}{L}q_z\right) \right]. \quad (5.5)$$

Here, D_+ represents the linear growing factor normalized to unity at the present time, and the vector $\mathbf{A} = (A_x, A_y, A_z)$, where $A_x \geq A_y \geq A_z \geq 0$, specifies the linear wave amplitudes along each axis. We adopt two configurations for this vector: $(A_x, A_y, A_z) = (24, 18, 12)$ and $(A_x, A_y, A_z) = (24, 4, 3)$, referred to as the anisotropic (ANI) and quasi-1D (Q1D) simulations, respectively. In the ANI case, the amplitudes are comparable across all axes, while in the Q1D case, one amplitude is significantly larger than the others. Both setups focus on the collapse of a single density peak, allowing for precise analysis of its density evolution, as all 512^3 particles represent the same peak. This contrasts with CDM simulations, where particle counts per prompt cusp are typically limited to the order of millions.

Snapshots of the anisotropic simulation are shown in Fig. 5.2 as an example. Initially, the collapse occurs along the x-axis at $a = 0.040$, marking the beginning of structure formation. Subsequently, the collapse extends to the other axes, resulting in a more isotropic configuration. Although this simulation setup is somewhat artificial, the density field evolution in the three-sine-wave simulations resembles that of prompt cusps in CDM simulations. The left panel of Fig. 5.3 shows the density profile evolution for the anisotropic simulation. After $a \geq 0.040$, following the first shell-crossing of the density peak (as seen in the top middle panel of Fig. 5.2), the density profile transitions to $\rho \propto r^{-1.5}$. A similar prompt cusp behavior is observed in the quasi-1D case as well. In summary, the three-sine-wave simulations serve as simplified models for CDM simulations, offering the advantage of a much larger number of simulation particles while maintaining comparable dynamics.

For the three-sine-wave simulations, we analyze three snapshots for each setup: $a = 0.125, 0.180$, and 0.450 for both the anisotropic (ANI) and quasi-1D (Q1D) cases. These snapshots are selected to represent (i) the collapse of the density peak and the emergence of prompt cusps following violent relaxation, (ii) an intermediate state dominated by the prompt cusps, and (iii) the onset of convergence to the NFW attractor (but with the prompt cusp still present at small radii), which coincides with the final snapshot of the simulation. The energy distributions of the particles at these three snapshots are depicted in Fig. 5.4. At the initial snapshot, $a = 0.050$ for ANI and $a = 0.125$ for Q1D, the coarse flow of accreted material is distinctly visible. Over time, phase-mixing occurs, dividing the coarse-grained phase-space distribution into increasingly fine streams of phase-space elements. Consequently, by the third snapshot, the energy distribution of the particles becomes significantly smoother. Although the figure only shows the three-sine-wave simulations, a similarly smooth energy distribution is observed in the CDM halos (Halo 1 and Halo 2) at their latest snapshots.

5.3 Density profiles of prompt cusps

In this section, we present the density profiles of particles classified by radial action J_r , in order to examine whether the universal power-law behavior found in the CDM and WDM simulations is also the case in the first halos or not. Following the method introduced in Section 4.3.2, we compute the radial action J_r of simulation particles in the three simulations introduced in the last section. This time we compute radial action from particles' specific energy and angular momentum by integrating equation (4.1).

In Figs. 5.5 to 5.8, we show the total density profiles (blue solid lines with errorbars at the top of each panel) and density profiles classified by J_r (lines with errorbars color-coded by J_r range) of the Halo 1 (Fig. 5.5) and Halo 2 (Fig. 5.6) in CDM simulation, and three-sine-wave simulations in the anisotropic case (Fig. 5.7) and the quasi 1D case (Fig. 5.8). Although the formation scenario of the first halos shown is significantly different from the massive halos investigated in the previous Chapters, the feature of the density profiles of particles classified by J_r exhibit a double-power law nature with the inner slope of -1 . However, for the outer part, their index of the slope is not -3.5 which is found in Chapter 4, but rather follows $\rho \propto r^{-2.5}$ in both the Halo 1 and 2 with fluctuations due to the caustics.

It is notable that we can see shallower inner profiles for the profiles of large radial action, in $J_r \geq 0.149$ cases in the middle and the right panel of Fig. 5.5 for example. This shallower slope also exists in other simulations, especially in both three-sine-wave simula-

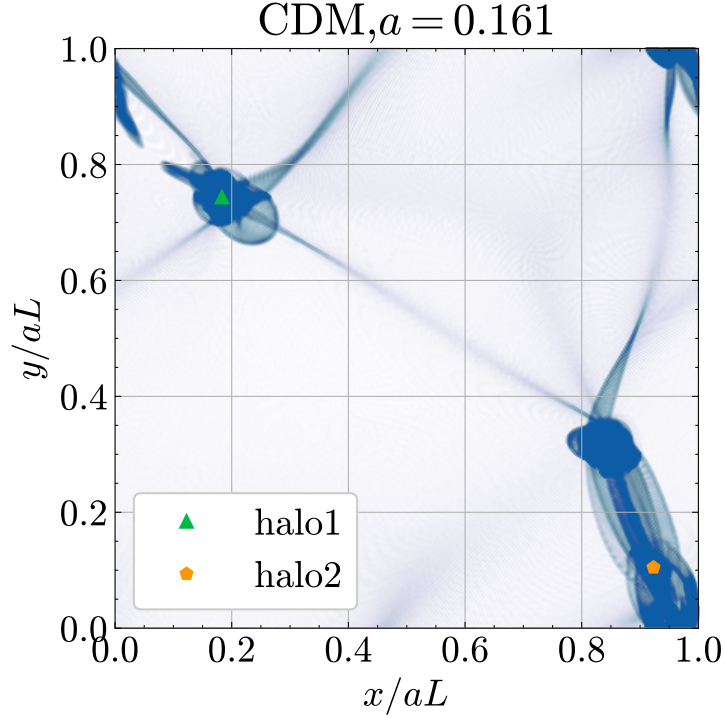


Figure 5.1: Snapshots of the CDM simulation at $a = 0.161$, which is the latest snapshot we have stored. The two halos under investigation are located at the indicated markers.

tions. Particles with large radial action generally have long orbital periods and would be less relaxed than those with small radial action. This reminds us of the similar shallower slope of $\rho(r; p \lesssim 5)$ found in Fig. 3.16 in Chapter 5, which could also be attributed to unrelaxed particles with a long orbital period that accreted recently. In Chapter 6, we will also discuss the possibility that all the profiles could exhibit shallower behavior in the $r \rightarrow 0$ limit, which is usually hidden by the softening length. Regarding the three-sine-wave simulation, we should point out that they only have a smaller amount of matter accreted than those of CDM simulations. This could also be a possible explanation for the shallower density profiles found in three-sine-wave simulations.

In the three-sine-wave simulation, we expected that we could observe similar double-power law behavior only in the latest snapshots because its phase space is phase-mixed well and similar to those of the massive CDM halos. However, apart from a large fluctuation due to caustics, we found that the double power-law feature is built up shortly after the formation of prompt cusp due to the violent relaxation. (see the leftmost panels in Figs. 5.7 and 5.8). This sheds light on the dynamical process that drives the profiles towards the nearly universal slope of the inner profiles, $\rho \propto r^{-1}$. However, this is beyond our current scope.

5.4 Conclusion

Unlike typical CDM halos, the initial halos are formed through the direct collapse of smooth density fluctuations on scales comparable to the dark matter free-streaming scale. To explore the conditions underlying the universality observed in the previous Chapters and gain further insight into its origins, it is worthwhile to investigate whether the density

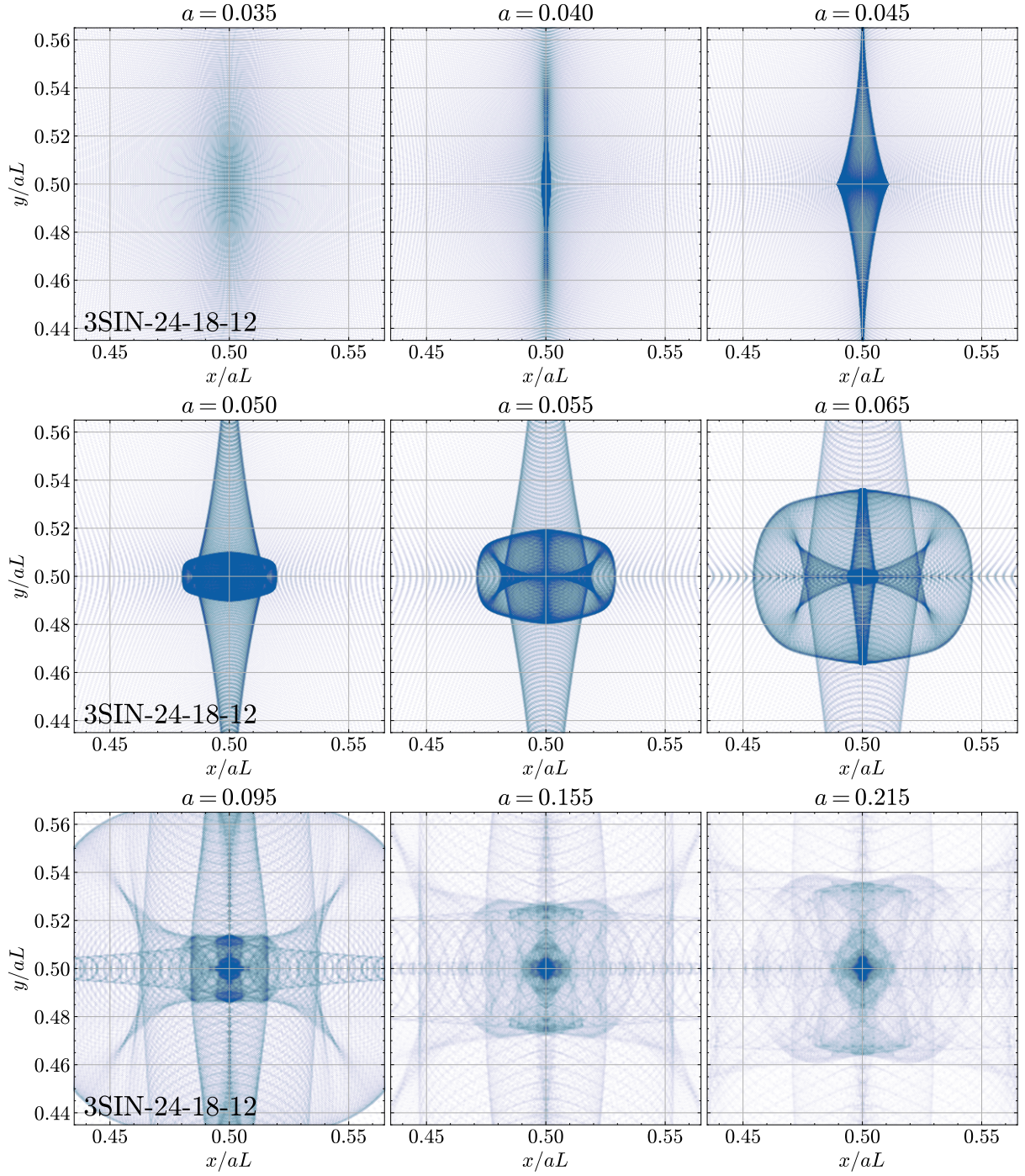


Figure 5.2: Snapshots from the anisotropic (ANI) simulation reveal the collapse dynamics of the density peak. The collapse first occurs along the x-axis at $a = 0.040$, marking the initial stage of structure formation. Shortly thereafter, the collapse proceeds in other directions, leading to a more isotropic configuration.

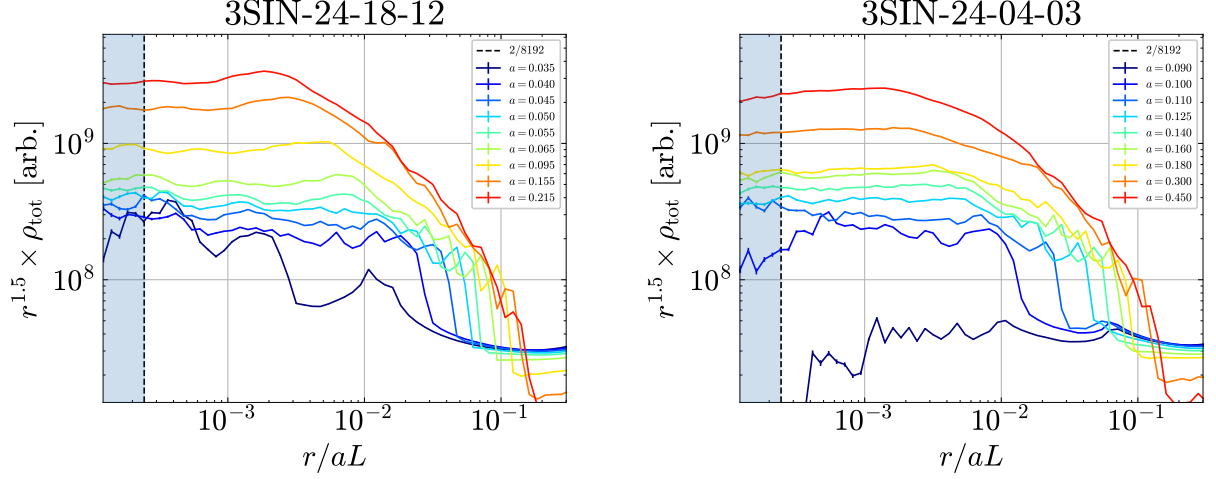


Figure 5.3: The evolution of the density profiles for the anisotropic simulation (left) and the quasi 1D simulation (right) is shown. In both cases, distinct features of the density profiles following $\rho \propto r^{-1.5}$ emerge shortly after the collapse of the initial density peak. Vertical line 2/8192 indicates the resolution limit of the simulations.

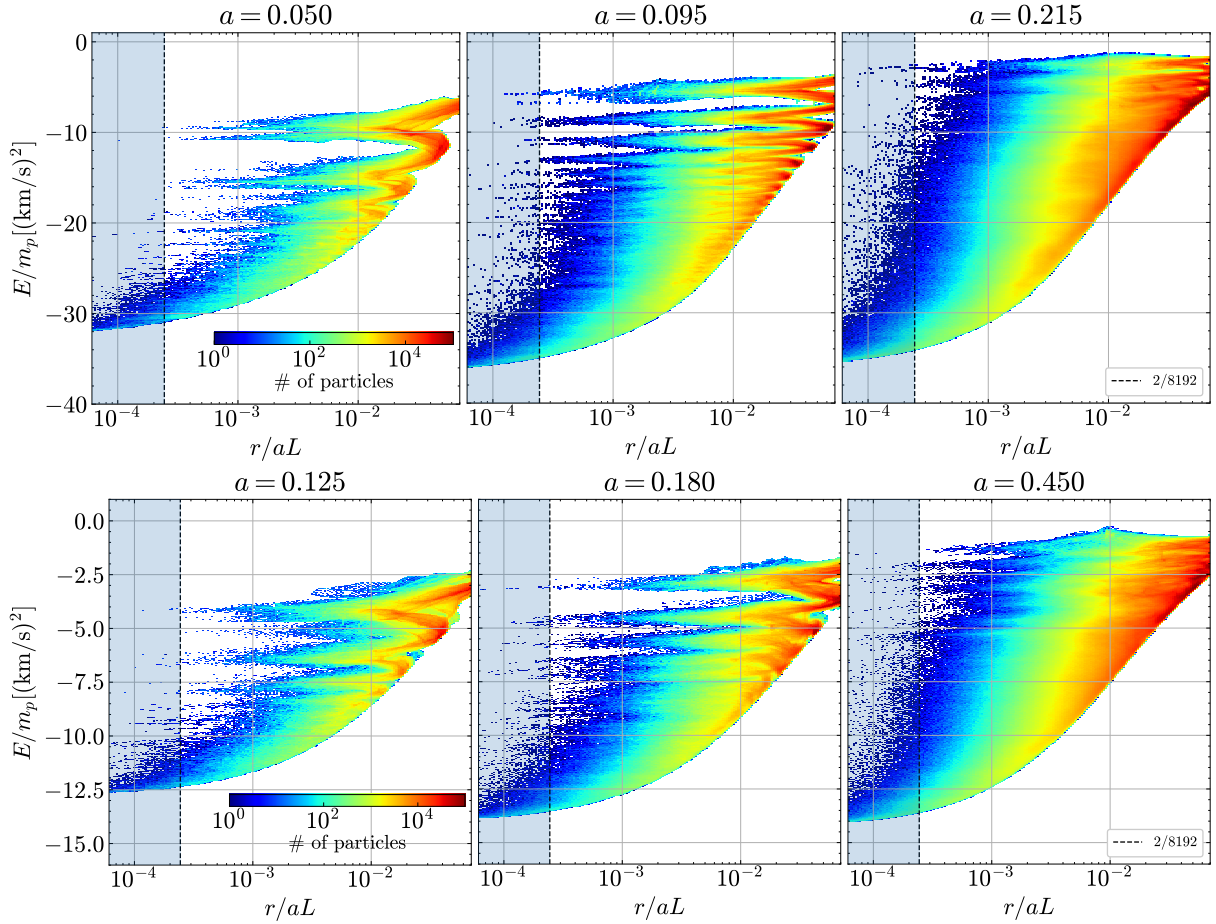


Figure 5.4: Energy distribution of particles in the anisotropic (top row) and quasi-1D (bottom row) simulations. Time progresses from left to right, and the three columns correspond to the snapshots analyzed in detail in the main text.

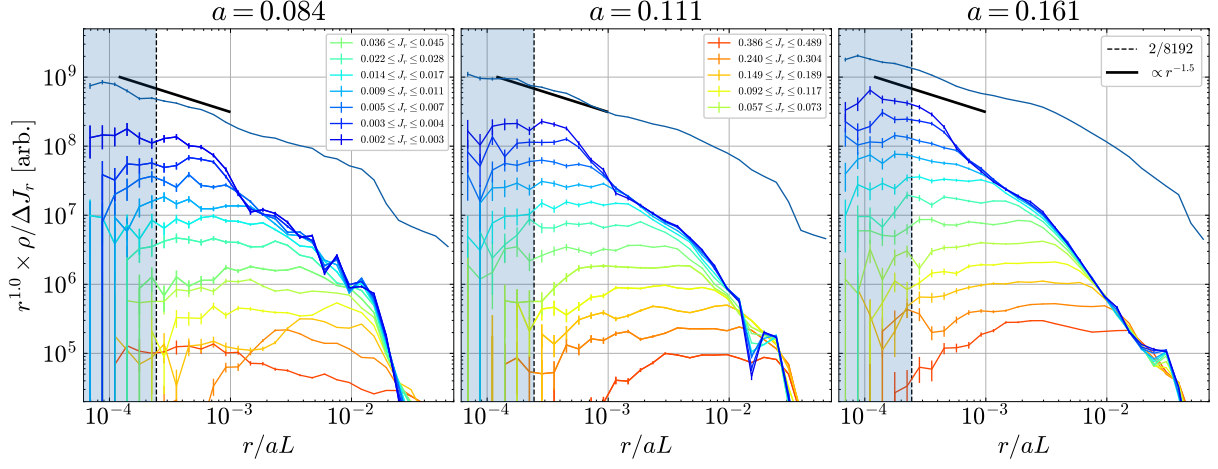


Figure 5.5: The radial density profiles of the Halo 1 in the CDM simulation at three redshifts we investigate. All the density profiles shown are multiplied with r as the label of the y-axis indicates. For profiles of particles binned by radial action color-coded by the value of radial action, they are normalized by the width of the radial action bin. The value of radial action shown in the legend is normalized by the unit of $[R_{\text{vir}} V_{\text{vir}}]$

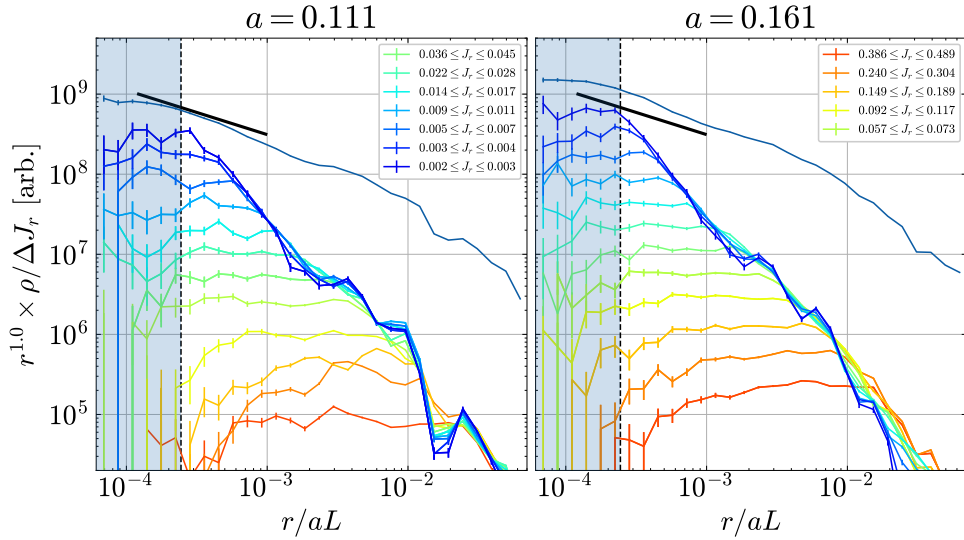


Figure 5.6: Same as Fig. 5.5, but the Halo 2 case in the CDM simulation.

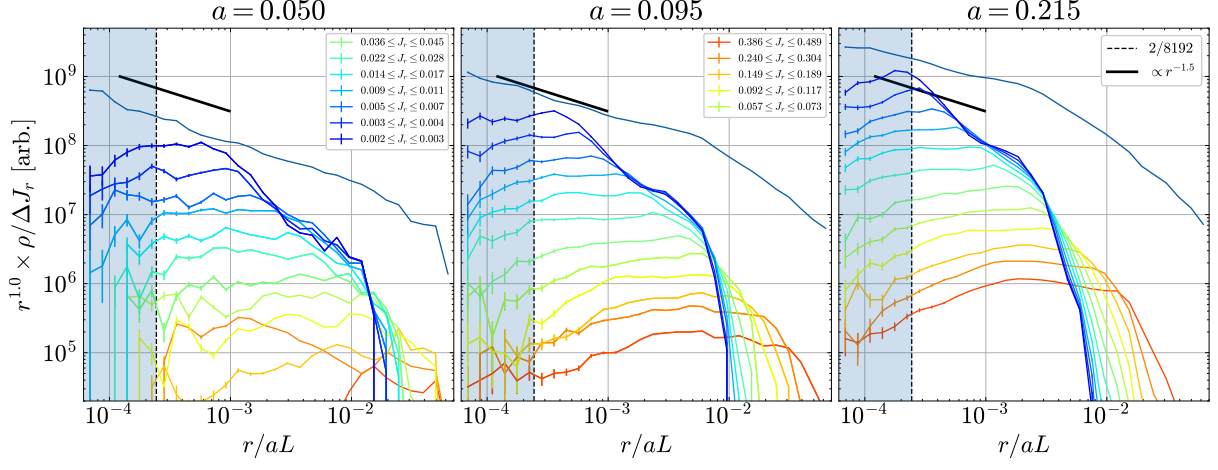


Figure 5.7: Same as Fig. 5.5, but the anisotropic case in the three-sine-wave simulation.

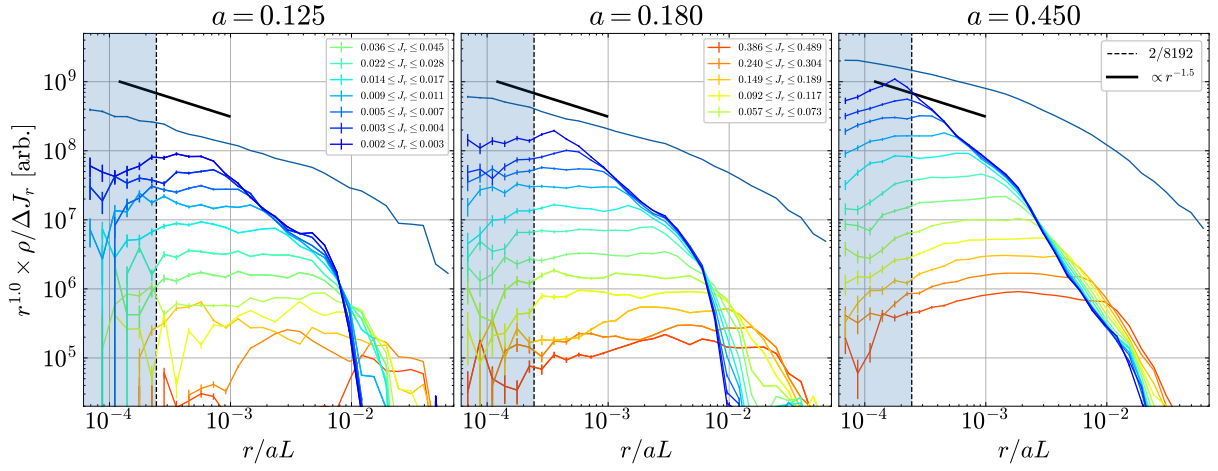


Figure 5.8: Same as Fig. 5.6, but quasi 1D case in the three-sine-wave simulation.

profiles of particles categorized by radial action exhibit power-law behavior in these early halos.

Using N -body simulations that follow the evolution of prompt cusps, we calculate the radial action for simulation particles and reassess $\rho(r; J_r)$. Our results demonstrate that akin to massive DM halos, prompt cusps also show a double-power-law structure in their $\rho(r; J_r)$ profiles with the index of the inner and outer slopes -1 and -2.5 with significant fluctuation due to caustics in the outer part. Although the power-law behavior seems to emerge shortly after the formation of prompt cusps—immediately following the process of violent relaxation—the theoretical explanation for the emergence of the double-power-law structure remains elusive. It is important to highlight, however, that this double-power-law characteristic is observed even in three-sine-wave simulations with varying initial conditions. This observation indicates that the physical origin of this feature may stem from the fundamental properties of self-gravitating systems.

Chapter 6

Towards the understanding of the universality

6.1 Introduction

In Chapters 4 and 5, we have shown the double-power law nature of the density profiles of N -body particles classified by their radial action J_r , $\rho(r; J_r)$. In most of cases $\rho(r; J_r)$ follows $\propto r^{-1}$ and $\propto r^{-3.5}$ in the inner and outer region respectively.

To discuss the origin of this universality further, in this Chapter, we shall connect the form of $\rho(r; J_r)$ with the distribution function of action variables, $f(\mathbf{J})$, using some approximations. Once the connection between $\rho(r; J_r)$ and $f(\mathbf{J})$ is established, our next interest would go on to the evolution of $f(\mathbf{J})$ in self-gravitational systems and uncovering its universal behavior. As we will demonstrate shortly, the power-law behavior of $\rho(r; J_r)$ in the inner regions of halos, where the potential is nearly flat, can indeed be attributed to the power-law behavior of the distribution function. In the outer regions, however, double-power-law profiles are observed, as discussed in previous sections. In these outer regions, the connection between the density profile and the distribution function remains unresolved and represents a compelling avenue for future research.

6.2 Setup and basic formulae

In this section, we show that the asymptotic slope of $\rho(r; J_r)$ in the inner region of halos where its potential is nearly flat is governed by the asymptotic behavior of the joint distribution function $p(j, J_r)$, defined as

$$p(j, J_r) \equiv \int f(\mathbf{J}) dj_z, \quad (6.1)$$

where $f(\mathbf{J})$ represents the distribution function of the action variables $\mathbf{J} = (J_r, j, j_z)$. It should be noted that, according to the Jeans theorem (e.g., Binney & Tremaine (2008)), in a steady-state system, the distribution function can be expressed as a function of integrals of motion, such as energy, angular momentum, or action variables. For modeling $f(\mathbf{J})$, functions proposed by Posti et al. (2015) and Williams & Evans (2015) are commonly utilized, providing a framework for understanding our Galaxy (e.g., Hattori et al. (2021); Binney & Vasiliev (2023, 2024)) or M31-like galaxies (Gherghinescu et al., 2024).

To establish the connection between the slope of $\rho(r; J_r)$ and $p(j, J_r)$, we begin with an equation for the internal mass of particles with a given J_r enclosed within radius r :

$$M(< r; J_r) = \int_0^\infty p(J_r, j) P(< r; J_r, j) dj \quad (6.2)$$

(Pontzen & Governato, 2013), where $P(< r; J_r, j)$ represents the fraction of time a particle with orbital parameters (J_r, j) spends within radius r . This is defined as:

$$\begin{aligned} P(< r; J_r, j) &\equiv \frac{2}{t_{\text{orb}}(J_r, j)} \int_0^r \frac{dr}{v_r(r; J_r, j)} \\ &= \begin{cases} 1 & (r_{\text{apo}} < r) \\ \frac{2}{t_{\text{orb}}} \int_{r_{\text{peri}}}^r \frac{dr}{v_r(r)} & (r_{\text{peri}} < r < r_{\text{apo}}) \\ 0 & (r < r_{\text{peri}}) \end{cases} \end{aligned} \quad (6.3)$$

Here, t_{orb} , r_{peri} , and r_{apo} are the orbital period, pericenter, and apocenter, respectively, determined by (J_r, j) . For simplicity, these arguments are omitted in the second line. Note that $\rho(r; J_r)$ is defined as

$$\rho(r; J_r) = \frac{1}{4\pi r^2} \frac{dM(< r; J_r)}{dr} \propto r^{-3} M(< r; J_r). \quad (6.4)$$

In order to investigate the connection between $\rho(r; J_r)$ and $p(J_r, j)$ from equation (6.3), we need to examine how $P(< r; J_r, j)$ depends on j . This requires understanding the j -dependence of $t_{\text{orb}}(J_r, j)$, $r_{\text{peri}}(J_r, j)$, and $r_{\text{apo}}(J_r, j)$ while keeping J_r fixed. In particular, at small radii, the primary contribution to the internal mass (equation (6.2)) arises from particles with lower j values, as their small pericenters allow them to contribute the internal mass. Hence, in the following analysis, we focus primarily on the behavior of these variables in the $j \rightarrow 0$ limit. For demonstrative purposes, we assume an NFW profile as the system's total density profile, with the density and potential given by:

$$\rho(r) = \rho_{\text{NFW}}(r) \equiv \frac{\rho_0}{(r/r_s)(1 + r/r_s)^2}, \quad (6.5)$$

and

$$\Phi(r) = \Phi_{\text{NFW}}(r) \equiv -4\pi G \rho_0 r_s^2 \frac{\ln(1 + r/r_s)}{r/r_s}, \quad (6.6)$$

where ρ_0 and r_s are free parameters representing the scale density and scale radius, respectively. In such a system, the radial action J_r depends on energy and angular momentum,

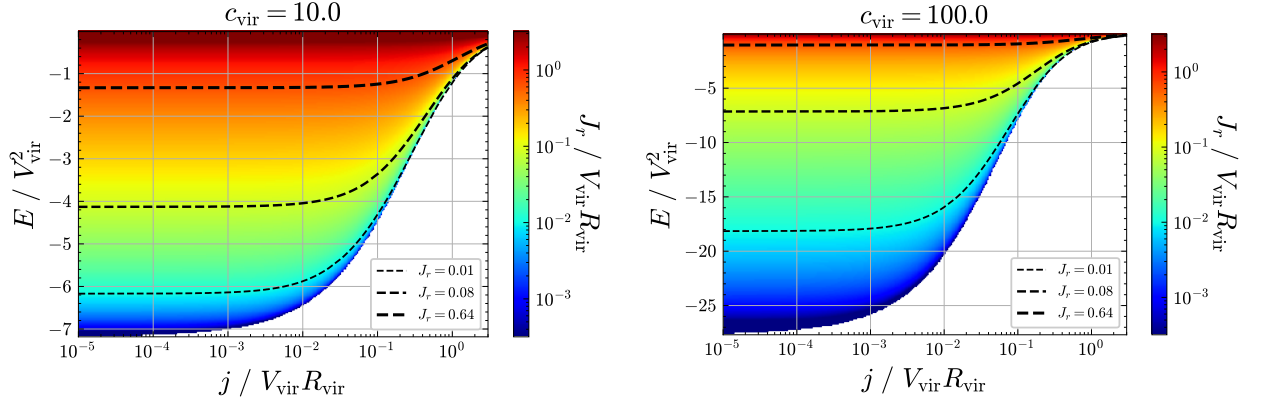


Figure 6.1: Radial action J_r as a function of angular momentum j and energy E in the NFW potential with $c_{\text{vir}} = 10$ (left) and $c_{\text{vir}} = 100$ (right). The units are normalized by the corresponding virial parameters. Dashed lines denote equilateral lines of radial action as indicated by the legends.

$J_r(E, j)$, but its functional form is only available in integral form¹:

$$J_r(E, j) = \frac{1}{\pi} \int_{r_{\text{peri}}}^{r_{\text{apo}}} \sqrt{2(E - \Phi(r)) - \frac{j^2}{r^2}} dr.$$

In Fig 6.1, we show $J_r(E, j)$ in $E - j$ plane color-coded by the value of J_r in potentials with $c_{\text{vir}} = 10$ (left) and $c_{\text{vir}} = 100$ (right). Each dashed line indicates the contour lines of $J_r = 0.01, 0.08, 0.64$. Figure shows that as decreasing j , the specific energy for a particle having a fixed value of J_r approaches an asymptotic value. The asymptotic value of E in $j \rightarrow 0$ limit is given by the inverse function of $J_r(E, j = 0)$, which we denote $E_c(J_r, c_{\text{vir}})$ below.

A more intuitive understanding of this behavior can be drawn from Fig. 6.2, which plots $-E(J_r, j)$ (blue), $t_{\text{orb}}(J_r, j)$ (green), $r_{\text{peri}}(J_r, j)$ (red), and $r_{\text{apo}}(J_r, j)$ (yellow) as functions of j , for the same J_r values as in Fig. 6.1. Each column corresponds to a different fixed J_r . Consider the $J_r = 0.08$ case and examine how these quantities behave as j decreases from $j = 1$. Initially, $r_{\text{peri}}(J_r, j)$ decreases monotonically with decreasing j , since j determines the minimum radius (=pericenter) allowed by the angular momentum barrier. To compensate for the inward shift of the pericenter, the apocenter $r_{\text{apo}}(J_r, j)$ also moves inward to keep J_r constant. However, suppose j goes down to around $J \sim 10^{-2}$, the amount of increase in J_r due to lowered pericenter is negligible compared with 0.08

¹In case of the isochrone potential

$$\Phi_{\text{iso}}(r) = -\frac{GM}{b} \frac{1}{1 + \sqrt{1 + (r/b)^2}}, \quad (6.7)$$

we can evaluate t_{orb} , J_r and r_{peri} explicitly in terms of E and j :

$$t_{\text{orb}}(E) = \frac{2\pi GM}{(-2E)^{3/2}}$$

and

$$J_r(E, j) = \frac{GM}{\sqrt{-2E}} - \frac{1}{2}(j + \sqrt{j^2 + 4GMb})$$

(Binney & Tremaine, 2008).

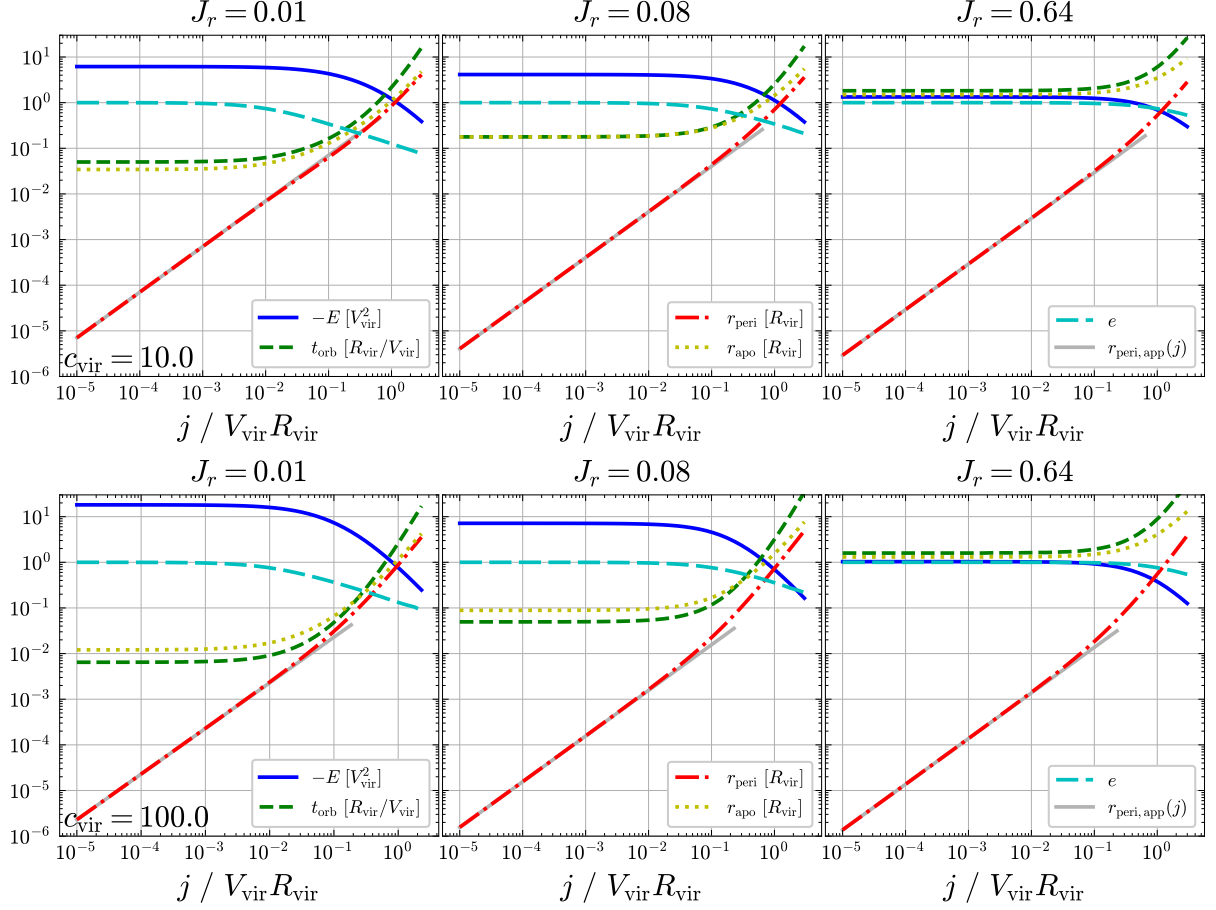


Figure 6.2: Behaviors of energy E (blue solid), orbital period t_{orb} (green dashed), pericenter r_{peri} (red dot-dashed), apocenter r_{apo} (yellow dotted), eccentricity e (cyan long-dashed), and pericenter approximated by equation (6.10) $r_{\text{peri,app}}$ (light gray solid) as functions of angular momentum j with fixing radial action J_r . Each column and each row corresponds to the fixed value of $J_r[V_{\text{vir}}R_{\text{vir}}]$ indicated as the titles and the concentration c_{vir} of halos respectively. The eccentricity e is defined by $e \equiv (r_{\text{apo}} - r_{\text{peri}})/(r_{\text{apo}} + r_{\text{peri}})$.

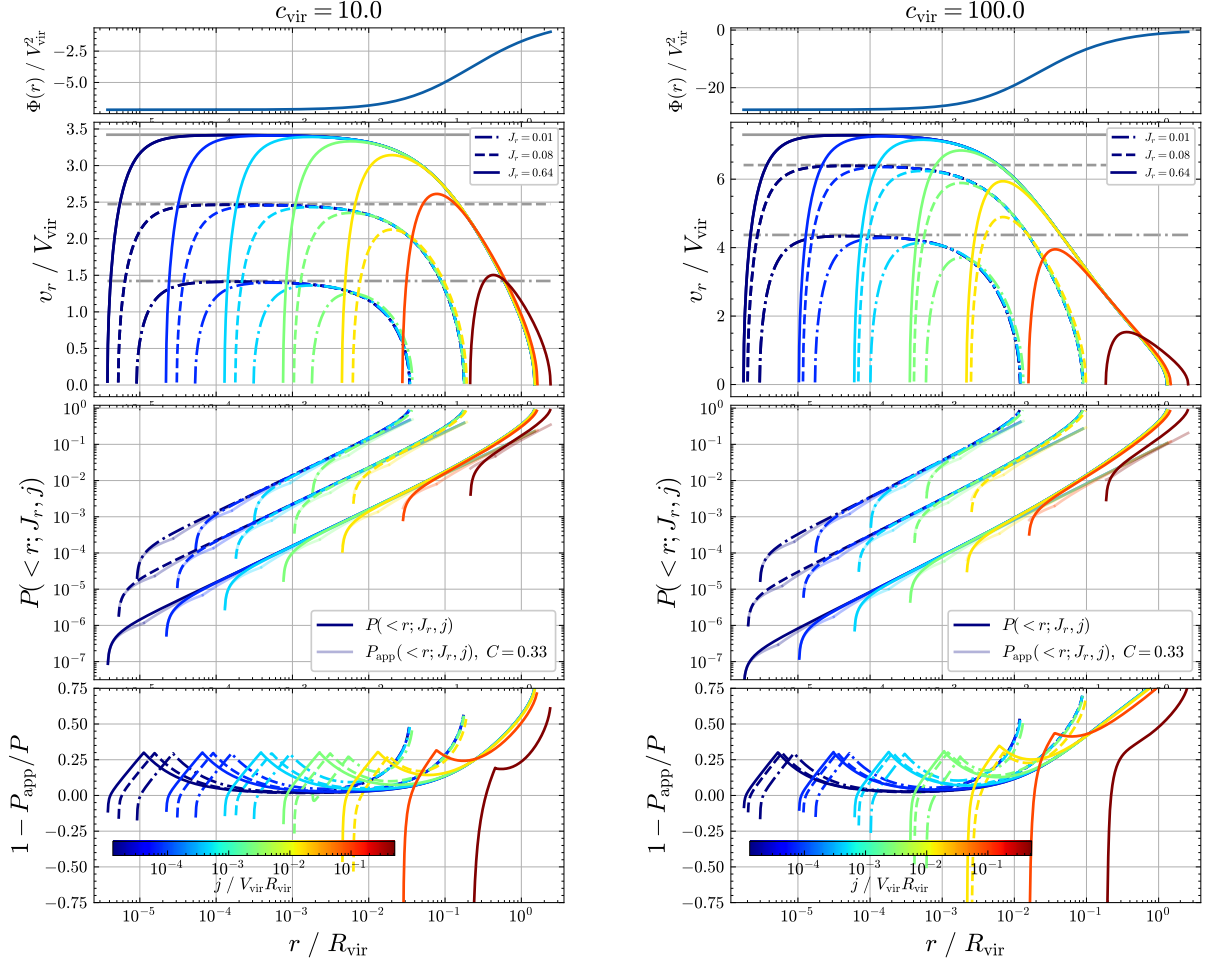


Figure 6.3: Phase-space trajectory and fractional time $P(< r; J_r, j)$ of particles with different radial action and angular momentum. (Top row): the NFW potential of the system. (Second row): the phase-space trajectory of particles for $J_r = 0.01$ (dot-dashed), 0.08 (dashed), 0.64 (solid). The trajectories are color-coded according to their angular momentum as indicated by the color bar in the bottom row. Horizontal light grey lines indicate $v_{r,c}(J_r, c_{\text{vir}})$. (Third row): the fraction of time a particle with orbital parameters (J_r, j) spends within radius r , $P(< r; J_r, j)$ (dark-colored lines). Approximated value for $P(< r; J_r, j)$, $P_{\text{app}}(< r; J_r, j)$ evaluated from equation (6.11) is also shown (light-colored lines). (Bottom row): the fractional difference between $P(< r; J_r, j)$ and $P_{\text{app}}(< r; J_r, j)$. In each column, the color and the kinds of lines represent the same trajectory in the other column.

and r_{apo} stops rapid shrinking and becomes nearly constant around $r_{\text{apo}} \sim 0.2R_{\text{vir}}$. We dub this convergent apocenter as $r_{\text{apo,c}}(J_r)$.

In line with this, the energy of the particle corresponds to its apocenter and become also a nearly constant value $E_c(J_r)$ in the low- j limit. The orbital period is mainly determined by the position of the apocenter, so t_{orb} roughly draws the same curve as r_{apo} as a function of angular momentum j . Note that both the energy and the apocenter do decrease as j is lowered; however, when j becomes sufficiently small, these changes are minimal and effectively negligible. In the $J_r = 0.01$ case (left panel), the angular momentum where energy and apocenter start to become nearly constant is lowered because the increase of radial action due to decreased pericenter is relatively large compared with the $J_r = 0.08$ case. These features are also the case for the highly-concentrated case shown in the lower panels of Fig. 6.2.

6.3 Asymptotic behavior of $\rho(r; J_r)$

Based on the preceding analysis, we can derive two approximate formulae for $P(< r; J_r, j)$ in the radial range $r_{\text{peri}} < r < r_{\text{apo}}$ (see equation (6.3)), specifically in the regime where r is smaller than the scale radius r_s . In this case, the radial dependence of the potential $\Phi(r)$ can be neglected, simplifying the problem. In the second top row in Fig. 6.3, we show the phase-space trajectories of particles with $J_r = 0.01, 0.08, 0.64$ varying their angular momentum as indicated in the color bar for $c_{\text{vir}} = 10$ case (left) and $c_{\text{vir}} = 100$ case (right). The key point here is that we are considering a small r limit where the radial dependence of the potential $\Phi(r)$ is weak because the slope of the total density profile is shallower than -2 . In those radii, because the potential is nearly flat with the value of $\Phi_{\text{min}}(c_{\text{vir}}) \equiv -c_{\text{vir}}[\log(1 + c_{\text{vir}}) - \frac{c_{\text{vir}}}{1+c_{\text{vir}}}]^{-1} [V_{\text{vir}}^2]$, radial velocities shown in Fig. 6.3 are nearly constant except for around their pericenters. That constant radial velocity can be written by $v_{r,c}(J_r, c_{\text{vir}}) \equiv \sqrt{2[E_c(J_r, c_{\text{vir}}) - \Phi_{\text{min}}(c_{\text{vir}})]}$, which is shown as horizontal thin lines in the second top row of Fig. 6.3 and remarkably matches with the constant radial velocity in every value of J_r . Therefore, we assume that the particles orbit by constant radial velocity $v_{r,c}(J_r, c_{\text{vir}})$ and we can obtain the following approximation from equation (6.3),

$$P(< r; J_r, j) \sim P_{\text{orb}}(< r; J_r, j) \equiv \frac{r - r_{\text{peri}}(J_r, j)}{t_{\text{orb,c}}(J_r) v_{r,c}(J_r)}, \quad (6.8)$$

where $t_{\text{orb,c}}$ is the asymptotic value of the orbital period in $j \rightarrow 0$ limit, and here we omit the dependence on c_{vir} for clarity. In Dalal et al. (2010) and Lithwick & Dalal (2011), a constant radial velocity approximation was similarly employed, resulting in analogous formulas that did not account for angular momentum. Therefore, our formula, equation (6.8), can be regarded as a generalized version of the expressions presented in those studies.

Around the pericenter and apocenter, this approximation is invalid because the radial velocity is no longer constant. Especially for current interests, considering small r limit of $\rho(r; J_r)$, a contribution from pericenter matters as we shall show soon. Here we simply consider the Taylor extension of the radial velocity $v_r(r) = \sqrt{2[E - \Phi(r)] - r^2/j^2}$ as a

function of radius and that the potential is nearly constant². We derive an approximation for $P(< r; J_r, j)$ from equation (6.3),

$$P(< r; J_r, j) \sim P_{\text{peri}}(< r; J_r, j) \equiv \frac{1}{t_{\text{orb,c}}(J_r)} \sqrt{\frac{2r_{\text{peri}}^3(J_r, j)}{j^2} [r - r_{\text{peri}}(J_r, j)]}. \quad (6.9)$$

This is the second approximation for $P(< r; J_r, j)$, and we have to combine equations (6.8) and (6.9) to compute the internal mass $M(< r; J_r)$ (equation (6.2)), which will be discussed soon.

With the approximations in equations (6.8) and (6.9), the only j -dependence of $P(< r; J_r, j)$ in the integral equation (6.2) comes from the j -dependence of $r_{\text{peri}}(J_r, j)$. The pericenter is determined as the smaller root of the equation $0 = \sqrt{2[E - \Phi(r)] - r^2/j^2}$, which can be approximated as

$$r_{\text{peri}}(J_r, j) \sim r_{\text{peri,app}}(J_r, j) \equiv \frac{j}{v_{r,c}(J_r, c_{\text{vir}})}. \quad (6.10)$$

The performance of this approximation is represented as the thin grey lines in Fig. 6.2, which closely match the pericenter values r_{peri} (red dot-dashed lines) for $r_{\text{peri}}/R_{\text{vir}} < c_{\text{vir}}^{-1}$. This demonstrates the validity of the approximation in the relevant regime.

In the bottom and second bottom row in Fig. 6.3, we show the fitting performance of P_{orb} and P_{peri} with using equation 6.10 as the approximation of the pericenter. In addition to $P(< r; J_r, j)$ which we numerically compute, here we show

$$P_{\text{app}}(< r; J_r, j) \equiv \begin{cases} P_{\text{peri}}(< r; J_r, j) & (r < r_{\text{peri,app}}(J_r, j)/C) \\ P_{\text{orb}}(< r; J_r, j) & (r > r_{\text{peri,app}}(J_r, j)/C) \end{cases} \quad (6.11)$$

where we introduced new parameter $0 < C < 1$ to handfully separate two ranges where P_{peri} is suitable and where P_{orb} is suitable. We found setting $C = 0.33$ provides the best fit to $P(< r; J_r, j)$. Within the range $r < r_s$, where the potential and radial velocities can be treated as nearly constant, P_{app} agrees with the numerically computed $P(< r; J_r, j)$ to within 30%. Conversely, near the radii corresponding to the apocenters of the trajectories, $r \sim r_{\text{apo,c}}(J_r)$, our approximations lose their validity. Additionally, it is worth noting that P_{peri} does not accurately represent $P(< r; J_r, j)$ at $r \sim r_{\text{peri}}$ due to the limited precision of the Taylor extension near the pericenter. However, since $P(< r; J_r, j) = 0$ at $r = r_{\text{peri}}$, the resulting discrepancy remains negligible. In summary, the approximation P_{app} for $P(< r; J_r, j)$ is sufficiently accurate within the range of radii satisfying both conditions: $r < r_s$, where the potential is approximately flat, and $r < r_{\text{apo,c}}(J_r)$, where the radial velocity deviates from being constant even in a nearly constant potential.

²Radial velocity $v_r(r)$ can be extended around its pericenter as

$$v_r(r) = \sqrt{\frac{2j^2}{r_{\text{peri}}^3}(r - r_{\text{peri}}) - \frac{3j^2}{r_{\text{peri}}^4}(r - r_{\text{peri}})^2 + \mathcal{O}([r - r_{\text{peri}}]^3)}$$

under the assumption of constant potential. Substituting this into equation (6.3), we can obtain

$$P_{\text{peri}}(< r; J_r, j) = \frac{1}{t_{\text{orb,c}}(J_r)} \left\{ \sqrt{\frac{2r_{\text{peri}}^3}{j^2} [r - r_{\text{peri}}]} + \sqrt{\frac{r_{\text{peri}}}{8j^2} [r - r_{\text{peri}}]^3} + \mathcal{O}([r - r_{\text{peri}}]^{5/2}) \right\}$$

for the second order. For simplicity, we mainly use until the first order. This does not change our conclusion in this Chapter significantly.

Now we start calculating the internal mass in the radii $r \ll r_s$. First of all, particles whose pericenters are larger than the radius considering, i.e., $j > j_{\text{th}}(r) \equiv v_{r,c}(J_r)r$, can not contribute to the internal mass:

$$M(< r; J_r) = \int_0^{j_{\text{th}}(r)} p(J_r, j) P(< r; J_r, j) dj. \quad (6.12)$$

Then we divide the range of this integral into two regimes corresponding to the approximations for $P(< r; J_r, j)$. In this integral, $P(< r; J_r, j)$ of the particles with small j can be approximated by $P_{\text{peri}}(< r; J_r, j)$ because the given radius r is close to their pericenters, while the contribution from particles with large j can be approximated by $P_{\text{orb}}(< r; J_r, j)$ because their pericenters are much smaller than the given r . This is done by substituting P_{app} (equation (6.11)) into $P(< r; J_r, j)$:

$$\begin{aligned} M(< r; J_r) &= \int_0^{j_{\text{th}}(r)} p(J_r, j) P_{\text{app}}(< r; J_r, j) dj \\ &= \int_0^{C j_{\text{th}}(r)} p(J_r, j) P_{\text{orb}}(< r; J_r, j) dj + \int_{C j_{\text{th}}(r)}^{j_{\text{th}}(r)} p(J_r, j) P_{\text{peri}}(< r; J_r, j) dj. \end{aligned} \quad (6.13)$$

We define the two terms as:

$$M(< r; J_r) = M_{\text{orb}}(< r, J_r) + M_{\text{peri}}(< r, J_r), \quad (6.14)$$

where: the first term $M_{\text{orb}}(< r, J_r)$ accounts for the contribution from particles with large j . The second term $M_{\text{peri}}(< r, J_r)$ accounts for the contribution from particles with small j . To proceed analytically, we assume a simple form for the joint distribution function:

$$p(J_r, j) = p_c(J_r) j^\alpha, \quad (6.15)$$

where $p_c(J_r)$ depends only on the radial action J_r , and α is a constant. This form approximates the behavior of $p(J_r, j)$ observed in N -body simulations for low- j limit, particularly for $\alpha = 1$, which provides a reasonable match to simulation results. From equations (6.8), (6.9), and (6.10), now we can directly integrate equation (6.13) and obtain

$$M_{\text{orb}}(< r, J_r) = \frac{p_c(J_r) v_{r,c}^\alpha(J_r)}{t_{\text{orb},c}(J_r)} \left\{ \frac{C^{\alpha+1}}{\alpha+1} - \frac{C^{\alpha+2}}{\alpha+2} \right\} r^{\alpha+2}, \quad (6.16)$$

and

$$M_{\text{peri}}(< r, J_r) = \frac{\sqrt{2} p_c(J_r) v_{r,c}^\alpha(J_r) f(C, \alpha)}{t_{\text{orb},c}(J_r)} r^{\alpha+2}, \quad (6.17)$$

where we introduced numerical factor

$$f(C, \alpha) = \int_C^1 x^{\alpha+1/2} \sqrt{1-x} dx. \quad (6.18)$$

For instance, $f(0.33, 0) = 0.280$, and $f(0.33, 1) = 0.174$. From equations (6.16) and (6.17), we can obtain the asymptotic slope of $\rho(r; J_r)$, which is proportional to r^α . Therefore, the slope is determined solely by the j -dependence of the joint distribution function $p(J_r, j)$

in the low- j regime. It is notable that we can also calculate the fraction between M_{peri} and M_{orb} from equations (6.16) and (6.17):

$$\begin{aligned}\frac{M_{\text{peri}}(< r; J_r)}{M_{\text{orb}}(< r; J_r)} &= \sqrt{2}f(C, \alpha) \left\{ \frac{C^{\alpha+1}}{\alpha+1} - \frac{C^{\alpha+2}}{\alpha+2} \right\}^{-1} \\ &= 2.65 \text{ (for } \alpha = 0, C = 0.33) \\ &= 15.5 \text{ (for } \alpha = 1, C = 0.33).\end{aligned}\tag{6.19}$$

This indicates that including the contribution from particles whose pericenters are close to the given r is important for calculating the internal mass precisely.

Finally, in Fig. 6.4, to validate our findings, we compare the internal mass of particles for a given radial action, $M(< r; J_r)$, obtained through numerical calculations (solid lines) with those derived using equations (6.16) and (6.17) (black dotted lines). For this comparison, we employ the joint distribution function $p(j, J_r) = j^\alpha \exp(-j/0.1)$ with $J_r = 0.08$, considering $\alpha = 0.0, 0.5, 1.0$. A total of 1×10^6 particles are sampled from $p(j, J_r)$, and the internal mass is computed numerically, as shown by the solid lines. For angular momenta below $j \lesssim 0.1$, this distribution follows a power-law $p(j, J_r) \propto j^\alpha$. Consequently, our approximated formulas are expected to perform well in the radial range below the pericenter of particles with $j = 0.1$, which is approximately $0.04 [R_{\text{vir}}]$, as inferred from the top middle panel of Fig. 6.2. It is important to note that the necessary conditions for our approximation—namely, $r/R_{\text{vir}} < c_{\text{vir}}^{-1}$ (flat potential condition) and $r < r_{\text{apo,c}}(J_r) \sim 0.2$ (pericenter approximation)—are satisfied within this range of radii. For all values of α , the solid and dotted lines show remarkable agreement, demonstrating that our formulas, equations (6.16) and (6.17), provide an excellent approximation for computing the internal mass and density profiles of particles sampled from nearly power-law distribution functions.

6.4 Comparison with N -body simulations

In the last section, we discussed the relationship between the power-law behavior of the density profile and the distribution function. In summary, the discussion was based on the hypothesis that if the left-hand side of equation (6.2) exhibits a power-law behavior, then both the distribution function $p(j, J_r)$ and the fraction of time $P(< r; j, J_r)$ might also follow a power law. For radii $r < r_s$, where the potential is approximately constant, $P(< r; j, J_r)$ behaves like a power-law function of radius (equations (6.8) and (6.9)). Consequently, the slope of the internal mass is directly related to the slope of the distribution function.

In this section, we investigate how these discussions can be extended to interpret the double power-law behavior observed in the previous Chapters. In Fig. 6.5, we show the density profiles of particles classified by their radial action (left columns), and their distribution function (right columns). We begin by examining particles within the range $0.036 \leq J_r \leq 0.046$ (green lines) of the halo shown in the top row. In this case, the distribution function $p(j)$ remains nearly constant across the range $0.01 \lesssim j \lesssim 0.2$. The scale radius of this halo is $r_s/R_{\text{vir}} = 0.10$, as noted in the title, and the pericenters of particles with angular momenta in this range are below $r_{\text{peri}}(0.01 \leq j \leq 0.2; J_r) \lesssim 0.1 [R_{\text{vir}}]$, as shown by the green solid line in the right panel.

This scenario corresponds to the $\alpha = 0$ case discussed in the previous section, where the distribution function $p(j, J_r)$ behaves as $p(j, J_r) \propto \text{const.}$, resulting in a density profile

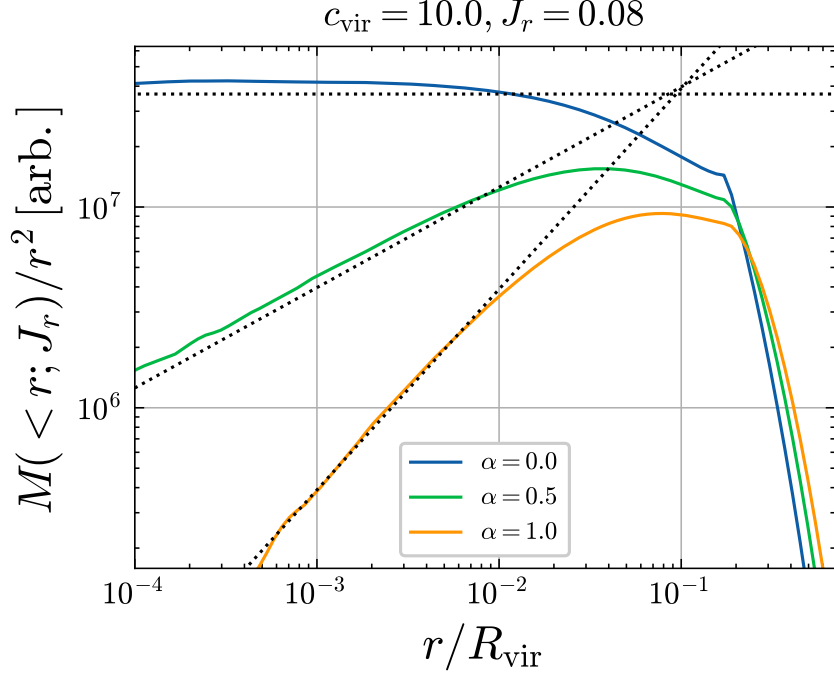


Figure 6.4: Internal mass distribution of 1×10^6 particles sampled from the distribution function $p(j, J_r) = j^\alpha \exp(-j/0.1)$ for $\alpha = 0.0, 0.5, 1.0$. Dotted lines indicate those computed from the sum of equations (6.16) and (6.17) with substituting the corresponding value of α , which coincide with solid lines very well.

$\rho(r; J_r) \propto r^{\alpha-1} \propto r^{-1}$. This is consistent with the density profiles shown by the green solid line in the left panel of the top row. Specifically, for radii $r < 0.1 [R_{\text{vir}}]$, the density profile follows $\rho(r; J_r) \propto r^{-1}$, as the density in this radial range is dominated by particles in the plateau of the distribution function. However, examining the range $j < 0.01$ in the right panel reveals that the distribution of particles within $0.036 \leq J_r \leq 0.046$ (green lines) follows approximately $p(j, J_r) \propto j^{0.7}$. Consequently, their density profiles are expected to follow $\rho \propto r^{-0.3}$ below $r < r_{\text{peri}}(j = 0.01, J_r) \sim 0.005$, although this falls below the resolution limit of our simulations.

A common feature across the halos shown in Fig. 6.5 is that the distribution function exhibits slow growth at small angular momenta and remains nearly flat in the middle range. In most cases, the density profiles show a $\rho \propto r^{-1}$ behavior at radii corresponding to the pericenter ranges where the distribution function is nearly flat or has an extremum. It is worth noting that some distribution functions lack a clear plateau (e.g., green markers in the right panel of the second row) and instead display spiky extrema, yet their corresponding density profiles still follow $\rho \propto r^{-1}$ distinctly. This suggests that a $\rho \propto r^{-1}$ behavior can arise even when the distribution function does not exhibit a clear power-law trend. While our discussion derives the density profile slopes under the assumption of a power-law behavior in the distribution function, the observed deviations highlight an important area for further investigation. Addressing this point, which is not covered in our current discussion, remains a subject for future studies.

We should also address the outer slope of the density profiles. Interestingly, for particles in the range $0.011 \leq J_r \leq 0.014$ (blue line in the left panel of the middle row), the density profile approximately follows $\rho \propto r^{-3}$ for $r \geq 0.04$. Examining the corresponding angular momentum range, i.e., $j \gtrsim 0.2$, the distribution function appears to follow

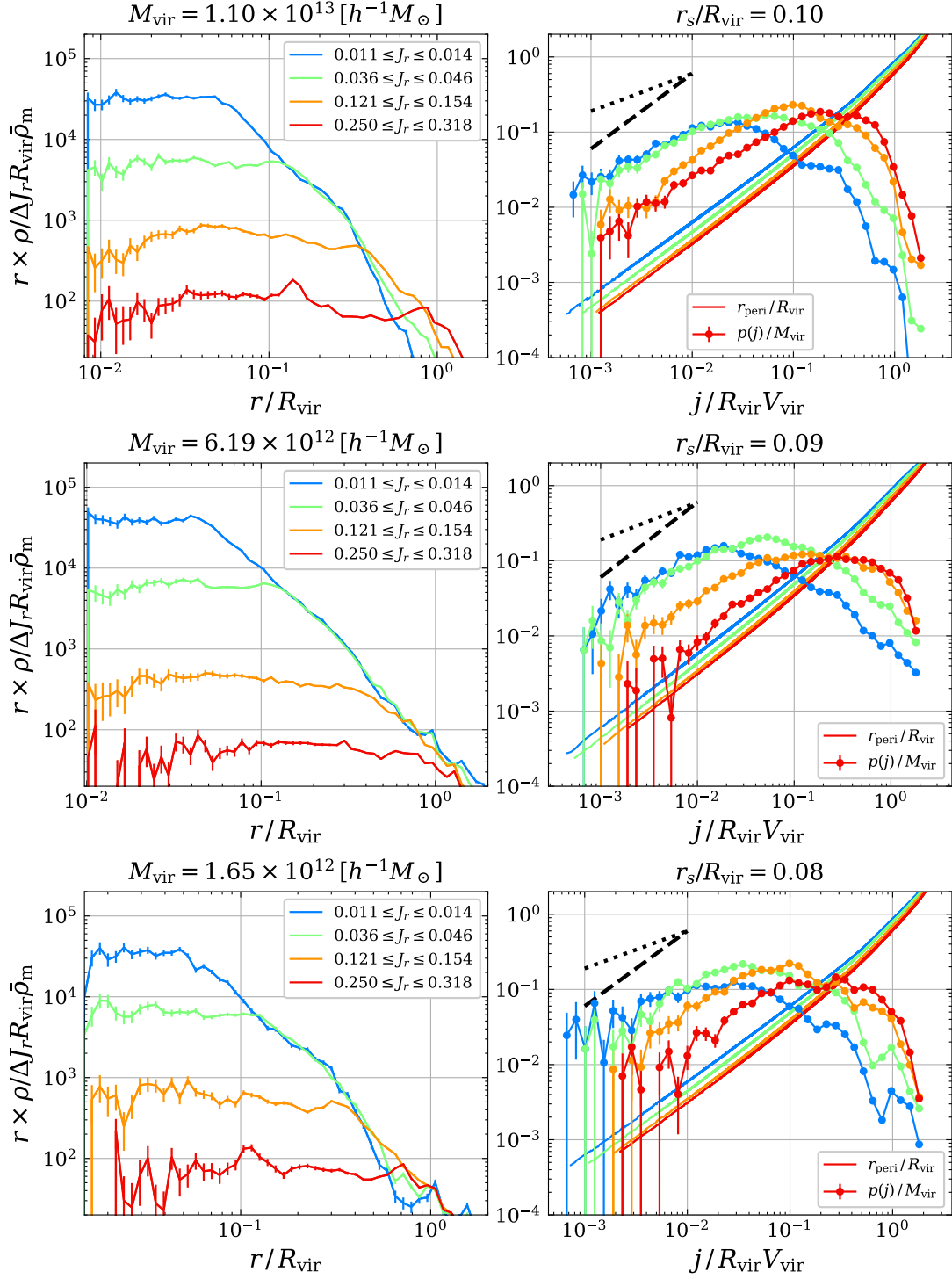


Figure 6.5: The density profiles of particles classified by their radial action (left), their distribution function (markers with errorbar in the right panel), and pericenter (solid lines in the right panel) as functions of angular momentum. In the right panels, we also show referential lines that are proportional to $j^{0.5}$ (dotted lines) and j (black dashed lines). Each row corresponds to halos whose mass and concentration are indicated in the title. The three halos shown are the same as those shown in the left column of Figs. 4.10 and 4.9 in Chapter 4. The lowest radii of the left column is three times the softening radius of the simulation.

$p(j, J_r) \propto j^{-1}$. By substituting $\alpha = -1$ into equations (6.8) and (6.9), we anticipate the density profiles should exhibit $\rho \propto r^{-2}$. However, this expectation is not reflected in the observed results. This discrepancy indicates that our theoretical framework cannot be applied to the outer slopes of the density profiles at radii $r \geq r_s$, where the flat potential approximation breaks down. Furthermore, understanding the outer slopes requires considering the behavior of the apocenters, which becomes relevant in these larger radii. The precise relationship between the outer density slopes and the distribution function in these regions remains an open question, warranting further investigation in future work.

From a theoretical standpoint, several studies have explored the functional form of the distribution function. For instance, Pontzen & Governato (2013) derived the distribution function using the maximum entropy principle for the phase-space distribution, proposing the form $p(j, J_r) = \sinh(\beta_z j) \exp(-\beta_r J_r - \beta_j j - \beta_E E(j, J_r))$, where $\beta_z, \beta_j, \beta_r, \beta_E$ are parameters. In the limit $j \rightarrow 0$, this expression implies $p(j, J_r) \propto j$. Similarly, Lu et al. (2006) suggested that the angular momentum distribution follows $\propto j$ in the $j \rightarrow 0$ limit.

In general, the shape of the angular momentum distribution reflects the available phase-space states of the angular momentum vector. If particles with the same energy exhibit an isotropic velocity distribution, their angular momentum distribution becomes proportional to j as $j \rightarrow 0$ (Lu et al., 2006) (see their equation 16). While this hypothesis should be further validated using N -body simulations, all radial action ranges shown in the right column of Fig. 6.5 display a $p(j, J_r) \propto j^{0.5 \sim 1.0}$ trend. At the very least, we can infer that the corresponding density profile $\rho(r; j, J_r)$ scales as $\rho(r; j, J_r) \propto r^{-0.5 \sim 0}$, which is notably shallower than the inner slopes derived earlier. Although this shallower slope is often masked by the resolution limit (e.g., hidden within the softening length in Fig. 6.5), it becomes evident in high-resolution simulations, as demonstrated in Figs. 5.7 and 5.8 in Chapter 5. These high-resolution results reveal the shallower slope more clearly.

In summary, the $\rho \propto r^{-1}$ feature represents a plateau behavior of the distribution function in both CDM and WDM halos. However, halos may not exhibit this cuspy behavior in the $r \rightarrow 0$ limit, as seen in prompt cusps in high-resolution simulations. This distinction underscores the need for additional studies to refine our understanding of these density profiles at small radii.

6.5 Conclusion

In this Chapter, we investigated the relationship between the density profiles of particles grouped by radial action and their angular momentum distribution functions. Using the flat potential approximation, we derived analytic expressions for the fraction of time particles spend within a given radius, $P(< r; j, J_r)$ (equations (6.8) and (6.9)), and formulated corresponding expressions for the internal mass profiles (equations (6.16) and (6.17)). These formulae demonstrated that the density slope of particles follows $\rho \propto r^{\alpha-1}$, where α is the slope of the distribution function, $p(j, J_r) \propto j^\alpha$, in the $j \rightarrow 0$ limit.

Applying this analytical framework to N -body simulation data, we showed that the $\rho \propto r^{-1}$ feature primarily stems from the nearly flat behavior of the distribution function. Notably, the $\rho \propto r^{-1}$ feature also emerges in cases where the distribution function displays a somewhat spiky structure. Additionally, we predict that shallower density profile slopes could become apparent upon analyzing radii smaller than the current resolution limit set by the softening length.

Regarding the outer slope of $\rho(r; J_r) \propto r^{-3.5}$, our analysis cannot be directly applied because the flat potential approximation breaks down in the radii where this feature is

observed. Specifically, the outer slope of $\rho(r; J_r)$ deviates from the expected relation $\rho(r; J_r) \propto r^{\alpha-1}$. The exact connection between the outer density slopes and the distribution function in these regions remains unresolved, highlighting the need for further exploration in future studies.

Chapter 7

Summary and Discussion

This thesis investigates the intricate phase-space structure of dark matter halos and its implications for understanding cosmic structure formation under the framework of the cold dark matter (CDM) model. By leveraging cosmological N -body simulations, we have advanced the theoretical comprehension of both configuration- and phase-space properties of dark matter halos. Additionally, our work extends these analyses to warm dark matter (WDM) models, providing a comparative perspective that could potentially aid in distinguishing dark matter paradigms. The main contributions and findings of this thesis are summarized as follows:

Multistream structure of CDM Halos (Chapter 3)

Dark matter halos, as collisionless self-gravitating systems, exhibit remarkable universalities in their density and pseudo-phase-space profiles. A central aspect of our investigation was the multi-stream nature of CDM halos, characterized by regions where dark matter velocities are multi-valued at a given spatial location. By classifying particles based on the number of apocenter passages, p , we quantified the phase-space distribution within the splashback radius, which is the outermost radius of the multi-stream region. Our key findings are:

- The density profiles for particles with a fixed p exhibit a double-power-law structure, with inner slopes of ~ -1 and steep outer slopes (-7 or steeper), universally across halo mass ranges.
- Stacked density profiles averaged over hundreds of halos reveal that the density structures are well-modeled by a fitting function parameterized by p , with minimal mass dependence. This model reproduces the total density profiles, aligning closely with the Einasto profile even below the softening length of the simulations we calibrated the fitting functions.
- Discrepancies between self-similar solutions and simulation results underscore the complexity of multi-stream dynamics and the need to incorporate realistic accretion and merger histories to construct a reliable model that can describe the phase-space structure of simulated halos.

Inner structure of CDM and WDM halos (Chapter 4)

We also explored WDM models to investigate how deviations from CDM properties manifest in the phase-space structure of halos. By employing the radial action, J_r , as an alternative classification criterion of the multistream region, we examined the density profiles of halos in WDM simulations with particle masses $m_{\text{DM}} = 0.5$ and 1 keV . Key results include:

- The double-power law nature of radial density profiles persists in WDM halos, with inner slopes of -1 and outer slopes of -3.5 when classified by J_r . This contrasts with the steeper outer slopes observed in p -based classifications.
- Systematic differences in the amplitude and characteristic scales of density profiles were observed across the three dark matter models. Higher WDM particle masses correspond to halos with increased inner densities, reduced outer densities, and smaller characteristic scales.
- Despite these differences, halos with similar concentration parameters exhibit nearly identical J_r -based density profiles, suggesting that halo concentration plays a key role in shaping these structures across dark matter models.
- There is a connection between the number of apocenter passages p and the radial action J_r . Especially for the inner part of the density profiles that exhibit r^{-1} feature, classified by J_r and p results in the same density slope because both methods classify particles in nearly the same way.

The difference in the density profiles between CDM and WDM offers a possible framework for testing alternative dark matter models by observing the phase-space structure of stars in our galaxy or local galaxies.

Inner structure of prompt cusps (Chapter 5)

We investigated the inner structure of prompt cusps—the first halos formed near the free-streaming scale corresponding to the mass of dark matter. Through high-resolution N -body simulations, we observed that prompt cusps exhibit a double-power-law structure in their $\rho(r; J_r)$ profiles. The indices of the inner and outer slopes were found to be approximately -1 and -2.5 , respectively, with significant fluctuations in the outer region caused by caustics. Although the power-law behavior emerges shortly after the formation of the prompt cusps, coinciding with the violent relaxation phase, a theoretical explanation for the origin of the double-power-law structure remains elusive.

Connecting density profiles to action variables (Chapter 6)

We explored the relationship between density slopes and distribution functions of action variables to unravel the origin of universal density profiles. This theoretical framework demonstrated that the asymptotic slopes are governed by the form of the distribution function. Through the analysis of N -body simulation data, we showed that the inner slope of $\rho(r; J_r)$ is dictated by the characteristics of the distribution function and is expected to be shallower than -1 . The findings presented in this chapter highlight the critical role of theoretical investigations in developing a framework to describe the distribution of action variables.

Outlook

This thesis demonstrates that the phase-space structure of dark matter halos has universal features as well as the density profiles and the pseudo-phase-space densities. While significant progress has been made, several open questions remain:

- The physical origin of the universal profiles of particles classified radial action J_r , particularly the behavior of the distribution function of action variables warrants further investigation.
- The discrepancies between self-similar models and numerical simulations highlight the need for improved models incorporating non-linear, asymmetric dynamics.
- Extending this analysis to other dark matter candidates such as self-interacting dark matter and fuzzy dark matter models or including baryonic effects could offer new insights into the interplay between dark matter and visible matter in galaxy formation.
- It would be also interesting to investigate the distribution of stars classified by radial action and compare them with those of dark matter, utilizing the cosmological hydrodynamics simulations. If they also exhibit power-law behavior, they could be a handful of probes of the nature of dark matter or the formation history of our galaxy.

This thesis has made significant strides in characterizing the phase-space structure of dark matter halos, yet several avenues for future research remain. Advancements in numerical methods, such as Vlasov-Poisson solvers (Yoshikawa et al., 2013; Hahn & Angulo, 2016; Sousbie & Colombi, 2016), could enable more detailed studies of multi-stream dynamics and phase-space folding. Additionally, observational tests of the phase-space features identified here, particularly those distinguishing CDM from WDM, hold great promise for constraining dark matter properties.

Overall, this study contributes to a growing body of work that bridges theoretical models, simulations, and observations, aiming to unravel the mysteries of dark matter and its role in cosmic structure formation.

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