

Around Golden-Thompson inequality

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1. INTRODUCTION

For $n \in \mathbb{N}$, $\mathbb{M}_n = \mathbb{M}_n(\mathbb{C})$ denotes the space of all $n \times n$ complex matrices. Let $A = (a_{ij}) \in \mathbb{M}_n$. The trace of A is the sum of the diagonal entries:

$$\mathrm{Tr}(A) = \sum_{i=1}^n a_{ii}.$$

A norm $\|\cdot\|$ on $\mathbb{M}_n$ is said to be unitarily invariant if

$$\|A\| = \|UAV\|$$

for all $A \in \mathbb{M}_n$ and all unitaries $U, V \in \mathbb{M}_n$. Let A and B be Hermitian matrices in $\mathbb{M}_n$. The partial ordering $A \geq B$ holds if $A - B$ is positive semi-definite, or equivalently

$$x^*Ax \geq x^*Bx$$

for all vectors $x \in \mathbb{C}^n$.

In the commutative case, if A and B are Hermitian matrices, then $e^{A+B} = e^A e^B$. However, in the noncommutative case, it is entirely no relation between e^{A+B} and $e^A e^B$ under the usual order. The celebrated Golden-Thompson trace inequality, independently proved by Golden[5] and Thompson[13], says as follows:

Theorem 1. *If A and B are Hermitian matrices in $\mathbb{M}_n$, then*

$$(1) \quad \mathrm{Tr}(e^{A+B}) \leq \mathrm{Tr}(e^A e^B).$$

Moreover, Hiai-Petz in [6] showed the following unitarily invariant norm version of Theorem 1:

Theorem 2. *If A and B are Hermitian matrices in $\mathbb{M}_n$, then*

$$(2) \quad \|e^{A+B}\| \leq \left\| (e^{pA/2} e^{pB} e^{pA/2})^{1/p} \right\| \quad \text{for all } p > 0$$

for every unitarily invariant norm $\|\cdot\|$, and the right hand side of (2) converges to $\|e^{A+B}\|$ as $p \downarrow 0$. In particular,

$$(3) \quad \|e^{A+B}\| \leq \|e^{A/2} e^B e^{A/2}\| \leq \|e^A e^B\|.$$

Let A and B be positive definite matrices in $\mathbb{M}_n$ and $\alpha \in [0, 1]$. The weight geometric matrix mean $A \sharp_\alpha B$ is defined as

$$A \sharp_\alpha B = A^{1/2} (A^{-1/2} B A^{-1/2})^\alpha A^{1/2}.$$

Ando-Hiai [2] showed the following complemented Golden-Thompson inequalities:

Theorem 3. If A and B are Hermitian matrices in $\mathbb{M}_n$ and $\alpha \in [0, 1]$, then

$$(4) \quad \left\| (e^{pA} \sharp_{\alpha} e^{pB})^{1/p} \right\| \leq \left\| e^{(1-\alpha)A + \alpha B} \right\|$$

for all $p > 0$ and the left hand side of (4) increases to $\left\| e^{(1-\alpha)A + \alpha B} \right\|$ as $p \downarrow 0$ for any unitarily invariant norm $\|\cdot\|$. In particular,

$$\mathrm{Tr} (e^{pA} \sharp_{\alpha} e^{pB})^{1/p} \leq \mathrm{Tr} (e^{(1-\alpha)A + \alpha B}) \quad \text{for all } p > 0.$$

Remark 4. If we put $p = 1$ and $\alpha = \frac{1}{2}$ in Theorem 3 and replacing A, B by $2A, 2B$ respectively, then we have the lower bound of the Golden-Thompson inequality (3):

$$\left\| e^{2A} \sharp e^{2B} \right\| \leq \left\| e^{A+B} \right\| \leq \left\| e^A e^B \right\|$$

for any unitarily invariant norm $\|\cdot\|$. In particular,

$$\mathrm{Tr} (e^{2A} \sharp e^{2B}) \leq \mathrm{Tr} (e^{A+B}) \leq \mathrm{Tr} (e^A e^B).$$

To show the reverse of Theorem 3, we need some preliminaries. We present an important constant due to Specht [12], who estimated the upper bound of the arithmetic mean by the geometric one for positive numbers: For $x_1, \dots, x_n \in [m, M]$

$$(5) \quad \sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n} \leq S(h) \sqrt[n]{x_1 x_2 \cdots x_n}$$

where $h = \frac{M}{m}$ and the Specht ratio is defined by

$$(6) \quad S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \quad (h \neq 1) \quad \text{and} \quad S(1) = 1.$$

We note that the Specht theorem (5) means a ratio type reverse inequality of the arithmetic-geometric mean inequality.

Now, in [4], we showed a noncommutative version of the Specht theorem (5):

Theorem 5. Let A be a positive definite matrix in $\mathbb{M}_n$ such that $0 < m \leq A \leq M$ for some scalars $0 < m < M$ and put $h = \frac{M}{m}$. Then

$$(7) \quad e^{\langle \log A, x, x \rangle} \leq \langle Ax, x \rangle \leq S(h) e^{\langle \log A, x, x \rangle}$$

holds for every unit vector $x \in \mathbb{C}^n$.

We mention some basic properties of the Specht ratio $S(h)$ in [3, Theorem 2.16, Theorem 2.17]:

Lemma 6. Let $h > 0$ and $\alpha \in \mathbb{R}$.

- (i) $S(1) = \lim_{h \rightarrow 1} S(h) = 1$.
- (ii) $S(h) = S(h^{-1})$.
- (iii) A function $S(h)$ is strictly decreasing for $0 < h < 1$ and strictly increasing for $h > 1$.
- (iv) $\lim_{\alpha \rightarrow 0} S(h^{\alpha})^{1/\alpha} = 1$.
- (v) $\lim_{\alpha \rightarrow \infty} S(h^{\alpha})^{1/\alpha} = h$ for $h > 1$ and $\lim_{\alpha \rightarrow \infty} S(h^{\alpha})^{1/\alpha} = h^{-1}$ for $0 < h < 1$.
- (vi) $\lim_{r \rightarrow 0} K(h^r, \frac{\alpha}{r}) = S(h^{\alpha})$.

We showed reverses of the complemented Golden-Thompson inequality (4) due to Ando-Hiai in terms of the Specht ratio in [11]:

Theorem 7. Let A and B be Hermitian matrices such that $m \leq A, B \leq M$ for some scalars $m < M$, and let $\alpha \in [0, 1]$. Then

$$(8) \quad (\| (e^{pA} \sharp_{\alpha} e^{pB})^{\frac{1}{p}} \| \leq \| e^{(1-\alpha)A + \alpha B} \| \leq S(e^{p(M-m)})^{\frac{1}{p}} \| (e^{pA} \sharp_{\alpha} e^{pB})^{\frac{1}{p}} \|$$

for all $p > 0$ and every unitarily invariant norm $\|\cdot\|$, and the right-hand side of (8) converges to the middle hand side as $p \downarrow 0$. In particular,

$$(\| e^{2A} \sharp e^{2B} \| \leq \| e^{A+B} \| \leq S(e^{2(M-m)}) \| e^{2A} \sharp e^{2B} \|$$

and

$$(\mathrm{Tr} (e^{2A} \sharp e^{2B}) \leq \mathrm{Tr} (e^{A+B}) \leq S(e^{2(M-m)}) \mathrm{Tr} (e^{2A} \sharp e^{2B})).$$

The obvious generalization of the Golden-Thompson trace inequality (1), namely,

$$\mathrm{Tr}(e^{A+B+C}) \leq \mathrm{Tr}(e^A e^B e^C)$$

is not true in general. We would like to consider a $k(\geq 3)$ -variable version of the Golden-Thompson trace inequality and its complements.

One is to consider the Hadamard product instead of the usual product. For $A = (a_{ij}), B = (b_{ij}) \in \mathbb{M}_n$, the Hadamard product is defined to be the entrywise product

$$A \circ B = (a_{ij} b_{ij}).$$

The following result due to Ando is already shown in [1]:

Theorem 8. Let $A_1, \dots, A_k$ be Hermitian matrices, and $\circ$ the Hadamard product. Then

$$\| e^{A_1 + \dots + A_k} \| \leq \| e^{U^* A_1 U} \circ \dots \circ e^{U^* A_k U} \|$$

for some unitary U and every unitarily invariant norm $\|\cdot\|$.

In the commutative case, we have

$$e^{A+B+C} = e^A e^B e^C = (e^{3A} e^{3B} e^{3C})^{1/3},$$

that is, the right hand side is regarded as the geometric mean of e^{3A}, e^{3B}, e^{3C} . Thus, the other is to consider a k -variable geometric mean version instead of the matrix geometric mean in Theorem 7.

In the next section, we will proceed with a discussion in this direction.

2. k -VARIABLE VERSION

First of all, we recall the k -variable version of the matrix geometric mean: We start with the Karcher mean of positive definite matrices in $\mathbb{M}_n$: In 2012, Lim and Pálfi [10] established the formulation of the geometric mean for $k (\geq 3)$ positive definite matrices which is a nice extension of the matrix geometric mean in the Kubo-Ando theory [8]. They showed that there exists the unique positive definite solution of the Karcher equation

$$(9) \quad \sum_{i=1}^k \omega_i \log X^{-\frac{1}{2}} A_i X^{-\frac{1}{2}} = 0$$

for given k positive definite matrices $A_1, \dots, A_k$, where $\omega = (\omega_1, \dots, \omega_k)$ is a weight vector, i.e., $\omega_1, \dots, \omega_k \geq 0$ and $\sum_{i=1}^k \omega_i = 1$. We say the solution X of (9) the Karcher mean for n positive definite matrices $A_1, \dots, A_k$ and denote it by $G_K(\omega; A_1, \dots, A_k)$. In the

case of $k = 2$, the Karcher mean $G_K((1 - \alpha, \alpha); A, B)$ coincides with the weighted matrix geometric mean

$$A \sharp_\alpha B = A^{1/2}(A^{-1/2}BA^{-1/2})^\alpha A^{1/2} \quad \text{for } \alpha \in [0, 1].$$

We list some properties of the Karcher mean which we need later, also see [9]:

- (P1) Consistency with scalars: $G_K(\omega; A_1, \dots, A_k) = A_1^{\omega_1} \cdots A_k^{\omega_k}$ if the A_i 's commute;
- (P2) Joint homogeneity: $G_K(\omega; a_1 A_1, \dots, a_k A_k) = a_1^{\omega_1} \cdots a_k^{\omega_k} G_K(\omega; A_1, \dots, A_k)$;
- (P3) Permutation invariance: $G_K(\omega_\sigma; A_{\sigma(1)}, \dots, A_{\sigma(k)}) = G_K(\omega; A_1, \dots, A_k)$ where $\omega_\sigma = (\omega_{\sigma(1)}, \dots, \omega_{\sigma(k)})$ and σ is any permutation;
- (P4) Transformer inequality: $T^* G_K(\omega; A_1, \dots, A_k) T \leq G_K(\omega; T^* A_1 T, \dots, T^* A_k T)$ for every operator T ;
- (P5) Self-duality: $G_K(\omega; A_1^{-1}, \dots, A_k^{-1})^{-1} = G_K(\omega; A_1, \dots, A_k)$;
- (P6) Information monotonicity: $\Phi(G_K(\omega; A_1, \dots, A_k)) \leq G_K(\omega; \Phi(A_1), \dots, \Phi(A_k))$ for any unital positive linear map Φ ;
- (P7) AGH weighted mean inequality:

$$\left(\sum_{i=1}^k \omega_i A_i^{-1} \right)^{-1} \leq G_K(\omega; A_1, \dots, A_k) \leq \sum_{i=1}^k \omega_i A_i.$$

- (P8) Determinant identity:

$$\det(G_K(\omega : A_1, \dots, A_k)) = \prod_{i=1}^k \det(A_i)^{\omega_i}.$$

Moreover, Yamazaki in [14] showed the following Ando-Hiai inequality for the Karcher mean:

Theorem 9. *Let $A_1, \dots, A_k$ be positive definite matrices and $\omega = (\omega_1, \dots, \omega_k)$ a weight vector. Then*

$$G_K(\omega : A_1, \dots, A_k) \leq I \quad \text{implies} \quad G_K(\omega : A_1^p, \dots, A_k^p) \leq I \quad \text{for all } p \geq 1.$$

By Theorem 9, we show a k -variable version of Theorem 3. Put $\|G\|_\infty = \|G_K(\omega : A_1, \dots, A_k)\|_\infty$, where $\|\cdot\|_\infty$ is matrix norm. Since

$$G_K(\omega : A_1, \dots, A_k) \leq \|G_K(\omega : A_1, \dots, A_k)\|_\infty I,$$

it follows from (P2) that

$$G_K(\omega : \frac{A_1}{\|G\|_\infty}, \dots, \frac{A_k}{\|G\|_\infty}) \leq I.$$

By Theorem 9, we have

$$G_K(\omega : \left(\frac{A_1}{\|G\|_\infty}\right)^p, \dots, \left(\frac{A_k}{\|G\|_\infty}\right)^p) \leq I \quad \text{for all } p \geq 1$$

and hence

$$G_K(\omega : A_1^p, \dots, A_k^p) \leq \|G_K(\omega : A_1, \dots, A_k)\|_\infty^p I.$$

Therefore we have

$$\|G_K(\omega : A_1^p, \dots, A_k^p)\|_\infty \leq \|G_K(\omega : A_1, \dots, A_k)\|_\infty^p.$$

For $0 < q < p$, since $p/q \geq 1$, the fact above implies

$$\left\| G_K(\omega : A_1^{p/q}, \dots, A_k^{p/q}) \right\|_{\infty} \leq \left\| G_K(\omega : A_1, \dots, A_k)^{p/q} \right\|_{\infty}.$$

Replacing A_i by A_i^q , we have

$$\left\| G_K(\omega : A_1^p, \dots, A_k^p)^{1/p} \right\|_{\infty} \leq \left\| G_K(\omega : A_1^q, \dots, A_k^q)^{1/q} \right\|_{\infty} \quad \text{for all } 0 < q < p.$$

Since Hiai-Petz in [7] showed the Lie-Trotter formula for the Karcher mean:

$$\lim_{q \rightarrow 0} G_K(\omega : A_1^q, \dots, A_k^q)^{1/q} = e^{\omega_1 \log A_1 + \dots + \omega_k \log A_k},$$

as $q \rightarrow 0$ we have

$$\left\| G_K(\omega : A_1^p, \dots, A_k^p)^{1/p} \right\|_{\infty} \leq \left\| e^{\omega_1 \log A_1 + \dots + \omega_k \log A_k} \right\|_{\infty}.$$

By antisymmetric tensor technique and (P8), we have

$$\left\| \left\| G_K(\omega : A_1^p, \dots, A_k^p)^{1/p} \right\| \right\| \leq \left\| \left\| e^{\omega_1 \log A_1 + \dots + \omega_k \log A_k} \right\| \right\|$$

for every unitarily invariant norm $\|\cdot\|$. See [2] for antisymmetric tensor technique. Hence we have the following Golden-Thompson inequality for the Karcher mean due to Hiai-Petz in [7]:

Theorem 10 (Hiai-Petz [7]). *Let $A_1, \dots, A_k$ be positive definite matrices and $\omega = (\omega_1, \dots, \omega_k)$ a weight vector. Then*

$$(10) \quad \text{Tr}[G_K(\omega : e^{pA_1}, \dots, e^{pA_k})^{1/p}] \leq \text{Tr}[e^{\omega_1 A_1 + \dots + \omega_k A_k}] \quad \text{for all } p > 0$$

and the left hand side of (10) converges to $\text{Tr}[e^{\omega_1 A_1 + \dots + \omega_k A_k}]$ as $p \downarrow 0$. In particular,

$$\text{Tr}[G_K(\tilde{\omega} : e^{kA_1}, \dots, e^{kA_k})] \leq \text{Tr}[e^{A_1 + \dots + A_k}],$$

where a weight vector $\tilde{\omega} = (1/k, \dots, 1/k)$.

Remark 11. *Theorem 10 is just a k -variable version of Theorem 3, that is, if we put $k = 2$ in Theorem 10, then we have Theorem 3.*

Next, we show a k -variable version of Theorem 7. For this, we need the following Lemma:

Lemma 12. *Let $A_1, \dots, A_k$ be positive definite matrices such that $m \leq A_i \leq M$ for some scalars $0 < m \leq M$ and $\omega = (\omega_1, \dots, \omega_k)$ a weight vector. Put $h = \frac{M}{m}$. Then*

$$(11) \quad \sum_{i=1}^k \omega_i A_i \leq S(h) e^{\sum_{i=1}^k \omega_i \log A_i}$$

where the Specht ratio $S(h)$ is defined by (6).

Proof. Put $\mathbb{A} = \text{diag}(A_1, \dots, A_k)$, $y = (\sqrt{\omega_1}x, \dots, \sqrt{\omega_k}x)^T$ for every unit vector $x \in \mathbb{C}^n$. By Theorem 5, since $m \leq \mathbb{A} \leq M$, we have

$$\langle \mathbb{A}y, y \rangle \leq S(h) e^{(\log \mathbb{A}y, y)}.$$

Hence it follows from the Jensen inequality that

$$\begin{aligned}
 \langle (\sum_{i=1}^k \omega_i A_i) x, x \rangle &= \langle \mathbb{A} y, y \rangle \\
 &\leq S(h) e^{\langle \log \mathbb{A} y, y \rangle} \\
 &= S(h) e^{\langle \sum_{i=1}^k \omega_i \log A_i x, x \rangle} \\
 &\leq S(h) \langle e^{\sum_{i=1}^k \omega_i \log A_i} x, x \rangle \quad \text{by (7)}
 \end{aligned}$$

for every unit vector $x \in \mathbb{C}^n$ and we get (11):

$$\sum_{i=1}^k \omega_i A_i \leq S(h) e^{\sum_{i=1}^k \omega_i \log A_i}.$$

□

Theorem 13. Let $A_1, \dots, A_k$ be positive definite matrices such that $m \leq A_i \leq M$ for some scalars $0 < m \leq M$ and $\omega = (\omega_1, \dots, \omega_k)$ a weight vector. Put $h = \frac{M}{m}$. Then

$$(12) \quad \left\| e^{\sum_{i=1}^k \omega_i A_i} \right\| \leq S(e^{p(M-m)})^{1/p} \left\| G_K(\omega : e^{pA_1}, \dots, e^{pA_k})^{1/p} \right\|$$

for all $p > 0$ and every unitarily invariant norm $\|\cdot\|$, and the right-hand side of (12) converges to the left hand side as $p \downarrow 0$. In particular,

$$\left\| e^{A_1 + \dots + A_k} \right\| \leq S(e^{(M-m)}) \left\| G_K(\tilde{\omega} : e^{A_1}, \dots, e^{A_k}) \right\|$$

where a weight vector $\tilde{\omega} = (1/k, \dots, 1/k)$, and

$$\text{Tr}[e^{A_1 + \dots + A_k}] \leq S(e^{(M-m)}) \text{Tr}[G_K(\tilde{\omega} : e^{A_1}, \dots, e^{A_k})].$$

Proof. By Lemma 12 and (P7), we have

$$G_K(\omega : A_1, \dots, A_k) \leq \sum_{i=1}^k \omega_i A_i \leq S(h) e^{\sum_{i=1}^k \omega_i \log A_i}.$$

Replacing A_i by e^{-pA_i} for $i = 1, \dots, k$ and $p > 0$, since $e^{-pM} \leq e^{-pA_i} \leq e^{-pm}$, it follows that

$$G_K(\omega : e^{-pA_1}, \dots, e^{-pA_k}) \leq S(e^{p(M-m)}) e^{\sum_{i=1}^k -\omega_i p A_i}.$$

Taking the inverse of both sides, we have

$$G_K(\omega : e^{-pA_1}, \dots, e^{-pA_k})^{-1} \geq S(e^{p(M-m)})^{-1} e^{\sum_{i=1}^k \omega_i p A_i}$$

and this and (P5) imply

$$e^{\sum_{i=1}^k \omega_i p A_i} \leq S(e^{p(M-m)}) G_K(\omega : e^{pA_1}, \dots, e^{pA_k})$$

for all $p > 0$ and there exists a unitary matrix U such that

$$\left(e^{\sum_{i=1}^k \omega_i p A_i} \right)^{1/p} \leq S(e^{p(M-m)})^{1/p} U^* G_K(\omega : e^{pA_1}, \dots, e^{pA_k})^{1/p} U.$$

Hence we have

$$\left\| e^{\sum_{i=1}^k \omega_i A_i} \right\| \leq S(e^{p(M-m)})^{1/p} \left\| G_K(\omega : e^{pA_1}, \dots, e^{pA_k})^{1/p} \right\|$$

for all $p > 0$ and every unitarily invariant norm $\|\cdot\|$. □

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