

Crossed homomorphisms and the Schur-Zassenhaus theorem

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1 Theorems

We can find several proofs, for example, in [6–13], of the following classical theorem of Frobenius:

Theorem 1.1 (Frobenius). *Let n be an integer and G a finite group. Then*

$$|\{g \in G \mid g^n = 1\}| \equiv 0 \pmod{\gcd(n, |G|)},$$

where $|X|$ denotes the cardinality of a set X .

This theorem is equivalent to the fact that

$$|\text{Hom}(C, G)| \equiv 0 \pmod{\gcd(|C|, |G|)}$$

for any finite cyclic group C , where Hom denotes the set of group homomorphisms. Yoshida has generalized the theorem as follows:

Theorem 1.2 (Yoshida [12]). *Let A be a finite abelian group and G a finite group. Then*

$$|\text{Hom}(A, G)| \equiv 0 \pmod{\gcd(|A|, |G|)}.$$

Another way of generalization is due to P. Hall:

Theorem 1.3 (P. Hall [10]). *Let G be a finite group and θ an automorphism of G . If the order of θ divides a positive integer n , then*

$$|\{g \in G \mid g \cdot \theta(g) \cdot \theta^2(g) \cdots \theta^{n-1}(g) = 1\}| \equiv 0 \pmod{\gcd(n, |G|)}.$$

The theorem of Frobenius corresponds to the case $\theta = 1$. We reform this Hall's generalization in terms of ' $Z^1(A, G)$ ' as well as Theorem 1.1 in terms of $\text{Hom}(A, G)$, as follows.

Let a group A act on a group G by a group homomorphism $\varphi: A \rightarrow \text{Aut}(G)$, where $\text{Aut}(G)$ is the automorphism group of G . For $a \in A$ and $g \in G$, we indicate $\varphi(a)(g)$ by ${}^a g$. A map $\lambda: A \rightarrow G$ is called a *crossed homomorphism* or a *derivation* (with respect to φ) provided

$$\lambda(ab) = \lambda(a) \cdot {}^a \lambda(b) \quad \text{for all } a, b \in A.$$

We denote by $Z^1(A, G)$ the set of crossed homomorphisms from A to G . For example, the zero map $0: A \rightarrow G$ sending all the elements of A onto $1 \in G$ is a crossed homomorphism. If the action φ is trivial, then $Z^1(A, G) = \text{Hom}(A, G)$. On the other hand, if G is abelian, then $Z^1(A, G)$ coincides with the first cocycle group of the $\mathbb{Z}A$ -module G with respect to the standard resolution of A . However, unless G is abelian, $Z^1(A, G)$ may be only a set; it may not have a group structure in general.

Now, Hall's theorem is equivalent to the fact that

$$|Z^1(C, G)| \equiv 0 \pmod{\gcd(|C|, |G|)}$$

for any finite cyclic group C and for any action of C on G . Yoshida and the first author of this report have conjectured the following:

Conjecture 1.4 ([5]). If a finite group A acts on a finite group G , then

$$|Z^1(A, G)| \equiv 0 \pmod{\gcd(|A/A'|, |G|)},$$

where A' denotes the commutator subgroup of A .

This conjecture is a generalization of all the theorems above, and is still open. Recent progress for this conjecture is found in [1–4]. In particular, in order to prove the conjecture completely, it suffices to prove the conjecture in the case where A is an abelian p -group and G is a p -group for a prime p ([1]). This reduction mainly owes to the functorial properties of $Z^1(A, G)$ on the variables A and G , where the latter is first observed by Brauer [6] in a certain case (see §3.3 for generalization). In addition, Brauer has based his alternative proof of the theorem of Frobenius on the following lemma:

Lemma 1.5 (Brauer [6]). Let G be a finite normal subgroup of a group E . Then, for any $g \in G$ and $x \in E$, $(gx)^{|G|}$ and $x^{|G|}$ is conjugate by an element of G .

In this report, we shall generalize this Brauer's lemma as the formula

$$\text{res}_{A, A|G|}(Z^1(A, G)) = B^1(A^{|G|}, G)$$

for abelian A (Theorem 4.1), where B^1 denotes the set of coboundaries, which will be introduced in the next section. Throughout the report, our main tools are the functorial properties of $Z^1(A, G)$, and our principle is to compare $Z^1(A, G)$ with $B^1(A, G)$. As a corollary of our arguments together with the Feit-Thompson theorem, we shall also prove Theorem 4.2 which is equivalent to the second statement of the following classical theorem:

Theorem 1.6 (Schur-Zassenhaus). Let G be a finite normal subgroup of a finite group E such that $\gcd(|E : G|, |G|) = 1$. Then

- (1) There exists a subgroup A of E such that $E = G \rtimes A$.
- (2) If $E = G \rtimes A = G \rtimes B$, then A and B are conjugate by an element of G .

Note that if G is abelian, then it is well known that the first statement of the Schur-Zassenhaus theorem is equivalent to $H^2(A, G) = 0$, and the second is so to $H^1(A, G) = 0$. In fact, we shall prove $Z^1(A, G) = B^1(A, G)$ for any finite group A and G whose orders are relatively prime.

Notation. For the remainder of the report, we fix the following notation: let A and G be groups, which need not be finite, and let A act on G by a group homomorphism $\varphi: A \rightarrow \text{Aut}(G)$. With respect to this action φ , we denote by $Z^1(A, G)$ the set of crossed homomorphisms from A to G , and by $G \rtimes A$ the semidirect product of G and A . For $x \in G \rtimes A$, we denote by $\text{Inn}(x)$ the inner automorphism associated with x , so that $\text{Inn}(x)(y) = {}^x y = xyx^{-1}$ for all $y \in G \rtimes A$.

2 Coboundaries

For a given map $\lambda: A \rightarrow G$, consider the map $\tilde{\lambda}: A \rightarrow G \rtimes A$ which is defined by

$$\tilde{\lambda}(a) = \lambda(a)a \quad \text{for all } a \in A.$$

It is easy to show that $\lambda \in Z^1(A, G)$ if and only if $\tilde{\lambda} \in \text{Hom}(A, G \rtimes A)$, and in this case, $\tilde{\lambda}$ becomes a splitting monomorphism of the canonical epimorphism $\pi: G \rtimes A \rightarrow A$. On the other hand, any splitting monomorphism θ of π defines a complement $\theta(A) \leq G \rtimes A$ of G , and vice versa. From these observations, we obtain the following well-known result:

Theorem 2.1. *There are two bijections*

$$\begin{aligned} Z^1(A, G) &\xrightarrow{\Phi} \{\theta \in \text{Hom}(A, G \rtimes A) \mid \pi \circ \theta = \text{id}_A\} \\ &\xrightarrow{\Psi} \{B \leq G \rtimes A \mid GB = G \rtimes A, G \cap B = 1\}, \end{aligned}$$

where $\Phi(\lambda) = \tilde{\lambda}$ and $\Psi(\theta) = \theta(A)$.

As in homological algebra, we introduce the concept of ‘coboundary’ as well as cocycle. For arbitrary $g \in G$ and $a \in A$, regarding them as elements in $G \rtimes A$, we consider their commutator $[g, a]$, where

$$[g, a] = gag^{-1}a^{-1} = g \cdot {}^a(g^{-1}) \in G.$$

Then this induces a map $[g, -]: A \rightarrow G$ sending $a \in A$ to $[g, a] \in G$. We call this map $[g, -]$ a *coboundary* or an *inner derivation* induced from g (with respect to φ), and set

$$B^1(A, G) = \{[g, -] \mid g \in G\}.$$

Easy calculation shows that $B^1(A, G) \subseteq Z^1(A, G)$. In fact, if G is abelian, then $B^1(A, G)$ coincides with the first coboundary group of the $\mathbb{Z}A$ -module G with respect to the standard resolution of A . However, in general cases, $B^1(A, G)$ may not have a group structure. Our principle of this report is to compare $B^1(A, G)$ with $Z^1(A, G)$. First we emphasize the following lemma on the relation between the coboundary $[g, -]$ and conjugation by g . Since $[g, a]a = {}^g a$ in $G \rtimes A$, we have

Lemma 2.2. *Given $g \in G$, set $\gamma = [g, -]$. Then $\tilde{\gamma}(a) = {}^g a$ for all $a \in A$.*

In other words, $\Phi([g, -]) = \text{Inn}(g)$ on A . Note that ${}^g A \neq A$ in general.

3 Parameters

Both $Z^1(A, G)$ and $B^1(A, G)$ have three parameters: groups A , G and action φ . We shall consider functorial properties on these parameters.

3.1 Change of actions

We fix $\lambda \in Z^1(A, G)$. For given $a \in A$, the inner automorphism $\text{Inn}(\tilde{\lambda}(a))$ on $G \rtimes A$ leaves the normal subgroup G invariant. This induces a new action $\text{Inn} \tilde{\lambda}: A \rightarrow \text{Aut}(G)$, namely,

$$(\text{Inn} \tilde{\lambda})(a)(g) = \tilde{\lambda}(a)g = \lambda(a)(^a g) \quad \text{for } a \in A \text{ and } g \in G.$$

We denote simply by $Z_\lambda^1(A, G)$ the set of crossed homomorphisms with respect to $\text{Inn} \tilde{\lambda}$.

Since $G \rtimes A = G \rtimes \tilde{\lambda}(A)$, Theorem 2.1 states that both $Z^1(A, G)$ and $Z_\lambda^1(A, G)$ correspond to the same set — the set of complements of G in $G \rtimes A$. This is a group-theoretic meaning of the following theorem.

Theorem 3.1 (Change of actions). *Let $\lambda \in Z^1(A, G)$. Then right multiplication by λ induces a bijection $\lambda_r: Z_\lambda^1(A, G) \rightarrow Z^1(A, G)$, which is defined by*

$$\lambda_r(\eta)(a) = \eta(a)\lambda(a) \quad \text{for all } \eta \in Z_\lambda^1(A, G) \text{ and } a \in A.$$

We often write $\lambda_r(\eta) = \eta \cdot \lambda$.

Let us determine the image of the coboundaries by this bijection λ_r . Set

$$B_\lambda^1(A, G) = \{[g, -]_\lambda \mid g \in G\},$$

where $[g, -]_\lambda: A \rightarrow G$ denotes the coboundary induced from g with respect to the action $\text{Inn} \tilde{\lambda}$, i.e.,

$$[g, a]_\lambda = g \cdot \tilde{\lambda}(a)(g^{-1}) \in G \leq G \rtimes A \quad \text{for all } a \in A.$$

We indicate $\lambda_r([g, -]_\lambda) = [g, -]_\lambda \cdot \lambda \in Z^1(A, G)$ by ${}^g \lambda$, so that

$$({}^g \lambda)(a) = [g, a]_\lambda \cdot \lambda(a) = {}^g(\tilde{\lambda}(a)) \cdot a^{-1}.$$

On the other hand, G acts on $\text{Hom}(A, G \rtimes A)$ by

$${}^g \theta = \text{Inn}(g) \circ \theta \quad \text{for } g \in G \text{ and } \theta \in \text{Hom}(A, G \rtimes A).$$

Lemma 3.2. *Let $\lambda \in Z^1(A, G)$. Then we have*

- (1) $\lambda_r(B_\lambda^1(A, G)) = \{{}^g \lambda \mid g \in G\}$.
- (2) $\widetilde{{}^g \lambda} = {}^g \tilde{\lambda}$ for any $g \in G$. (In other words, ${}^g \lambda$ is the 'G-part' of ${}^g \tilde{\lambda}$.)

As the easiest case, we consider the zero map.

Lemma 3.3. *Let $0 \in Z^1(A, G)$ be the zero map. Then we have*

- (1) $\tilde{0}: A \rightarrow G \rtimes A$ is the inclusion map (the canonical monomorphism).
- (2) ${}^g 0 = [g, -]$ and ${}^g \tilde{0} = \text{Inn}(g)$ on A for any $g \in G$.

This implies the following at once:

Corollary 3.4. *All the complements of G in $G \rtimes A$ are conjugate if and only if $B^1(A, G) = Z^1(A, G)$.*

Note that any two conjugate complements of G in $G \rtimes A$ are conjugate by an element of G . We can also show the following by easy calculation:

Lemma 3.5. *For any $g, h \in G$, we have*

$${}^g[h, -] = [g, -]_{[h, -]} \cdot [h, -] = [gh, -].$$

3.2 Contravariant parameter A

Suppose that there is a short exact sequence of groups $1 \rightarrow B \rightarrow A \rightarrow \bar{A} \rightarrow 1$. We consider a problem whether there exists an *exact* sequence such as

$$1 \rightarrow Z^1(\bar{A}, G_?) \xrightarrow{\text{incl}} Z^1(A, G) \xrightarrow{\text{res}_{A,B}} Z^1(B, G),$$

where $G_?$ is some subgroup of G on which B acts trivially, incl is the inclusion map, and $\text{res}_{A,B}$ is the restriction map (although exactness of a sequence is not defined in the category of sets). Whereas we can not find such a common subgroup $G_?$, we can locally do as follows:

Theorem 3.6. *Suppose that $\mu \in Z^1(B, G)$ lies in $\text{res}_{A,B}(Z^1(A, G))$, namely, $\mu = \text{res}_{A,B}(\lambda)$ for some $\lambda \in Z^1(A, G)$. Then $\lambda_r: Z^1_\lambda(A, G) \rightarrow Z^1(A, G)$ induces a bijection*

$$\lambda_r: Z^1_\lambda(\bar{A}, C_G(\tilde{\mu}(B))) \rightarrow Z^1(A, G; B, \mu),$$

where we regard $Z^1_\lambda(\bar{A}, C_G(\tilde{\mu}(B))) \subseteq Z^1_\lambda(A, G)$ in a natural way, and where we set

$$Z^1(A, G; B, \mu) = \text{res}_{A,B}^{-1}(\mu) = \{\tau \in Z^1(A, G) \mid \text{res}_{A,B}(\tau) = \mu\}.$$

By Lemma 3.2, we have

Corollary 3.7. *Under the notation in Theorem 3.6, we have*

$$\lambda_r(B^1_\lambda(\bar{A}, C_G(\tilde{\mu}(B)))) = \{^h\lambda \mid h \in C_G(\tilde{\mu}(B))\}.$$

3.3 Covariant parameter G — Brauer's argument

Suppose that there is a short exact sequence of groups $1 \rightarrow K \rightarrow G \rightarrow K \backslash G \rightarrow 1$. We consider a similar problem whether there exists an exact sequence such as

$$1 \rightarrow Z^1(A, K_?) \xrightarrow{\text{incl}} Z^1(A, G) \xrightarrow{\text{mod } K} \text{Map}(A, K \backslash G),$$

where $K_?$ is some subgroup of G , and Map denotes the set of maps, which may be replaced by Z^1 if K is A -invariant. For this problem, Brauer [6] gave an answer in the case where A is cyclic with trivial action on G , i.e., $Z^1(A, G) = \text{Hom}(A, G)$. Moreover, it is remarkable that he assumed K is neither normal nor A -invariant. We can generalize his answer as follows.

For $K \leq G$ and $\lambda \in Z^1(A, G)$, let K_λ be the maximal $\tilde{\lambda}(A)$ -invariant subgroup of K , namely,

$$K_\lambda = \bigcap_{a \in A} \tilde{\lambda}(a)K.$$

Theorem 3.8. *Let K be a subgroup of G , and $\lambda \in Z^1(A, G)$. Then $\lambda_r: Z^1_\lambda(A, G) \rightarrow Z^1(A, G)$ induces a bijection*

$$\lambda_r: Z^1_\lambda(A, K_\lambda) \rightarrow \{\eta \in Z^1(A, G) \mid K\eta(a) = K\lambda(a) \text{ for all } a \in A\}.$$

By Lemma 3.2, we have

Corollary 3.9. *Under the notation in Theorem 3.8, we have*

$$\lambda_r(B^1_\lambda(A, K_\lambda)) = \{^k\lambda \mid k \in K_\lambda\}.$$

4 Applications

For given $B \leq A$ and $g \in G$, we indicate the coboundary $[g, -]: B \rightarrow G$ by $[g, -]_B$ to avoid ambiguities, so that $\text{res}_{A,B}([g, -]_A) = [g, -]_B$. Note that it always holds that

$$\text{res}_{A,B}(B^1(A, G)) = B^1(B, G). \quad (*)$$

If n is an integer and A is abelian, then $A^n = \{a^n \mid a \in A\}$ is a subgroup of A . The following is a generalization of Brauer's lemma (Lemma 1.5).

Theorem 4.1. *Let A be a finitely generated abelian group and let G be a finite group. Then*

$$\text{res}_{A, A^{|G|}}(Z^1(A, G)) = B^1(A^{|G|}, G).$$

Proof. We use induction on the rank of A .

(1) Suppose that A is cyclic. We reduce this case to Hall's theorem (Theorem 1.3) as follows. Taking an epimorphism $F \simeq \mathbb{Z} \rightarrow A$, we have a commutative diagram

$$\begin{array}{ccc} Z^1(A, G) & \xrightarrow{\text{res}} & Z^1(A^{|G|}, G) \\ \inf \downarrow & & \inf \downarrow \\ Z^1(F, G) & \xrightarrow{\text{res}} & Z^1(F^{|G|}, G). \end{array}$$

This allows us to assume that $A = F$. Since $F \simeq \mathbb{Z}$, we have $|F : F^{|G|}| = |G| = |Z^1(F, G)|$. On the other hand, we have $B^1(F^{|G|}, G) = \{[g, -]_{F^{|G|}} \mid g \in [G/C_G(F^{|G|})]\}$, where $[G/H]$ denotes a set of representatives for left cosets in G modulo a subgroup H . Thus, by definition,

$$\text{res}_{F, F^{|G|}}^{-1}(B^1(F^{|G|}, G)) = \bigoplus_{g \in [G/C_G(F^{|G|})]} Z^1(F, G; F^{|G|}, [g, -]_{F^{|G|}}).$$

However, Theorem 3.6 and usual argument for conjugation yield that

$$Z^1(F, G; F^{|G|}, [g, -]_{F^{|G|}}) \simeq Z^1_{[g, -]}(F/F^{|G|}, C_G(g(F^{|G|}))) \simeq Z^1(F/F^{|G|}, C_G(F^{|G|})).$$

Therefore Hall's theorem implies that

$$\left| \text{res}_{F, F^{|G|}}^{-1}(B^1(F^{|G|}, G)) \right| = |G : C_G(F^{|G|})| \cdot |Z^1(F/F^{|G|}, C_G(F^{|G|}))| \equiv 0 \pmod{|G|},$$

which forces $\left| \text{res}_{F, F^{|G|}}^{-1}(B^1(F^{|G|}, G)) \right| = |G| = |Z^1(F, G)|$, as desired.

(2) Suppose that $A = B \times C$ for nontrivial subgroups B and C , and $\lambda \in Z^1(A, G)$. By the equation (*) and the inductive assumption, we have

$$\begin{aligned} B^1(B^{|G|}, G) &= \text{res}_{A, B^{|G|}}(B^1(A, G)) \\ &\subseteq \text{res}_{A, B^{|G|}}(Z^1(A, G)) \subseteq \text{res}_{B, B^{|G|}}(Z^1(B, G)) = B^1(B^{|G|}, G), \end{aligned} \quad (**)$$

so that $\text{res}_{A, B^{|G|}}(Z^1(A, G)) = B^1(B^{|G|}, G)$. Hence $\lambda \in Z^1(A, G; B^{|G|}, [h, -]_{B^{|G|}})$ for some $h \in G$. However, we have also $[h, -]_A \in Z^1(A, G; B^{|G|}, [h, -]_{B^{|G|}})$. Theorem 3.6 yields that

$$[h, -]_r: Z^1_{[h, -]}(A/B^{|G|}, C_G(h(B^{|G|}))) \rightarrow Z^1(A, G; B^{|G|}, [h, -]_{B^{|G|}})$$

is bijective. Thus $\lambda = \eta \cdot [h, -]_A$ for some $\eta \in Z^1_{[h, -]}(A/B^{|G|}, C_G({}^h(B^{|G|})))$. Again applying induction to $C^{|G|} \leq A/B^{|G|} \simeq (B/B^{|G|}) \times C$ as in (**), we have

$$\text{res}_{A/B^{|G|}, C^{|G|}}(Z^1_{[h, -]}(A/B^{|G|}, C_G({}^h(B^{|G|})))) = B^1_{[h, -]}(C^{|G|}, C_G({}^h(B^{|G|}))).$$

Hence there exists $g \in C_G({}^h(B^{|G|}))$ such that $\text{res}_{A/B^{|G|}, C^{|G|}}(\eta) = [g, -]_{[h, -]}$, the commutator of g with respect to the action $\text{Inn}[h, -] \sim$. This means that

$$\lambda(bc) = \eta(c) \cdot [h, bc] = [g, c]_{[h, -]} \cdot [h, bc] = [g, bc]_{[h, -]} \cdot [h, bc] \quad \text{for all } b \in B^{|G|}, c \in C^{|G|}.$$

Consequently, $\text{res}_{A, A^{|G|}}(\lambda) = [g, -]_{[h, -]} \cdot [h, -] = [gh, -]$ on $A^{|G|}$ by Lemma 3.5, as desired. $\square$

As observed in Corollary 3.4, the second statement of the Schur-Zassenhaus theorem (Theorem 1.6) is equivalent to the following theorem, which can be reduced to the case where either A or G is abelian by the Feit-Thompson theorem and by our arguments.

Theorem 4.2. *If A and G are finite groups with $\gcd(|A|, |G|) = 1$, then $Z^1(A, G) = B^1(A, G)$.*

Proof. We use induction on $|A|$ and $|G|$. By the Feit-Thompson theorem, we may assume that either $A' \leq A$ or $G' \leq G$.

(1) Suppose that $A' \leq A$, and consider the short exact sequence $1 \rightarrow A' \rightarrow A \rightarrow A/A' \rightarrow 1$. By induction, we have $Z^1(A', G) = B^1(A', G)$, so that

$$Z^1(A, G) = \bigoplus_{h \in [G/C_G(A')]} Z^1(A, G; A', [h, -]_{A'}).$$

By applying Theorem 3.6 to $[h, -]_A \in Z^1(A, G; A', [h, -]_{A'})$,

$$[h, -]_r: Z^1_{[h, -]}(A/A', C_G({}^h A')) \rightarrow Z^1(A, G; A', [h, -]_{A'})$$

is bijective. However, A/A' is abelian and $(A/A')^{|H|} = A/A'$ for all $H \leq G$ by hypothesis. Hence Theorem 4.1 implies that

$$Z^1_{[h, -]}(A/A', C_G({}^h A')) = B^1_{[h, -]}(A/A', C_G({}^h A')).$$

Consequently, it follows from Lemma 3.5 that every element of $Z^1(A, G)$ is of the form $[g, -]_{[h, -]} \cdot [h, -] = [gh, -]$ for some $g, h \in G$.

(2) Suppose that $G' \leq G$, and consider the short exact sequence $1 \rightarrow G' \rightarrow G \rightarrow G/G' \rightarrow 1$. We have a natural map $Z^1(A, G) \rightarrow Z^1(A, G/G')$. However, G/G' is an A -module of order relatively prime to $|A|$. Hence it is well known in cohomology theory that $Z^1(A, G/G') = B^1(A, G/G')$. Therefore, for each $\lambda \in Z^1(A, G)$, there exists some $h \in G$ such that $G'\lambda(a) = G'[h, a]$ for all $a \in A$. By Theorem 3.8,

$$[h, -]_r: Z^1_{[h, -]}(A, G') \rightarrow \{\eta \in Z^1(A, G) \mid G'\eta(a) = G'[h, a] \text{ for all } a \in A\}$$

is a bijection. However, $Z^1_{[h, -]}(A, G') = B^1_{[h, -]}(A, G')$ by induction. Consequently, it follows from Lemma 3.5 that $\lambda = [g, -]_{[h, -]} \cdot [h, -] = [gh, -]$ for some $g \in G'$. $\square$

As stated in the proof, this theorem is a generalization of a well known theorem in cohomology theory for A -modules G . Although we have used the Feit-Thompson theorem, the arguments of (1) and (2) in the proof are very parallel.

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