

GLOBAL CONTINUATION BEYOND SINGULARITY ON THE BOUNDARY

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1. INTRODUCTION

We consider problems of the form

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0, t) &= 0, & 0 < t < T, \\ u_x(1, t) &= f(u(1, t)), & 0 < t < T, \\ u(x, 0) &= u_0(x) > 0, & 0 \leq x \leq 1, \end{aligned}$$

where $f(u) = -u^{-p}$, $p > 0$, or $f(u) = u^p$, $p > 1$. We shall call them Problem (Q) and Problem (B), respectively. We discuss them separately.

1.1. Problem (Q) ($f(u) = -u^{-p}$). This problem was studied before by Fila & Levine(1993) where it was shown that that every solution quenches in a finite time $T = T(u_0)$ in the sense that $u > 0$ in $[0, 1] \times [0, T)$ and $u(1, t) \rightarrow 0$ as $t \rightarrow T$. The behavior of u near $(1, T)$ for $t \leq T$ was also studied.

The question whether it is possible to continue the solution beyond $t = T$ (in some suitable sense) was raised by Levine(1993). Since $u(\cdot, T) \in C([0, 1])$ and $u(1, T) = 0$, an obvious possibility of continuing the solution is to extend it for $t > T$ by $\tilde{u}$ which solves

$$\begin{aligned} \tilde{u}_t &= \tilde{u}_{xx}, & 0 < x < 1, \quad t > T, \\ \tilde{u}_x(0, t) &= 0, & t > T, \\ \tilde{u}(1, t) &= 0, & t > T, \\ \tilde{u}(x, T) &= u(x, T), & 0 \leq x \leq 1. \end{aligned}$$

We show that this continuation is natural since it can be obtained as a limit of a sequence of solutions of regularized problems. More precisely, if $\varepsilon > 0$ and $f_\varepsilon \in C^1([0, \infty))$ is such that $f_\varepsilon(0) = 0$ and

$$\begin{aligned} f_\varepsilon(s) &= -s^{-p} & \text{for } s \geq \varepsilon, \\ f(s) &\leq f_{\varepsilon_1}(s) \leq f_{\varepsilon_2}(s) & \text{for } s > 0 \quad \text{and} \quad \varepsilon_1 < \varepsilon_2, \end{aligned}$$

then the solutions of (Q_ε) :

$$\begin{cases} u_t^\varepsilon = u_{xx}^\varepsilon, & 0 < x < 1, \quad 0 < t < \infty, \\ u_x^\varepsilon(0, t) = 0, & 0 < t < \infty, \\ u_x^\varepsilon(1, t) = f_\varepsilon(u^\varepsilon(1, t)), & 0 < t < \infty, \\ u^\varepsilon(x, 0) = u_0(x), & 0 \leq x \leq 1, \end{cases}$$

converge to the extension of u by $\tilde{u}$.

The fact that solutions of Problem (Q) can be continued beyond $t = T$ for all $p > 0$ is in contrast with the situation when quenching occurs in the interior. Namely, for the problem

$$\begin{aligned} u_t &= u_{xx} - u^{-p}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0, t) &= 0, & 0 < t < T, \\ u(1, t) &= 1, & 0 < t < T, \\ u(x, 0) &= u_0(x), & 0 \leq x \leq 1, \end{aligned}$$

solutions can be continued beyond quenching if and only if $0 < p < 1$ (cf. Phillips(1987), Galaktionov & Vazquez(1995)).

Let us also mention here that a similar phenomenon when the continuation beyond gradient blow-up does not satisfy the original boundary condition was observed by Fila & Lieberman(1994).

1.2. Problem (B) ($f(u) = u^p$). The study of blow-up of solutions of the heat equation with a nonlinear boundary condition was initiated by Levine & Payne(1974) and it has attracted considerable attention (see a survey paper of Fila & Filo(1996)). It was shown by Fila(1989) that every solution of Problem (B) blows up in a finite time $T = T(u_0)$ and it is also known (cf. López Gómez, Márquez, & Wolanski(1991)) that the only blow-up point is $x = 1$.

(By a blow-up point we mean a point $a \in [0, 1]$ such that there are $\{x_n\} \subset [0, 1]$ and $t_n \rightarrow T$ such that $x_n \rightarrow a$ and $u(x_n, t_n) \rightarrow \infty$ as $n \rightarrow \infty$.)

We show that for Problem (B) blow-up is always complete in the following sense. If

$$f^n(s) = \min\{s^p, n^p\}, \quad s \geq 0, \quad n \in \mathbb{N}, \quad (1.1)$$

and u^n is the solution of (B^n) :

$$\begin{cases} u_t^n = u_{xx}^n, & 0 < x < 1, \quad 0 < t < \infty, \\ u_x^n(0, t) = 0, & 0 < t < \infty, \\ u_x^n(1, t) = f^n(u^n(1, t)), & 0 < t < \infty, \\ u^n(x, 0) = u_0(x), & 0 \leq x \leq 1, \end{cases}$$

then $u^n(x, t) \rightarrow \infty$ for $(x, t) \in [0, 1] \times (T, \infty)$.

For results on complete blow-up for the problem when the nonlinearity occurs in the equation we refer to the papers of Baras & Cohen(1987), Lacey & Tzanetis(1988), Galaktionov & Vazquez(1995, 1997), Martel(1998), etc.

Our method is different and it is restricted to one space dimension since we are using an intersection-comparison (or zero number(cf. [14])) argument.

2. INCOMPLETE QUENCHING

In this section we show that if $u(x, t)$ is the solution of the problem

$$\begin{cases} u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0, t) = 0, & 0 < t < T, \\ u_x(1, t) = -u^{-p}(1, t), & 0 < t < T, \\ u(x, 0) = u_0(x) > 0, & 0 \leq x \leq 1, \end{cases} \quad (Q)$$

where $p > 0$ and T is the quenching time for u then there is a natural continuation of u beyond T . We shall assume that $u_0 \in C^1([0, 1])$ and the compatibility conditions

$$u'_0(0) = 0, \quad u'_0(1) = -u_0^{-p}(1)$$

are satisfied.

Assume that $0 < \varepsilon < u_0(1)$. Then there exists a unique global (in time) solution u^ε of (Q_ε) such that $u^\varepsilon \in C^{2,1}([0, 1] \times [0, \tau])$ for any $\tau > 0$ and

- (i) $u^\varepsilon > 0$ for $(x, t) \in [0, 1] \times [0, \infty)$,
- (ii) $u^{\varepsilon_1} \leq u^{\varepsilon_2}$ for $0 < \varepsilon_1 < \varepsilon_2$ and $(x, t) \in [0, 1] \times [0, \infty)$,
- (iii) $u^\varepsilon \geq u$ for $(x, t) \in [0, 1] \times [0, T)$.

Also, by the maximum principle, it is clear that

$$u^\varepsilon \leq K \equiv \max_{0 \leq x \leq 1} u_0(x)$$

for all $\varepsilon > 0$.

Now, let

$$v(x, t) = \lim_{\varepsilon \rightarrow 0} u^\varepsilon(x, t), \quad (x, t) \in [0, 1] \times [0, \infty). \quad (2.1)$$

Then v is well-defined and $0 \leq v \leq K$ in $[0, 1] \times [0, \infty)$. It follows from the regularity theory for parabolic equations that v satisfies the heat equation in $(0, 1) \times (0, \infty)$. By the maximum principle, $v > 0$ in $(0, 1) \times (0, \infty)$. Also, it is clear that $v_x(0, t) = 0$ for $t > 0$. Furthermore, if $t \in (0, T)$, then

$$v_x(1, t) = -v^{-p}(1, t).$$

It follows that v is a solution of (Q). By uniqueness, $v = u$ in $[0, 1] \times [0, T)$. For the boundary condition for v on $\{x = 1, t > T\}$, it can be shown that $v(1, t) = 0$ for $t \geq T$.

We summarize the above results as follows:

Theorem 2.1[15]. *The function v defined by (2.1) satisfies*

$$\begin{aligned} v_t &= v_{xx}, & 0 < x < 1, \quad t > 0, \\ v_x(0, t) &= 0, & t > 0, \\ v_x(1, t) &= -v^{-p}(1, t), & 0 < t < T, \\ v(1, t) &= 0, & t \geq T, \\ v(x, 0) &= u_0(x), & 0 \leq x \leq 1. \end{aligned}$$

It coincides with the solution u of Problem (Q) for $t \leq T$.

3. COMPLETE BLOW-UP

Consider the problem

$$\left\{ \begin{array}{ll} u_t = u_{xx}, & 0 < x < 1, \quad 0 < t < T, \\ u_x(0, t) = 0, & 0 < t < T, \\ u_x(1, t) = u^p(1, t), & 0 < t < T, \\ u(x, 0) = u_0(x) > 0, & 0 \leq x \leq 1, \end{array} \right. \quad (\text{B})$$

where $p > 1$, and T is the blow-up time for u . We assume further that $u'_0(0) = 0$ and $u'_0(1) = u_0^p(1)$.

Let $K = \max_{0 \leq x \leq 1} u_0(x)$. For any $n > K$, $n \in \mathbb{N}$, we define f^n as in (1.1). Note that f^n is Lipschitz and $u'_0(1) = f^n(u_0(1))$ if $n > K$. Hence, the solution of (B^n) is C^1 up to the boundary. We show that there exists a unique global (in time) solution u^n of (B^n) such that

- (i) $u^n > 0$ for $(x, t) \in [0, 1] \times [0, \infty)$,
- (ii) $u^n \leq u^{n+1}$ for $(x, t) \in [0, 1] \times [0, \infty)$,
- (iii) $u^n \leq u$ for $(x, t) \in [0, 1] \times [0, T)$.

Define

$$v(x, t) = \lim_{n \rightarrow \infty} u^n(x, t), \quad 0 \leq x \leq 1, \quad t \geq 0. \quad (3.1)$$

Similarly, one can show that $v_x(1, t) = v^p(1, t)$ for $t \in (0, T)$. Then it is clear that $v(x, t) = u(x, t)$ for $0 < t < T$. Note that $v(1, T) = \infty$. Furthermore, there holds $v(1, t) = \infty$ for $t \geq T$.

This proves the following:

Theorem 3.1[15]. *The function v defined in (3.1) coincides with the solution u of Problem (B) for $t \leq T$ and $v(x, t) = \infty$ for $(x, t) \in [0, 1] \times (T, \infty)$.*

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