

Analysis of the singular sets of the Landau-Lifshitz system *

Liu Xiangao

Institute of Mathematics, Fudan University, Shanghai 200433, China
email: xgliuk@online.sh.cn

1 Introduction

• Physical model

In 1935, L.D. Landau and E.M. Lifshitz derived the following equation, the so called Landau-Lifshitz system, which describes evolution of spin fields in continuum ferrimagnetism (see [LL]).

$$\partial_t u = -\alpha_1 u \times (u \times F_{eff}) + \alpha_2 u \times F_{eff},$$

where $u = (u^1, u^2, u^3) : R^m \times R_+ \rightarrow S^2 \subset R^3$ is the spin field; “ $\times$ ” denotes the vector cross product in R^3 ; $\alpha_1 > 0$ is a Gilbert damping constant, α_2 is a exchange constant, and F_{eff} is the effective field containing contributions from exchange interaction crystalline anisotropy, magneto-static self energy, external magnetic field, etc (see [LN]).

In particular, taking $F_{eff} = \Delta u$, corresponding to the pure isotropic case and without external magnetic fields, Landau-Lifshitz equation reads

$$\partial_t u = -\alpha_1 u \times (u \times \Delta u) + \alpha_2 u \times \Delta u.$$

When $\alpha_1 = 0$, the system is called the Heisenberg system.

• The equivalent equation

Using the following formula $a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$, and the fact that $|u| = 1$ implies $u \Delta u = -|\nabla u|^2 u$, we have the following equivalent equation

$$\begin{cases} \lambda_1 \partial_t u - \lambda_2 u \times \partial_t u = \Delta u + |\nabla u|^2 u & (x, t) \in R^m \times R_+ \\ u(x, 0) = u_0(x), & x \in R^m. \end{cases} \quad (1.1)$$

where $\lambda_1 = \frac{\alpha_1}{\alpha_1^2 + \alpha_2^2}$, $\lambda_2 = \frac{\alpha_2}{\alpha_1^2 + \alpha_2^2}$, and $|u_0(x)| = 1$,

*Project 10071013 supported by NSFC.

- Well known results

- **1-Dimension** A lot work contributed to the study of solutions of L-L system has been made by some physicists and mathematicians such as H.C.Fogedby [Fo], M.Laksmanan, K. Nakamura [LN], K.Nakamura, T.Sasada [NS], L.A.Takhtalian [T], J.Tjon, J.Wright [TW], Y.Zhou, B.Guo S.Tan [ZGT] and so on.
- **2-Dimension:** Singular points in the finite time are finite, because of, well-known, the conformal in-variation of the energy in the 2-dimension.
- **High-Dimension** The global existence of weak solutions for the equation has been established by F.Alouges A.Soyeur [AS](1992) and B.Guo, M.Hong [GA](1993).

- The problems concerned and their difficulties

- **What is the regularity to (1.1)?** This is a basic problem to any nonlinear equations considered in the space of the generalized functions such as Sobolev space. Furthermore we hope to know the behavior of the solution at a singular point, that is, one ask what happen at a singular point?
- **The characteristics of the equation (1.1)**
 - * The landau-Lifshitz equation is a parabolic type equation with the natural increasing term $|\nabla u|^2$.
 - * In appearance, (1.1) is similar to the heat flow of harmonic map into sphere (if $\alpha_2 = 0$), however there is an anti-symmetric term $u \times \partial_t u$. In other words, the Landau-Lifshitz system is a more general system, which contains in particularly the equations of harmonic map into sphere and its heat flow.
- **The difficulties in getting regularity** The natural increasing term and the anti-symmetric term are both difficult to regularity (no existence). The classical methods can't be used for the first one, and the second one breaks down the monotone property which is a main tool to deal with the harmonic maps and its heat flows as known. We can define the **stationary condition** in an analogous way as in harmonic maps by R.Schoen, the so called the variation of domain.
In the case of harmonic map and its heat flow, we have the following fact:

$$\text{The stationary condition} \implies \text{monotonicity.}$$

But In our case, the stationary condition does not imply the monotonicity, we will see it in the following. Maybe this is the crucial reason as which up to now one could not to get the regularity of Landau-Lifshitz system. It is well known that the weak harmonic map, without monotone property, may be almost discontinuous in three dimension (see T. Riviera).

• Our main results

- **Part one:** We prove the stationary weak solution is smooth except $\mathcal{H}_p^m$ -zero set, i.e.,

$$\mathcal{H}_p^m(\text{Sing}(u)) = 0.$$

The following results largely depend on the regularity got in the first part.

- **Part two:** Let u_k be a sequence of the stationary weak solutions of (1.1) with the initial data u_{k0} and $\int_{R^m} |\nabla u_{k0}|^2 \leq \Lambda$. For fixed t let Σ^t be the **blow up set** of the sequence, then we have
 1. Σ^t is rectifiable, i.e., almost C^1 smooth.
 2. Σ^t moves by quasi-mean curvature if u_k are the strong stationary weak solutions of (1.1) and $\lambda_1 \geq 2|\lambda_2|$.
- **Part three:** Let (x_0, t_0) be a singular point of u , by scaling we obtain the two blow up formulas.

2 Stationary weak solutions

In this section, we introduce the notions of the stationary weak solutions of Landau-Lifshitz system, and show some generalized monotonicity inequalities.

DEFINITION 2.1 $u(x, t) \in W^{1,2}(R^m \times R_+, S^2)$ is called a **stationary weak solution** of (1.1), if it is a weak solution of (1.1) and satisfies the following two assumptions:

$$\int_{R^m} 2(\lambda_1 u_t - \lambda_2 u \times u_t) \zeta \cdot \nabla u - |\nabla u|^2 \text{div} \zeta + 2 \partial_j u \partial_k u \partial_j \zeta^k = 0, \quad (2.1)$$

$$\begin{aligned} & \left(\int_{R^m \times t_2} - \int_{R^m \times t_1} \right) |\nabla u|^2 \vartheta dx \\ & \leq - \int_{t_1}^{t_2} \int_{R^m} [2(\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t) - |\nabla u|^2 \vartheta_t] dx dt, \end{aligned} \quad (2.2)$$

where $t_2 > t_1 > 0$, the functions ζ, ϑ are smooth, and $\vartheta \geq 0$

u is called a **strong stationary weak solution** if the equality in (2.2) holds.

REMARK 2.1 If a weak solution u of (1.1) satisfies the stability hypothesis defined by the requirement that, similar to heat flow in [Fe],

$$\begin{aligned} & \int_0^\infty \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t) \partial_\tau \hat{u}^\tau |_{\tau=0} \\ & + \partial_\tau^+ \int_0^\infty \int_{R^m} |\nabla \hat{u}^\tau|^2 dx dt |_{\tau=0} \leq 0 \end{aligned} \quad (2.3)$$

holds for each family $\hat{u}^\tau$ of the domain variation defined by $\hat{u}^\tau = u(F_\tau(x, t))$, where $F_\tau = (x + \tau \tilde{\zeta}, t + \tau \tilde{\vartheta})$, $\tilde{\zeta}, \tilde{\vartheta}$ are smooth functions and $\tilde{\vartheta} \geq 0$. Then the assumptions (2.1) and (2.2) hold (see Proposition 7 and 8 in [Fe]).

Certainly smooth solution is strong stationary.

Notations:

• Point Sets and Fundamental Solution

$$z = (x, t) \in R^m \times R, \quad z_0 = (x_0, t_0);$$

$$B_r(x_0) = \{x \in R^m : |x_0 - x| < r\}, \quad S_r(t_0) = \{(x, t) : t = t_0 - r^2\},$$

$$P_r(z_0) = \{(x, t) \in R^m \times R : |x - x_0| < r, |t - t_0| < r^2\},$$

$$T_r(z_0) = \{(x, t) \in R^m \times R : t_0 - 4r^2 < t < t_0 - r^2\}.$$

$$G_{z_0} = \frac{1}{[4\pi(t_0 - t)]^{m/2}} \exp\left(-\frac{|x - x_0|^2 \lambda_1}{4(t_0 - t)}\right), \quad t < t_0,$$

is the standard fundamental solution of the backward heat equation $\partial_t G - \lambda_1^{-1} \Delta G = 0$;

• Functionals

$$\Phi_{z_0}(R, u) = R^2 \int_{S_R(t_0)} |\nabla u|^2 G_{z_0} dx; \quad \Psi_{z_0}(R, u) = \int_{T_R(t_0)} |\nabla u|^2 G_{z_0} dx dt;$$

$$\Theta_{z_0}(R_1, R_2, u) = \frac{\lambda_2^2}{\lambda_1} \int_{t_0 - R_2^2}^{t_0 - R_1^2} \int_{R^m} u_t^2(t_0 - t) G_{z_0} dx dt;$$

$$E(r, u, z) = r^{2-m} \int_{B_r(z)} |\nabla u|^2 dy; \quad I(r, u, z) = r^{2-m} \int_{B_r(x)} |\partial_t u|^2 dy;$$

$$E_p(r, u, z) = r^{-m} \int_{P_r(z)} |\nabla u|^2 dy dt; \quad \mathcal{E}(r, u, z) = E_p(r, u, z) + r^\beta.$$

The stationary condition

$\Downarrow$

• Energy inequality

$$\begin{aligned} & \int_0^T \int_{R^m} 2\lambda_1 u_t^2 dx dt + \int_{R^m} |\nabla u|^2(x, T) dx \\ & \leq \int_{R^m} |\nabla u_0|^2 dx = E_0. \end{aligned} \quad (2.4)$$

• Generalized monotone inequality I

$$\begin{aligned} & \int_{t_0 - R_2^2}^{t_0 - R_1^2} \int_{R^m} \lambda_1 \left[\frac{r u_r}{\sqrt{2(t_0 - t)}} \right. \\ & \quad \left. - \sqrt{2(t_0 - t)} \left(u_t - \frac{\lambda_2}{2\lambda_1} u \times u_t \right) \right]^2 G_{z_0} dx dt \\ & \leq \Phi_{z_0}(R_2, u) - \Phi_{z_0}(R_1, u) + \Theta_{z_0}(R_1, R_2, u). \end{aligned} \quad (2.5)$$

The equality holds if and only if u is strong stationary weak solution, i.e. the equality in (2.2) (or (2.3)) holds.

• **Generalized monotone inequality II**

Suppose that $I(r, u, z) \leq \frac{K}{\lambda_1^2 + \lambda_2^2} r^{-\alpha}$ for $0 < r \leq r_0$, where $K > 0, 0 < \alpha < 1/2$. Then $e^r(E(r, u, z) + \frac{K}{1-\alpha} r^{1-\alpha})$ is non-decreasing with respect to r , precisely

$$\begin{aligned} & \frac{d(e^r(E(r, u, z) + \frac{K}{1-\alpha} r^{1-\alpha}))}{dr} \\ & \geq 2e^r r^{2-m} \int_{\partial B_r} |\partial_r u|^2 d\sigma + \frac{Ke^r}{1-\alpha} r^{1-\alpha}. \end{aligned} \quad (2.6)$$

REMARK 2.2 In the heat flow case, $\Theta_{z_0}(R_1, R_2, u) = 0$ since $\alpha_2 = 0$. So $\Phi_{z_0}(R, u)$ is increasing with respect to R .

Our idea is to hope that $\Theta_{z_0}(R, u)$ is small whenever R is small, which enables us to define the following set Σ_β . For any fixed $1/2 > \alpha, \beta > 0$, and fixed constant c_0 , we define

$$\begin{aligned} \Sigma_\beta &= \left\{ z \in R^m \times R_+ : \limsup_{R \rightarrow 0} R^{-(m-2+\beta)} \int_{t-R^2}^{t+R^2} \int_{B_{R\sqrt{|\log R|}}(x)} u_t^2(x, t) dx dt = \infty \right\} \\ S_\alpha(u, t) &= \left\{ x \in R^m : \limsup_{r \rightarrow 0} r^\alpha I(r, z, u) \geq c_0 > 0 \right\}. \end{aligned}$$

The following lemma is true (see [Liu] [LT]).

LEMMA 2.1

$$\mathcal{H}_\rho^{m-2+\beta+\epsilon}(\Sigma_\beta) = 0,$$

where $\mathcal{H}_\rho$ denotes the parabolic metric Hausdorff measure, and $\epsilon > 0$ is any positive constant.

If $\int_{R^m} |\partial_t u|^2 < \infty$, then for any $0 < \alpha < 1/2$, we have

$$\mathcal{H}^{m-2-\alpha}(S_\alpha(u, t)) < \infty.$$

The following lemma enables our idea to become possible, which is the key part of getting regularity.

LEMMA 2.2 Let $z_0 \notin \Sigma_\beta$, i.e. there exist constants $C_0(z_0)$ and $R_0(z_0) > 0$, such that whenever $R_0 \geq R > 0$,

$$\frac{1}{R^{m-2+\beta}} \int_{t_0-R^2}^{t_0+R^2} \int_{B_{\sqrt{|\log R|}R}(x_0)} u_t^2(x, t) dx \leq C_0.$$

Then for any $0 < R < R_1 \leq R_0$, there exists a constant C_1 , depending only on C_0, E_0, m, β and $\alpha_i, i = 1, 2$, such that

$$\Theta_{z_0}(R, R_1, u) \leq C_1 R_1^\beta.$$

REMARK 2.3 If $z_0 \notin \Sigma_\beta$, then for large $\sqrt{t_0} \geq R_1 \geq R_0$ the same inequality holds from energy estimate (2.4).

From the lemma above we have the following estimate for the normalized energy

PROPOSITION 2.1 Let $z_0 \notin \Sigma_\beta$, corresponding $C_0, R_0 > 0$, and let $R_2 < 1/3$, and $\mathcal{E}(R_2, u, z_0) \leq \epsilon_1$, where ϵ_1 small enough (e.g. $\epsilon_1 \leq \exp(-(80R_0^2)^{-1}\lambda_1)$), then there exists a constant C_3 , depending on C_0, E_0, m , and β and $\alpha_i, i = 1, 2$, such that for any $0 < R < R_2$, the following inequality

$$\mathcal{E}(R, u, z_0) \leq C_3 |\log \mathcal{E}(R_2, u, z_0)|^{m/2} \mathcal{E}(R_2, u, z_0).$$

holds.

REMARK 2.4 Let $z_0 \notin \Sigma_\beta$. Then

$$\mathcal{E}(R, u, z) \leq C_2(R_0)$$

3 Partial regularity of the stationary solutions

We have

THEOREM 3.1 Let $u \in W^{1,2}(R^m \times R_+, S^2)$ be a global stationary weak solution of the Landau-Lifshitz system (1.1) with $E(u_0) < \infty$, where $E(u_0) = \int_{R^m} |\nabla u_0|^2 dV$. Then singular set of u , $\text{Sing}(u)$, is a closed set with $\mathcal{H}_\rho^m(\text{Sing}(u)) = 0$. More precisely, $\text{Sing}(u) \subset \Sigma_\beta \cup \{z : \limsup_{r \rightarrow 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}$ and $\mathcal{H}_\rho^{m-2+\beta+\alpha}(\Sigma_\beta) = 0, \mathcal{H}_\rho^m(\{z : \limsup_{r \rightarrow 0} \mathcal{E}(r, u, z) \geq \epsilon_0\}) = 0$ for any $1/2 > \beta, \alpha > 0$, where $\mathcal{H}_\rho$ denotes the parabolic metric Hausdorff measure.

As the usual blow-up argument, the key part to the proof of Theorem 3.1 is the following small energy decay lemma.

LEMMA 3.1 There exist constants $0 < \epsilon_0, \tau < 1$ such that if $\mathcal{E}(r, u, z) \leq \epsilon_0 \leq \epsilon_1 \leq 1/2$, then

$$\mathcal{E}(\tau r, u, z) \leq \frac{1}{2} \mathcal{E}(r, u, z) \tag{3.1}$$

for any $z \in R^m \times R_+$ and $z \notin \Sigma_\beta$, and $0 < r < \sqrt{t}$ small enough.

The compact Lemma 3.1 can be proved by using Proposition 2.1 and famous compensated compactness principle [CLMS] and Hélein's trick.

4 Analysis of the blow up sets

4.1 Problems

Let u_k be a sequence of stationary weak solutions of (1.1) with initial data $u_k(x, 0)$ and $\int_{R^m} |\nabla u_k(x, 0)|^2 \leq \Lambda$. By the energy inequality we have

$$\begin{aligned} & \int_0^T \int_{R^m} 2\lambda_1 \partial_t u_k^2 dx dt + \int_{R^m} |\nabla u_k|^2(x, T) dx \\ & \leq \int_{R^m} |\nabla u_k(x, 0)|^2 dx = E_{k0} \leq \Lambda. \end{aligned} \quad (4.1)$$

Therefore we may assume that $u_k \rightarrow u$ weakly in $W^{1,2}(R^m \times R_+, S^2)$. We set

$$\begin{aligned} \Sigma_{\epsilon_0}^t &= \cap_{r>0} \left\{ x \in R^m \mid \liminf_{k \rightarrow \infty} \mathcal{E}(r, u_k, z) \geq \epsilon_0 \right\}, \\ \Sigma_\beta^t &= \left\{ x \in R^m \mid \limsup_{r \rightarrow 0} \liminf_{k \rightarrow \infty} \frac{1}{r^{m-2+\beta}} \int_{t-r^2}^{t+r^2} \int_{B_{r|\log r|^{1/2}}(x)} |\partial_t u_k|^2 = \infty \right\}, \end{aligned}$$

where $\epsilon_0 > 0$ defined in Theorem 3.1 and $1/2 > \beta > 0$ are the fixed constants. We call the set $\Sigma^t = \Sigma_{\epsilon_0}^t \cup \Sigma_\beta^t$ the **blow up set** for the sequence u_k at t and $\Sigma = \cup_{0 < t < \infty} \Sigma^t \times \{t\}$ the **total blow up set** for the sequence u_k .

The purpose in this part is to analyze the blow up set Σ^t . In view of the geometric measure theory we first hope to know that

- is Σ^t rectifiable?
- Furthermore, one ask how to move Σ^t with respect to t ?
- What happens at a singular point?

4.2 Rectifiable

Define

$$\mathcal{T}_\infty = \left\{ t \in R_+ \mid \liminf_{k \rightarrow \infty} \int_{R^m} |\partial_t u_k|^2 = \infty \right\}.$$

Then we have

$$\begin{aligned} \mathcal{H}^1(\mathcal{T}_\infty) &= 0; \\ \mathcal{H}^{m-2}(\Sigma_{\epsilon_0}^t) &< \infty; \end{aligned}$$

If $t \notin \mathcal{T}_\infty$ then for any $\epsilon > 0$, $\mathcal{H}^{m-4+\beta+\epsilon}(\Sigma_\beta^t) = 0$.

REMARK 4.1 Here we can not expect that $\mathcal{H}^{m-2}(\Sigma_{\epsilon_0}^t) = 0$, as we do not know if the sequence $\{|\nabla u_k|^2\}$ is of the equivalent continuous in the sense of integration, or equivalently, strong convergence.

The small energy regularity and (4.1) imply we may assume that

$$\begin{aligned} |\nabla u_k|^2(\cdot, t) dx dt &\rightharpoonup |\nabla u|^2(\cdot, t) dx dt + \nu_t dt, \\ |\partial_t u_k|^2(\cdot, t) dx dt &\rightharpoonup |\partial_t u|^2(\cdot, t) dx dt + \mu, \end{aligned}$$

in the sense of measure as $k \rightarrow \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t , μ is a nonnegative Radon measure in $R^m \times R_+$ supported in Σ .

We have from the following monotonicity (2.6).

LEMMA 4.1 *If $\liminf_{k \rightarrow \infty} I(r, u_k, z) \leq \frac{r^{-\alpha}}{\lambda_1^2 + \lambda_2^2}$ for $0 < r \leq r_0$, then $e^r(E(r, u, z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})$ is non-decreasing. Precisely*

$$\frac{d}{dr}(e^r(E(r, u, z) + r^{2-m}\nu_t(B_r(x)) + \frac{r^{1-\alpha}}{1-\alpha})) \geq \frac{e^r r^{1-\alpha}}{1-\alpha}.$$

Now we present our main theorem in this section.

THEOREM 4.1 *Let u_k be a strong stationary weak solution of (1.1) with the initial energy $E_{k0} \leq \Lambda$. Then for almost every $t \in R_+$, ν_t is $\mathcal{H}^{m-2}$ -rectifiable, therefore Σ^t is $\mathcal{H}^{m-2}$ -rectifiable.*

We obvious have the following properties for $\theta(x, t)$.

LEMMA 4.2 *For almost every t , $\nu_t = \theta(x, t)\mathcal{H}^{m-2}|_{\Sigma^t}$, and $\theta(x, t)$ is upper semi-continuous in $\Sigma^t/S_\alpha(t)$ with $C(\epsilon_0) \leq \theta(x, t) \leq C(\Lambda)$ for $\mathcal{H}^{m-2}$ -a.e. $x \in \Sigma^t$, and therefore $\theta(x, t)$ is $\mathcal{H}^{m-2}$ approximate continuous for $\mathcal{H}^{m-2}$ a.e. $x \in \Sigma^t$.*

4.3 Quasi-mean curvature flow

We know that the blow up sets of the heat flow for harmonic maps move by the mean curvature (see [LT]). In this section we will calculate the curvature of Σ^t at the point $t \notin \mathcal{T}_\infty$ and verify Σ^t moves by the quasi-mean curvature. The difference from the heat flows is that the blow up set Σ^t in Landau-Lifshitz case moves by the quasi-mean curvature defined in the following, no the mean curvature except $\alpha_2 = 0$.

Using the monotonicity inequality (2.6) we can obtain the following important lemmas.

LEMMA 4.3 *let $T \in T\Sigma^t$, where $T\Sigma^t$ is the tangent bundle on Σ^t . If $t \notin \mathcal{T}_\infty$, then we have*

$$\lim_{\epsilon \rightarrow 0} \liminf_{k \rightarrow \infty} \int_{B_\epsilon(\Sigma^t)} |\nabla_T u_k|^2 = 0.$$

Here and in the sequel we denote by

$$B_\epsilon(X) = \{x \in R^m | \text{dist}(x, X) < \epsilon\}.$$

LEMMA 4.4

$$\lim_{\epsilon \rightarrow 0} \lim_{k \rightarrow \infty} \int_{B_\epsilon(\Sigma^t)} \partial_j u_k \partial_i u_k \partial_j \zeta^i = \frac{1}{2} \int_{\Sigma^t} \operatorname{div}_{(\Sigma^t)^\perp}(\zeta) \nu_t.$$

We also define the induced Radon measures $\vec{w}_t$, $\vec{w}_t^*$, and $\overrightarrow{w_t^{*\perp}}$ on Σ^t by the following respectively,

$$\begin{aligned} & \lim_{k \rightarrow \infty} 2 \int_{R^m} (\lambda_1 \partial_t u_k - \lambda_2 u_k \times \partial_t u_k) \nabla u_k \zeta \\ &= 2 \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t) \zeta \cdot \nabla u + 2 \int_{\Sigma^t} \vec{w}_t \zeta \\ & \lim_{k \rightarrow \infty} \int_{R^m} (\zeta \cdot \nabla u_k) \partial_t u_k \\ &= \int_{R^m} (\zeta \cdot \nabla u) u_t + \int_{\Sigma^t} \zeta \vec{w}_t^* \\ & \lim_{k \rightarrow \infty} \int_{R^m} (u_k \times \partial_t u_k) (\zeta \cdot \nabla u_k) \\ &= \int_{R^m} (u \times \partial_t u) (\zeta \cdot \nabla u) + \int_{R^m} \zeta \cdot \overrightarrow{w_t^{*\perp}} \end{aligned}$$

where ζ is a smooth vector field with compact support in R^m , and $\vec{w}_t = \vec{w}_t^1 - \vec{w}_t^2$ and every component $\vec{w}_t^i, i = 1, 2$ is a nonnegative Radon measure on Σ^t . Analogously for $\vec{w}_t^*$ and $\overrightarrow{w_t^{*\perp}}$. It is obvious that

$$\vec{w}_t = \lambda_1 \vec{w}_t^* - \lambda_2 \overrightarrow{w_t^{*\perp}}.$$

Then we obtain from the definition of the stationary weak solutions and the two lemmas above

THEOREM 4.2 *Let ζ be a smooth vector field with compact support in R^m . If $t \notin T_\infty$, then*

$$\begin{aligned} & \int_{\Sigma^t} \operatorname{div}_{\Sigma^t}(\zeta) \nu_t + \int_{R^m} |\nabla u|^2 \operatorname{div} \zeta - 2 \partial_j u \partial_k u \partial_j \zeta^k \\ &= 2 \int_{R^m} (\lambda_1 u_t - \lambda_2 u \times u_t) \zeta \cdot \nabla u + 2 \int_{\Sigma^t} \vec{w}_t \zeta. \end{aligned}$$

We also have the following theorem.

THEOREM 4.3 *Suppose that u_k is a stationary weak solution of (1.1). Then for any $\vartheta \in C_0^\infty(R^m, R_+)$, we have*

$$\begin{aligned} & \int_{R^m} \vartheta \nu_t - \int_{R^m} \vartheta \nu_{t_0} + \left(\int_{R^m \times t} - \int_{R^m \times t_0} \right) |\nabla u|^2 \vartheta \\ & \leq -2 \int_{t_0}^t \int_{\Sigma^t} (\lambda_1 \vartheta \mu + \nabla \vartheta \vec{w}_t^*) \\ & \quad -2 \int_{t_0}^t \int_{R^m} (\lambda_1 u_t^2 \vartheta + \nabla u \nabla \vartheta u_t). \end{aligned}$$

And if u_k is strong stationary weak solution of (1.1). Then the equality above holds.

We are in the position to introduce the curvature of Σ^t . Suppose that u_k are the strong stationary weak solutions, and the limiting map u is a strong stationary weak solution. We then have from Theorem 4.2 that, for any $t \notin \mathcal{T}_\infty$,

$$\int_{\Sigma^t} \operatorname{div}_{\Sigma^t}(\zeta) \nu_t = 2 \int_{\Sigma^t} \overrightarrow{w_t} \zeta = - \int_{\Sigma^t} \overrightarrow{H_t} \zeta \nu_t$$

where $\overrightarrow{H_t} = -2 \frac{d\overrightarrow{w_t}}{d\nu_t}$. Clearly for $\mathcal{H}^{m-2}$ -a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t}(x) = -2 \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{B_r(x)} (\lambda_1 \partial_t u_k - \lambda_2 u_k \times \partial_t u_k) \nabla u_k}{\int_{B_r(x)} |\nabla u_k|^2}.$$

Applying Lemma 4.3 we see that

$$\overrightarrow{H_t}(x) \in S^\perp(x).$$

Where $S = S(x) = T_x \nu_t$. Thus we obtain the first variation of varifold V_{ν_t} :

$$\delta V_{\nu_t} = -\nu_t \lfloor \overrightarrow{H_t}.$$

And $\overrightarrow{H_t}$ is the **generalized mean curvature** of Σ^t (see 4.3 in [A]). We define the total variation $||\delta V_{\nu_t}||$ of δV_{ν_t} by the requirement that

$$||\delta V_{\nu_t}|| (U) = \sup \{ \delta V_{\nu_t}(\zeta) : \zeta \in T\Sigma^t, \operatorname{spt} \zeta \subset U \text{ and } |\zeta| \leq 1 \}.$$

Then by the representation theorems (see 2.5 in [F])

$$\overrightarrow{H_t} = - \frac{d||\delta V_{\nu_t}||}{d\nu_t} \eta_t, \quad (4.2)$$

where $\eta_t(x) \in S^{m-1}$. We also define the varifolds $\delta^* V_{\nu_t}$ and $\delta^{*\perp} V_{\nu_t}$ respectively by

$$\delta^* V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta \overrightarrow{w_t}^* \nu_t,$$

$$\delta^{*\perp} V_{\nu_t}(\zeta) = \int_{\Sigma^t} 2\zeta \overrightarrow{w_t^{*\perp}} \nu_t.$$

Then we have

$$||\delta^* V_{\nu_t}|| = ||\delta^{*\perp} V_{\nu_t}||, \quad (4.3)$$

$$|\overrightarrow{H_t}^*| = |\overrightarrow{H_t^{*\perp}}|, \quad (4.4)$$

$$\overrightarrow{H_t} = \lambda_1 \overrightarrow{H_t}^* - \lambda_2 \overrightarrow{H_t^{*\perp}}, \quad (4.5)$$

$$|\overrightarrow{H_t}| \leq (\lambda_1 + |\lambda_2|) |\overrightarrow{H_t}^*|. \quad (4.6)$$

Evidently $\overrightarrow{H_t}^* = -2 \frac{d\overrightarrow{w_t}^*}{d\nu_t}$ and $\overrightarrow{H_t^{*\perp}} = -2 \frac{d\overrightarrow{w_t^{*\perp}}}{d\nu_t}$, and for $\mathcal{H}^{m-2}$ -a.e. $x \in \Sigma^t$

$$\overrightarrow{H_t}^*(x) = -2 \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{B_r(x)} \partial_t u_k \nabla u_k}{\int_{B_r(x)} |\nabla u_k|^2}.$$

$$\overrightarrow{H_t^*}(x) = -2 \lim_{r \rightarrow 0} \lim_{k \rightarrow \infty} \frac{\int_{B_r(x)} (u_k \times \partial_t u_k) \nabla u_k}{\int_{B_r(x)} |\nabla u_k|^2}.$$

Applying Lemma 4.3 again we have

$$\overrightarrow{H_t^*}(x), \overrightarrow{H_t^*}(x) \in S^\perp(x).$$

$\overrightarrow{H_t^*}$ is called the **quasi-mean curvature** of Σ^t .

Now we introduce the Brakke's quantity $\mathcal{B}^*(\nu_t, \vartheta)$.

DEFINITION 4.1 Let $\vartheta \in C_0^2(R^m, R_+)$. Define

$$\mathcal{B}^*(\nu_t, \vartheta) = - \int_{R^m} \frac{\lambda_1}{2} \vartheta |\overrightarrow{H_t^*}|^2 \nu_t + \int_{R^m} \nabla \vartheta \cdot S^\perp \cdot \overrightarrow{H_t^*} \nu_t,$$

where $S = S(x) \equiv T_x \nu_t$ for H^{n-2} a.e. $x \in \{\vartheta > 0\}$.

DEFINITION 4.2 We say $\{\nu_t\}_{t \geq 0}$ a generalized Brakke's motion provided that for a.e. $t \geq 0$ and all $\vartheta \in C_0^2(R^m, R_+)$,

$$d_t^+ \nu_t(\vartheta) \leq \mathcal{B}^*(\nu_t, \vartheta),$$

where we denote by

$$d_t^+ f = \limsup_{s \rightarrow t} \frac{f(s) - f(t)}{s - t}.$$

We also say the measure family $\{\nu_t\}_{t \geq 0}$ (or surface Σ^t equivalently) moving by the quasi-mean curvature in the case.

We have

THEOREM 4.4 Suppose that u_k are the strong stationary weak solutions of (1.1) and the limiting map u is also a strong stationary weak solution of (1.1), then the blow up measure $\{\nu_t\}$ is a generalized Brakke's motion.

The following theorem asserts that the singular set Σ^t of Landau-Lifshitz system is a quasi-mean curvature flow.

THEOREM 4.5 Suppose that blow up set Σ^t of Landau-Lifshitz system (1.1) with $\lambda_1 \geq 2|\lambda_2|$ is a smooth family of sub-manifolds in R^m and assume that it is a generalized Brakke's flow in the sense of Theorem 4.4. Then Σ^t is a quasi-mean curvature flow.

Proof. We write $\Sigma^t = F(\cdot, t)(M_0^{m-2})$, which is a parametric representation of Σ^t . Suppose that

$$\partial_t F(x, t) = \vec{Y}(x, t)$$

with $\vec{Y} \in T\Sigma^{t^\perp}$. By the first variation of a varifold with respect to integrands (see 4.9 in [A] or 2.4 in [I]), we have

$$\delta V_{\nu_t}(\vartheta)(\vec{Y}) = d_t^+ \int_{\Sigma^t} \vartheta \nu_t = \int_{\Sigma^t} (-\vartheta \vec{H}_t + \nabla \vartheta \cdot S^\perp) \cdot \vec{Y} \nu_t, \quad (4.7)$$

where $\vec{H}_t$ defined in (4.2) is the mean curvature of Σ^t . On the other hand, by the Theorem 4.4, we have

$$d_t^+ \int_{\Sigma^t} \vartheta \nu_t \leq \int_{\Sigma^t} \left(-\frac{\lambda_1}{2} \vartheta |\vec{H}_t^*|^2 + \nabla \vartheta \cdot S^\perp \cdot \vec{H}_t^* \right) \nu_t. \quad (4.8)$$

Therefore we obtain that from (4.7) (4.8) and (4.5)

$$\int_{\Sigma^t} \vartheta [\lambda_1 \vec{H}_t^* \cdot (\vec{Y} - \frac{1}{2} \vec{H}_t^*) - \lambda_2 \vec{H}_t^{*\perp} \cdot \vec{Y}] \nu_t \geq \int_{\Sigma^t} \nabla \vartheta \cdot (\vec{H}_t^* - \vec{Y}) \nu_t.$$

Now since $\vartheta > 0$ is arbitrary and $|\vec{H}_t^* \cdot \vec{H}_t^{*\perp}| \leq |\vec{H}_t^*|^2$ and $\lambda_1 \geq 2|\lambda_2|$, we have to have $\vec{H}_t^* = \vec{Y}$, which is the desired result.

4.4 Blow up analysis at a singular point

Let $z_0 = (x_0, t_0) \notin \Sigma_\beta$ be a singular point of u such that

$$\lim_{r \rightarrow 0} \mathcal{E}(r, u, z_0) \geq \epsilon_0$$

by Theorem 3.1. Set $u_k(z) = u(x_0 + r_k x, t_0 + r_k^2 t)$ where $x \in R^m$ and $t \in R_-$, then u_k satisfies (1.1) and by scaling, for any $z \in R^m \times R_-$,

$$\mathcal{E}(r_k, u, z) = \int_{P_1(z)} |\nabla u_k|^2.$$

By Proposition 2.1 we see that for small r_k

$$\int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0),$$

and from the energy inequality we have

$$\int_{P_{1/2}(z)} |\partial_t u_k|^2 \leq c \int_{P_1(z)} |\nabla u_k|^2 \leq C(R_0).$$

Denote, for fixed constant $\delta > 0$,

$$D_k = \left\{ z \in R^m \times R_- : \begin{array}{l} x_0 + r_k x \in B_\delta(x_0), \\ t_0 + r_k^2 t \in [t_0 - \delta^2, t_0 + \delta^2] \end{array} \right\},$$

then $D_k \rightarrow R^m \times R_-$ as $k \rightarrow \infty$, since $r_k \rightarrow 0$. There is therefore a subsequence (still denoted by r_k) $r_k \rightarrow 0$ such that $u_k(x, t) \rightarrow v(x, t)$ weakly in $W_{loc}^{1,2}(R^m \times R_-, S^2)$,

$$|\nabla u_k|^2(\cdot, t) dx dt \rightarrow |\nabla v|^2(\cdot, t) dx dt + \nu_t dt$$

$$|\partial_t u_k|^2(\cdot, t) dx dt \rightarrow |\partial_t v|^2(\cdot, t) dx dt + \mu$$

in the sense of measure as $k \rightarrow \infty$, where ν_t is a nonnegative Radon measure in R^m supported in Σ^t , μ is a nonnegative Radon measure in $R^m \times R_-$ supported in Σ , here and in the sequel we use the same notations and results in the sections above.

We have by scaling

$$\begin{aligned} & \int_{t_0-(r_k R_2)^2}^{t_0-(r_k R_1)^2} \int_{R^m} \lambda_1 \left[\frac{r \partial_r u}{\sqrt{2(t_0 - t)}} \right. \\ & \quad \left. - \sqrt{2(t_0 - t)} \left(\partial_t u - \frac{\lambda_2}{2\lambda_1} u \times \partial_t u \right) \right]^2 G_{z_0} dx dt \\ & \leq \Phi_{z_0}(r_k R_2, u) - \Phi_{z_0}(r_k R_1, u) + \Theta_{z_0}(r_k R_1, r_k R_2, u). \end{aligned}$$

Since $z_0 \notin \Sigma_\beta$, we have $\Theta_{z_0}(r_k R_1, r_k R_2, u) \leq C_1(r_k R_2)^\beta$ by Lemma 2.1. Then $\Theta_{z_0}(r_k R_1, r_k R_2, u) \rightarrow 0$ as $k \rightarrow \infty$. Obviously, for any $R > 0$, $\Phi_{z_0}(r_k R, u) \rightarrow \lambda_1^{-m/2} \theta(x_0, t_0)$ as $k \rightarrow \infty$. We thus obtain

$$\begin{aligned} & \int_{-R_2^2}^{-R_1^2} \int_{R^m} \lambda_1 \left[\frac{r \partial_r u_k}{\sqrt{2(-t)}} - \sqrt{2(-t)} \left(\partial_t u_k \right. \right. \\ & \quad \left. \left. - \frac{\lambda_2}{2\lambda_1} u_k \times \partial_t u_k \right) \right]^2 G_0 dx dt \rightarrow 0 \end{aligned}$$

as $k \rightarrow \infty$. That is,

$$\lambda_1 \partial_t u_k - \frac{\lambda_2}{2} (u_k \times \partial_t u_k) - \frac{\lambda_1 r \partial_r u_k}{2(-t)} \rightarrow 0 \quad (4.9)$$

strongly in $L^2(R^m \times [-R_2, -R_1], R^3)$ as $k \rightarrow \infty$. Furthermore

$$u_k \times \partial_t u_k - u_k \times \frac{x \cdot \nabla u_k}{2(-t)} \rightarrow 0 \quad (4.10)$$

strongly in $L^2(R^m \times [-R_2, -R_1], R^3)$. Applying the results in section 4.2 we obtain the blow up formulas

THEOREM 4.6 *If $t_0 \notin \mathcal{T}_\infty$, and $z_0 = (x_0, t_0) \notin \Sigma_\beta$ is a blow up point, then we have two blow up formulas*

$$\begin{aligned} & \int_{R^m} (|\nabla u|^2 \operatorname{div}(\zeta) - 2 \partial_j v \partial_k v \partial_j \zeta^k) \\ & - 2 \int_{R^m} (\lambda_1 \partial_t v - \lambda_2 v \times \partial_t v) \nabla v \zeta \\ & = \int_{\Sigma^t} \frac{1}{2(-t)} (\lambda_1 x_{\bar{n}} \cdot \zeta_{\bar{n}} - \frac{\lambda_2}{2} \sigma(x) |x_{\bar{n}} \times \zeta_{\bar{n}}|) \nu_t - \int_{\Sigma^t} \operatorname{div}_{(\Sigma^t)}(\zeta) \nu_t, \end{aligned} \quad (4.11)$$

$$\begin{aligned}
& \int_{t_1}^{t_2} \int_{R^m} 2(\lambda_1 v_t^2 \vartheta + \nabla v \nabla \vartheta v_t) dx dt \\
& + \left(\int_{R^m \times t_2} - \int_{R^m \times t_1} \right) |\nabla v|^2 \vartheta dx \\
& = \int_{\Sigma^t} \vartheta \nu_{t_1} - \int_{\Sigma^t} \vartheta \nu_{t_2} - \frac{\lambda_1^3}{4\lambda_1^2 + \lambda_2^2} \int_{t_1}^{t_2} \int_{\Sigma^t} \frac{\vartheta |x_{\bar{n}}|^2}{t^2} \nu_t \\
& \quad - \int_{t_1}^{t_2} \int_{\Sigma^t} \frac{1}{2(-t)} \left(\lambda_1 x_{\bar{n}} \cdot \zeta_{\bar{n}} - \frac{\lambda_2}{2} \sigma(x) |x_{\bar{n}} \times \zeta_{\bar{n}}| \right) \nu_t, \tag{4.12}
\end{aligned}$$

where $\zeta \in C_0^\infty(R^m, R^m)$ and $\vartheta \in C_0^\infty(R^m, R_+)$, $y_{\bar{n}}(x)$ denotes the projective vector of the vector $y(x)$ on the normal plane of Σ^t at x , and $\sigma(x) = \pm 1$ is a sign function determined by some Radon measure.

References

- [A] W. Allard, On the first variation of a varifold, *Ann. Math.* 95(1972), 417-491
- [ASo] L. Ambrosio and H.M. Soner, Level set approach to mean curvature flow in arbitrary codimension, *J. Diff. Geom.* 43(1996), 693-737
- [AS] F. Alouges and A. Soyeur, On global weak solutions for Landau-Lifshitz equations: existence and nonuniqueness. *Nonlinear Anal., TMA*, vol 18(11), 1071-1084(1992).
- [C] Y. Chen, Bubbling phenomena and energy identity for Landau-Lifschitz equations, preprint
- [CLL] Y. Chen, J. Li, F.H. Lin, Partial regularity for weak heat flows into spheres, *Comm. Pure and Applied Math.* 4(1995), 429-448
- [Ch] X. Cheng, Estimate of the singular set of the evolution problem for harmonic maps, *J. D. G.*, 34(1991), 169-174
- [E] L.C. Evans, Partial regularity for stationary harmonic maps into spheres, *Arch. Rational Mech. Anal* 116(1991), 101-113
- [F] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969
- [Fe] M. Feldman, Partial regularity for harmonic maps of evolution into sphere, *Comm. P. D. E.* 19(5&6)(1994), 761-790
- [Fo] H.C. Fogedby, *Theoretical aspects of mainly low dimensional magnetic systems. Lecture Notes Phys. Vol 131*, Berlin, Heidelberg, New York, Springer 1980.

- [GH] B.Guo and M.Hong, The Landau-Lifshitz equation of the ferromagnetic spin chain and harmonic maps, *Calc. Var.* 1, 1993, 311-331.
- [H] F.Helein, Regularity des applications faiblement harmoniques entre une surface et une sphere, *C. R. Acad.Sci.Pari.* 311(1990), 519-524
- [I] T.Ilmanen, Elliptic regularization and partial regularity for motion by mean curvature, *Memoirs of the Amer. Math. Soc.*, 520, 994
- [LN] M.Lakshmanan and K.Nakamura, Landau-Lifshitz equation of ferromagnetism exact treatment of the Gilbert damping, *Phys. Rev. Letter* 53(26), 2497-2499(1984).
- [LL] L.D.Landau and E.M.Lifshitz, On the theory of the dispersion of magnetic permeability in ferromagnetic bodies. *Phys. Z.Sowj.* 8, 153 (1935).
- [LT] J.Li, G.Tian, The blow up locus of heat flows for harmonic maps, *Acta Math. Sinica (English version)* 16(2000), 29-62
- [LT1] J.Li, G.Tian, A blow up formula for stationary harmonic maps, *IMRN* 14(1998), 735-755
- [Liu] X.Liu, Partial regularity for the Landau-Lifshitz system of the ferromagnetic spin chain, to appear in *Comm. PDE*
- [Liu1] X.Liu, Blow up sets of the Landau-Lifshitz system and quasi-mean curvature flows, preprint
- [LW] F.H.Lin and C.Y.Wang, Harmonic and qusi-harmonic spheres, *Comm. Anal. Geom.*, to appear
- [P] D.Priess, Geometry of measure in R^n ; Distribution, rectifiability, and densities; *Ann. Math.*, 125(1987), 537-643
- [Si] L.Simon, Lectures on geometric measure theory, *Proc. Center Math. Anal.* 3(1983), Australian National Univ. Press
- [T] L.A.Takhtalian, Integration of the continuous Heisenberg spin chain through the inverse scattering method, *Phys. Letter* 15B,3470-3476(1977).
- [TW] J.Tjon and J.Wright, Soliton in the continuous Heisenberg chain. *Phys. Rev.*
- [ZGT] Y.Zhou, B.Guo and S.Tan, Existence and uniqueness of smooth solution for system of ferromagnetic chain. *Scientia Sinica Ser. A* 34, 257-266(1991).