

The Borsuk-Ulam Theorem and Combinatorics

By

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Abstract

The Borsuk-Ulam antipodal theorem is studied by many mathematicians and generalized in many ways. On the other hand, the Borsuk-Ulam theorem has applications in many mathematical fields. In this paper, we will see some generalization and combinatorial applications of the Borsuk-Ulam theorem.

§ 1. The Borsuk-Ulam Theorem

Let S^n be the unit sphere in $\mathbf{R}^{n+1}$. The (Borsuk-Ulam type) antipodal theorem can be stated in several different, but equivalent ways.

1.1. If $f: S^m \rightarrow S^n$ satisfies $f(-x) = -f(x)$, then $m \leq n$.

1.2. If $f: S^n \rightarrow \mathbf{R}^n$ satisfies $f(-x) = -f(x)$, then $f^{-1}(0) \neq \emptyset$.

1.3. For every $f: S^n \rightarrow \mathbf{R}^n$ there exists an $x \in S^n$ with $f(-x) = f(x)$.

1.4. For every closed covering $\{M_1, \dots, M_{n+1}\}$ of S^n , there exists an $i \in \{1, \dots, n+1\}$ with $M_i \cap (-M_i) \neq \emptyset$.

1.5. $\text{cat} \mathbf{R}P^n = n+1$, where $\text{cat} \mathbf{R}P^n$ denotes the Ljusternik-Schnirelmann category of the n -dimensional real projective space $\mathbf{R}P^n$.

For a topological space X , the Ljusternik-Schnirelmann category $\text{cat} X$ of X is defined by

$$\text{cat} X = \min\{k \in \mathbf{N} \mid \text{there exists a closed cover } \{A_1, \dots, A_k\} \text{ of } X \\ \text{such that all } A_i \text{ are contractible in } X\}.$$

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There are many way to prove the Borsuk-Ulam theorem. For example, we can prove it by using the cohomology algebra of $\mathbf{R}P^n$ (see [22]).

The Borsuk-Ulam theorem is also related with the degree theory of maps between manifolds. Let M and N be closed connected oriented manifolds of dimension n . Then for a continuous map $f: M \rightarrow N$, we have the induced homomorphism $f_*: H_n(M) \rightarrow H_n(N)$. Let $[M]$ and $[N]$ be the fundamental homology classes of M and N respectively. We define the degree $\deg f$ of f by $f_*[M] = (\deg f)[N]$. Then the following theorem is known.

1.6. If a continuous map $f: S^n \rightarrow S^n$ satisfies $f(-x) = -f(x)$, then $\deg f$ is odd.

We can also prove the Borsuk-Ulam theorem by using this theorem. By considering the antipodal $\mathbf{Z}_2$ -action on S^n , 1.6 is considered as a theorem of equivariant maps. We can prove 1.6 by using the Gysin-Smith (Thom-Gysin) exact sequence([22], [18]).

The Borsuk-Ulam theorem has been generalized and extended in many ways from the view point of transformation group theory (see [5], [7], [12], [23], [24]). Fadell-Husseini and Jaworowski introduced ideal-valued index theory and generalized the Borsuk-Ulam theorem (see [7], [12]). Let G be a compact Lie group and X a G -space. Let $EG \rightarrow BG$ be a universal principal G -bundle. We denote by $\bar{H}(-; \mathbf{K})$ the Alexander-Spanier cohomology theory with coefficients in a field $\mathbf{K}$. We set $\bar{H}_G^*(X; \mathbf{K}) = \bar{H}^*(EG \times_G X; \mathbf{K})$. The G -index $\text{Ind}^G(X; \mathbf{K})$ of X is defined to be the kernel of the G -cohomology homomorphism induced by the constant map $c_X: X \rightarrow pt$;

$$\text{Ind}^G(X; \mathbf{K}) = \text{Ker}(c_X^*: \bar{H}_G^*(pt; \mathbf{K}) \rightarrow \bar{H}_G^*(X; \mathbf{K})),$$

where pt is a one-point space. Since $\bar{H}_G^*(pt; \mathbf{K}) = \bar{H}^*(BG; \mathbf{K})$, $\text{Ind}^G(X; \mathbf{K})$ is an ideal of $\bar{H}^*(BG; \mathbf{K})$. The following proposition was proved in [7] and [12].

Proposition([7], [12]). *If $f: X \rightarrow Y$ is a G -map, then $\text{Ind}^G(X; \mathbf{K}) \supset \text{Ind}^G(Y; \mathbf{K})$ in $\bar{H}^*(BG)$.*

Proposition([7], [12]). *Let X and Y be G -spaces, and W a G -invariant closed subspace of Y . If $f: X \rightarrow Y$ is a G -map, then $\text{Ind}^G f^{-1}(W) \cdot \text{Ind}^G(Y - W; \mathbf{K}) \subset \text{Ind}^G(X; \mathbf{K})$ in $\bar{H}^*(BG; \mathbf{K})$, where $\cdot$ represents the product of ideals.*

By using these propositions, we have generalized Borsuk-Ulam theorems (see [7], [9], [12], [14]).

§ 2. Applications of the Borsuk-Ulam Theorem in Combinatorics

In this section, we introduce combinatorial applications of the Borsuk-Ulam theorem.

§ 2.1. Tucker's Lemma

Let T be some triangulation of the n -dimensional ball B^n . We call T *antipodally symmetric on the boundary* if the set of simplices of T contained in $S^{n-1} = \partial B^n$ is a triangulation of S^{n-1} and it is antipodally symmetric; that is, if $\sigma \subset S^{n-1}$ is a simplex of T , then $-\sigma$ is also a simplex of T .

Tucker's lemma. *Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Let*

$$\lambda: V(T) \rightarrow \{+1, -1, +2, -2, \dots, +n, -n\}$$

be a labelling of the vertices of T such that $\lambda(-v) = -\lambda(v)$ for every vertex $v \in \partial B^n$. Then there exists a 1-simplex $\{v_1, v_2\}$ in T such that $\lambda(v_1) = -\lambda(v_2)$.

Let $\diamond^n = \text{conv}\{\pm e_1, \dots, \pm e_n\}$ be the n -dimensional cross-polytope and $\partial(\diamond^n) \cong S^{n-1}$ its boundary with $\mathbf{Z}_2$ -invariant triangulation.

It is easily seen that the following theorem is equivalent to Tucker's lemma.

Theorem. *Let T be a triangulation of B^n that is antipodally symmetric on the boundary. Then there is no map $\lambda: V(T) \rightarrow V(\partial(\diamond^n))$ that is a simplicial map of T into $\partial(\diamond^n)$ and is antipodal on the boundary.*

This theorem follows from the fact that the degree of any antipodal map $f: S^n \rightarrow S^n$ is odd (1.6). Ky Fan generalized this fact from the combinatorial viewpoint.

Ky Fan's theorem ([8]). *Let T be a $\mathbf{Z}_2$ -invariant triangulation of S^n . If $f: T \rightarrow \partial(\diamond^m)$ is a simplicial $\mathbf{Z}_2$ -map, then $n < m$ and*

$$\sum_{1 \leq k_1 < k_2 < \dots < k_{n+1} \leq m} \alpha(k_1, -k_2, k_3, -k_4, \dots, (-1)^n k_{n+1}) \equiv 1 \pmod{2},$$

where $\alpha(j_1, j_2, \dots, j_{n+1})$ is the number of n -simplices in T mapped to the simplex spanned by vectors $e_{j_1}, e_{j_2}, \dots, e_{j_{n+1}}$ and by definition $e_{-j} = -e_j$.

§ 2.2. Lovász-Kneser Theorem

First we prepare basic definitions and notations. A graph is a pair (V, E) , where V is a set (the *vertex set*) and $E \subset \binom{V}{2}$ is the *edge set*, where $\binom{V}{2}$ denotes the set of all subsets of V of cardinality exactly 2. We denote by $[n]$ the finite set $\{1, 2, \dots, n\}$. A k -coloring of a graph $G = (V, E)$ is a map $c: V \rightarrow [k]$ such that $c(u) \neq c(v)$ whenever $\{u, v\} \in E$. The *chromatic number* of G , denote by $\chi(G)$, is the smallest k such that G has a k -coloring.

Let X be a finite set and let $\mathcal{F} \subset 2^X$ be a set system. The *Kneser graph* of $\mathcal{F}$, denoted by $\text{KG}(\mathcal{F})$, has $\mathcal{F}$ as the vertex set, and two sets $F_1, F_2 \in \mathcal{F}$ are adjacent if and

only if $F_1 \cap F_2 = \emptyset$. Let $\text{KG}_{n,k}$ denote the Kneser graph of the system $\mathcal{F} = \binom{[n]}{k}$ (all k -element subsets of $[n]$). The following theorem was expected by Kneser and proved by Lovász ([15]).

Lovász-Kneser theorem. *For all $k > 0$ and $n \geq 2k - 1$, the chromatic number of the Kneser graph $\text{KG}_{n,k}$ is $n - 2k + 2$.*

It is easy to prove $\chi(\text{KG}_{n,k}) \leq n - 2k + 2$. We define a coloring $c: \binom{[n]}{k} \rightarrow [n - 2k + 2]$ of the Kneser graph $\text{KG}_{n,k} = \text{KG}(\binom{[n]}{k})$ by

$$c(F) = \min\{\min(F), n - 2k + 2\}.$$

If two sets F_1, F_2 get the same color $c(F_1) = c(F_2) = i < n - 2k + 2$, then they cannot be disjoint, since they both contain the element i . If the two sets both get color $n - 2k + 2$, then they are both contained in the set $\{n - 2k + 2, n - 2k + 3, \dots, n\}$. Since $|\{n - 2k + 2, n - 2k + 3, \dots, n\}| = 2k - 1$, they can not be disjoint either.

Lovász used the neighborhood complex of a graph to prove the Lovász-Kneser theorem. The neighborhood complex $\mathcal{N}(G)$ of a graph G is the simplicial complex whose vertices are the vertices of G and whose simplices are those subsets of $V(G)$ which have a common neighbor. Denote by $\bar{\mathcal{N}}(G)$ the polyhedron determined by $\mathcal{N}(G)$. Lovász proved that if $\bar{\mathcal{N}}(G)$ is i -connected, then $\chi(G) > i + 2$ by using the Borsuk-Ulam theorem (1.4). Moreover he proved $\bar{\mathcal{N}}(\text{KG}_{n,k})$ is $(n - 2k - 1)$ -connected and therefore he proved the Lovász-Kneser theorem. The Lovász-Kneser theorem was proved in other ways after Lovász proved it (see [3], [17], [20]).

A *hypergraph* is a pair $(X, \mathcal{F})$, where X is a finite set and $\mathcal{F} \subset 2^X$ is a system of subsets of X . The element of $\mathcal{F}$ are called the *edges* or *hyperedges*. A hypergraph H is m -colorable if its vertices can be colored by m colors such that no hyperedge becomes monochromatic. We define the m -colorability defect $\text{cd}_m(\mathcal{F})$ of a set system $\emptyset \notin \mathcal{F}$ by

$$\text{cd}_m(\mathcal{F}) = \min\{|Y| : (X - Y, \{F \in \mathcal{F} \mid F \cap Y = \emptyset\}) \text{ is } m\text{-colorable}\}.$$

Dol'nikov prove the following theorem in [6].

Dol'nikov's theorem. *For any set system $\emptyset \notin \mathcal{F}$, the inequality*

$$\text{cd}_2(\mathcal{F}) \leq \chi(\text{KG}(\mathcal{F}))$$

holds.

This theorem generalizes the Lovász-Kneser theorem. Because it is easy to prove that if $\mathcal{F}$ consists of all the k -subsets of an n -set with $k \leq n/2$, then $\text{cd}_2(\mathcal{F}) = n - 2k + 2$.

We say that a graph is *completely multicolored* in a coloring if all its vertices receive different colors. For $x \in \mathbf{R}$, let $\lfloor x \rfloor = \max\{n \in \mathbf{Z} \mid n \leq x\}$ and $\lceil x \rceil = \min\{n \in \mathbf{Z} \mid n \geq x\}$. Symonyi and Tardos prove the following theorem.

Theorem([21]). *Let $\mathcal{F}$ be a finite family of sets, $\emptyset \notin \mathcal{F}$ and $\text{KG}(\mathcal{F})$ its general Kneser graph. Let $r = \text{cd}_2(\mathcal{F})$. Then any proper coloring of $\text{KG}(\mathcal{F})$ with colors $1, \dots, m$ (m arbitrary) must contain a completely multicolored complete bipartite graph $K_{\lceil r/2 \rceil, \lfloor r/2 \rfloor}$ such that the r different colors occur alternating on the two sides of the bipartite graph with respect to their natural order.*

This theorem generalizes Dol'nikov's theorem, because it implies that any proper coloring must use at least $\text{cd}_2(\mathcal{F})$ different colors. In the proof of the above theorem, Ky Fan's theorem is used.

$S \in \binom{[n]}{k}$ is said to be *stable* if it does not contain any two adjacent elements modulo n . In other words, S corresponds to an independent set in the cycle C_n . We denote by $\binom{[n]}{k}_{\text{stab}}$ the family of stable k -subsets of $[n]$. We define the *Schrijver graph* by

$$\text{SG}_{n,k} = \text{KG} \left(\binom{[n]}{k}_{\text{stab}} \right).$$

It is an induced subgraph of the Kneser graph $\text{KG}_{n,k}$. In [19], Schrijver defined the Schrijver graph and proved $\chi(\text{SG}_{n,k}) = \chi(\text{KG}_{n,k}) = n - 2k + 2$ for all $n \geq 2k \geq 0$.

§ 2.3. Necklace Theorem

Two thieves have stolen a precious necklace, which has n beads. These beads belong to t different types. Assume that there is an even number of beads of each type, say $2a_i$ beads of type i , for each $i \in \{1, 2, \dots, t\}$, where a_i is a nonzero integer. Remark that we have $2 \sum_{i=1}^t a_i = n$. The beads are fixed on an open chain made of gold.

As we do not know the exact value of each type of bead, a fair division of the necklace consists of giving the same number of beads of each type to each thief. The number of beads of each type is even, hence such a division is always possible: cut the chain at the $n - 1$ possible positions. But we want to do the division with fewer cuts. The following theorem was proved by Goldberg and West.

Theorem. *A fair division of the necklace with t types of beads between two thieves can be done with no more than t cuts.*

Alon and West gave a simpler proof by using the Borsuk-Ulam theorem in [2].

In 1987, Alon proved the following generalization, for a necklace having qa_i beads for each type i , a_i integer, using a generalized Borsuk-Ulam theorem:

Theorem([1]). *A fair division of the necklace with t types of beads between q thieves can be done with no more than $t(q - 1)$ cuts.*

The following theorem is considered as a continuous version of the necklace theorem.

Hobby-Rice theorem([11]). *Let $\mu_1, \mu_2, \dots, \mu_d$ be continuous probability measures on the unit interval. Then there is a partition of $[0, 1]$ into $d + 1$ intervals $I_0, I_1, \dots, I_d$ and signs $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_d \in \{-1, +1\}$ with*

$$\sum_{j=0}^d \varepsilon_j \mu_i(I_j) = 0 \text{ for } i = 1, 2, \dots, d.$$

Alon generalized Hobby-Rice theorem as follows.

Theorem([1]). *Let $\mu_1, \mu_2, \dots, \mu_t$ be t continuous probability measures on the unit interval. Then it is possible to cut the interval in $(k - 1) \cdot t$ places and partition the $(k - 1) \cdot t + 1$ resulting intervals into k families $F_1, F_2, \dots, F_k$ such that $\mu_i(\cup F_j) = l/k$ for all $l \leq i \leq t, l \leq j \leq k$. The number $(k - 1) \cdot t$ is best possible.*

Recently de Longueville and Živaljeić generalize the theorem above and got a higher-dimensional necklace theorem in [16].

§ 2.4. Tverberg's Theorem

H. Tverberg showed the following theorem in [25]

Theorem([25]). *Consider a finite set $X \subset \mathbf{R}^d$ with $|X| = (d + 1)(r - 1) + 1$. Then X can be partitioned into r subset $X_1, \dots, X_r$ so that*

$$\bigcap_{i=1}^r \text{conv} X_i \neq \emptyset.$$

The following theorem is a topological generalization of Tverberg's theorem.

Theorem([4], [26]). *Let $q = p^r$ be a prime power and $d \geq 1$. Put $N = (d + 1)(q - 1)$. For every continuous map $f: \|\sigma^N\| \rightarrow \mathbf{R}^d$ there are q disjoint faces $F_1, F_2, \dots, F_q$ of the standard N -simplex σ^N whose images under f intersect: $\bigcap_{i=1}^q f(\|F_i\|) \neq \emptyset$.*

This theorem was proved by using a Borsuk-Ulam type theorem. It is still unknown whether such a theorem holds for q not equal to a prime power.

§ 3. A Generalization of Tucker's Lemma

Tucker's lemma is equivalent to the Borsuk-Ulam theorem (see [17]) We shall consider a generalization of Tucker's lemma as analogous as generalizations of the Borsuk-Ulam theorem.

Let K be a simplicial complex and A a subcomplex of K . Suppose that A has a simplicial $\mathbf{Z}_2$ -action and $\mathbf{Z}_2$ -action on $|A|$ is free, where $|A|$ denotes the polyhedron determined by A . Then we can consider the first Stiefel-Whitney class $w_1(|A|/\mathbf{Z}_2)$ of the double covering $\pi: |A| \rightarrow |A|/\mathbf{Z}_2$. Let $\pi^!: H^*(|A|; \mathbf{Z}_2) \rightarrow H^*(|A|/\mathbf{Z}_2; \mathbf{Z}_2)$ be the transfer. We denote by $i: |A| \rightarrow |K|$ the inclusion. Then we have the following.

Theorem. *Suppose that $w_1(|A|/\mathbf{Z}_2)^{n-1} \notin \pi^! \circ i^*(H^{n-1}(|K|; \mathbf{Z}_2))$. If*

$$\lambda: V(K) \rightarrow \{\pm 1, \pm 2, \dots, \pm n\}$$

satisfies $\lambda(-v) = -\lambda(v)$ for $v \in V(A)$, then there exists a 1-simplex $\{v_1, v_2\}$ in K such that $\lambda(v_1) = -\lambda(v_2)$.

Proof. Suppose that there is no 1-simplex $\{v_1, v_2\}$ in K such that $\lambda(v_1) = -\lambda(v_2)$. Then we have a simplicial map $f_\lambda: K \rightarrow \partial(\diamond^n)$, where $\partial(\diamond^n)$ denotes a triangulation of S^{n-1} in section 2.1. Since $f_\lambda \circ i: A \rightarrow \partial(\diamond^n)$ is a $\mathbf{Z}_2$ -map, we have a continuous map $f: |A|/\mathbf{Z}_2 \rightarrow |\partial(\diamond^n)|/\mathbf{Z}_2$ such that $f \circ \pi = \pi_S \circ f_\lambda \circ i$, where $\pi_S: |\partial(\diamond^n)| \rightarrow |\partial(\diamond^n)|/\mathbf{Z}_2$ is a projection. Let α be the generator of $H^{n-1}(\partial(\diamond^n); \mathbf{Z}_2)$. Since $\pi^! \circ (f_\lambda \circ i)^*(\alpha) = f^* \circ \pi_S^!(\alpha) = f^*(w_1(S^{n-1}/\mathbf{Z}_2)^{n-1}) = w_1(|A|/\mathbf{Z}_2)^{n-1}$, $\pi^! \circ i^* \circ f_\lambda^*(\alpha) = w_1(|A|/\mathbf{Z}_2)^{n-1}$. This contradicts $w_1(|A|/\mathbf{Z}_2)^{n-1} \notin \pi^! \circ i^*(H^{n-1}(|K|; \mathbf{Z}_2))$. $\square$

Remark. In the above theorem, if $w_1(|A|/\mathbf{Z}_2)^n \neq 0$, then there exists a 1-simplex $\{v_1, v_2\}$ in A such that $\lambda(v_1) = -\lambda(v_2)$. Because if there is no 1-simplex $\{v_1, v_2\}$ in A such that $\lambda(v_1) = -\lambda(v_2)$, then we have an equivariant map $f_\lambda: |A| \rightarrow |\partial(\diamond^n)|$ from λ . Since $|\partial(\diamond^n)|/\mathbf{Z}_2 \cong \mathbf{RP}^{n-1}$, $w_1(|A|/\mathbf{Z}_2)^n = \bar{f}_\lambda^*(w_1(\mathbf{RP}^{n-1})) = 0$, where $\bar{f}_\lambda: |A|/\mathbf{Z}_2 \rightarrow \mathbf{RP}^{n-1}$ is a map determined by f_λ .

In the above theorem we consider S^{n-1} and its triangulation $\partial(\diamond^n)$. In [7], [12] and [14], we see Borsuk-Ulam type theorems on Stiefel manifolds.

Problem. Consider a generalization of Tucker's lemma on Stiefel manifolds.

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