

Semicontinuity of set valued mappings and duality formulas of integral functionals

NAOTO KOMURO

Mathematics Laboratory, Asahikawa Campus,

Hokkaido University of Education

Asahikawa 070

測度の凸関数による汎関数の双対公式と集合値写像の半連続性

§1 DUALITY FORMULAS

Let X be a metric space, and let f be a real valued function defined on $X \times \mathbb{R}^d$. Suppose that for each $x \in X$, $f_x(p) = f(x, p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. By K_x , we denote the subdifferential of f_x at 0;

$$\begin{aligned} K_x &= \partial f_x(0) \\ &= \{q \in \mathbb{R}^d \mid \langle q, p \rangle \leq f_x(p), \quad p \in \mathbb{R}^d\} \end{aligned}$$

For every $x \in X$ the set K_x is convex in $\mathbb{R}^d$, and since $f_x(p)$ is finite for all $p \in \mathbb{R}^d$, K_x is compact. Let $\mu = (\mu_1, \dots, \mu_n)$ be a $\mathbb{R}^d$ -valued finite Borel regular measure on X . The finite Borel measure $f(x, \mu)$ on X is defined by

$$\int_A f(x, \mu) = \int_A f(x, \overrightarrow{\mu(x)}) d|\mu| \quad \text{for a Borel set } A \subset X$$

where $|\mu|$ is the total variation measure of μ and $\overrightarrow{\mu(x)} = \frac{d\mu}{d|\mu|}(x)$ is the Radon Nikodym derivative of μ with respect to $|\mu|$. The measure $f(x, \mu)$ is independent of the choice of a norm in $\mathbb{R}^d$.

THEOREM 1. Suppose that f satisfies

- (1) f is lower semicontinuous (l.s.c.) on $X \times \mathbb{R}^d$,
- (2) for each $x \in X$, $f_x(p) = f(x, p)$ is convex, positively homogeneous in p ,
- (3) $f(x, p) \leq c|p|$ ($x \in X, p \in \mathbb{R}^d$) with some constant c .

Then for every bounded $|\mu|$ -measurable function $\varphi \geq 0$ on X ,

$$(F,1) \quad \int_X f(x, \mu) \varphi = \sup \left\{ \int_X \langle \overrightarrow{\mu(x)}, v(x) \rangle \varphi(x) d|\mu|(x) \mid \right. \\ \left. v \in C(X, \mathbb{R}^d), v(x) \in K_x \text{ for all } x \in X \right\}.$$

Next we consider the case when $f_x(\cdot)$ is only convex in $p \in \mathbb{R}^d$, and is not necessarily positively homogeneous. For defining the measure $f(x, \mu)$ in this case, we introduce the homogenization $F(x, p_0, p)$ of $f(x, p)$ defined by

$$F(x, p_0, p) = \begin{cases} f_\infty(x, p) & p_0 = 0 \\ f(x, \frac{p}{p_0}) p_0 & p_0 > 0 \\ \infty & p_0 < 0 \end{cases}$$

where f_∞ is the recession function of f , i.e.,

$$f_\infty(x, p) = \lim_{t \downarrow 0} f(x, \frac{p}{t}) t.$$

If f satisfies $f(x, p) \leq c(1 + |p|)$ ($x \in X, p \in \mathbb{R}^d$) with some constant c , F is well-defined real valued function on $X \times C$ with $C = [0, \infty) \times \mathbb{R}^d$ and $F = \infty$ on $X \times (\mathbb{R}^{d+1} \setminus C)$. Moreover, F is convex and positively homogeneous in $(p_0, p) \in \mathbb{R}^{d+1}$. (See [8, §8])

Let α be a nonnegative finite Borel regular measure on X . We fix this measure and now define the measure $f(x, \mu)$ by

$$f(x, \mu) = F(x, \alpha, \mu),$$

where F is the homogenization of f . Here (α, μ) is a $C = [0, \infty) \times \mathbb{R}^d$ valued Borel regular measure, and since F is positively homogeneous, $f(x, \mu)$ is a finite Borel regular measure.

It is easy to see that

$$\begin{aligned} f(x, \mu) &= F(x, \alpha, \mu) \\ &= F(x, 1, \overrightarrow{h(x)})\alpha + F(x, 0, \mu^s) \\ &= f(x, \overrightarrow{h(x)})\alpha + f_\infty(x, \overrightarrow{\mu^s(x)})|\mu^s| \end{aligned}$$

where $\overrightarrow{h(x)}\alpha$ is the absolutely continuous part of μ , and μ^s is the singular part with respect to α .

THEOREM 2. Suppose that f satisfies

- (1) for every $x_0 \in X$ and $\varepsilon > 0$, there is $\delta > 0$ such that $d(x, x_0) < \delta$ implies

$$f(x_0, p) - f(x, p) < \varepsilon(1 + |p|),$$
- (2) for each $x \in X$, $f_x(p)$ is convex in p ,
- (3) $f(x, p) \leq c(1 + |p|)$ ($x \in X, p \in \mathbb{R}^d$) with some constant c .

Then for every bounded $|\mu|$ -measurable function $\varphi \geq 0$ on X ,

$$\begin{aligned} (F, 2) \quad \int_X f(x, \mu) \varphi &= \sup \left\{ \int_X \langle \overrightarrow{\mu(x)}, v(x) \rangle \varphi(x) d|\mu|(x) - \int_X \varphi(x) f^*(x, v(x)) d\alpha \mid \right. \\ &\quad \left. v \in C(X, \mathbb{R}^d), f^*(x, v(x)) \in L^1(X, d\alpha) \right\}. \end{aligned}$$

Similar results can be seen in [2], [3], [6]. In the proof of Rockafellar [6], it is assumed that K_x has an interior point and the assumption on the regularity of f in x is slightly stronger than ours. In [2], it is assumed that f is continuous on $X \times \mathbb{R}^d$. We have weakened these assumptions by some arguments of the continuous selection.

We consider the set valued mapping K which carries each $x \in X$ to the compact convex set $K_x \subset \mathbb{R}^d$. K is said to be lower semicontinuous (l.s.c.) if $x_n \rightarrow x_0$ in X and $q_0 \in K_{x_0}$ implies the existence of a sequence $\{q_n\}$ such that $q_n \in K_{x_n}$ and $q_n \rightarrow q_0$. K_x is said to be upper semicontinuous (u.s.c.) if for any sequence $\{x_n\}$ tends to x_0 and $\varepsilon > 0$, $K_{x_n} \subset K_{x_0} + \varepsilon B$ holds for sufficiently large n , where $K_{x_0} + \varepsilon B = \{q + q' \in \mathbb{R}^d \mid q \in K_{x_0}, |q'| \leq \varepsilon\}$. Furthermore, when K_x is both l.s.c. and u.s.c., K is said to be continuous. One can find some other definitions of this semicontinuity in [1], [5], and [6] for instance.

However, in our case, most of them are all equivalent because K_x is always compact. The importance of the lower semicontinuity is that this allows us to take continuous selection of K_x . For example, In [6], the lower semicontinuity of K_x and the continuous selection theorem ([5]) are applied to prove a type of duality formula. Also in [2], the conditions for the same formula are given in terms of the function $f(x, p)$. However, the relation between the conditions of these two theorems is unclear. In this note, we investigate the conditions of f under which K_x is lower semicontinuous. Moreover, we will consider the upper semicontinuity and derive some duality of these two notions.

§2 SEMI CONTINUITY OF K_x

LEMMA 3. Let $f(x, p)$ be a function on $X \times \mathbb{R}^d$, and suppose that $f_x(p) = f(x, p)$ is convex and positively homogeneous in $p \in \mathbb{R}^d$. Put $K_x = \partial f_x(0)$, then the following conditions are equivalent.

- (l, 1) f is l.s.c. on $X \times \mathbb{R}^d$.
- (l, 2) For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies

$$f(x_0, p) - f(x, p) < \varepsilon |p|, \quad \text{for all } p \in \mathbb{R}^d.$$

- (l, 3) $K : x \longrightarrow K_x$ is l.s.c. on X .

REMARK: When f is l.s.c. only in x , these conditions do not hold though f is convex (and hence continuous) in p . This fact is the only thing that the symmetry of Lemma 3 and Proposition 6 fails. The space $\mathbb{R}^d$ in this theorem can be replaced by any closed convex cone in $\mathbb{R}^d$, but not by any infinite dimensional space. Moreover, positively homogeneity of f is essential in this lemma even if K_x can be defined as the subdifferential of f .

PROOF: (l, 1) $\Rightarrow$ (l, 2)

It suffices to show that $\{f(\cdot, p) \mid |p| = 1\}$ is equi l.s.c.. If not, there exists $x_0 \in X$, $\varepsilon > 0$, and sequences $\{x_n\} \subset X$ and $\{p_n\} \subset \mathbb{R}^d$, such that $x_n \longrightarrow x_0$, $|p_n| = 1$, and $f(x_0, p_n) - f(x_n, p_n) \geq \varepsilon$ for every n . Since $\{p \in \mathbb{R}^d \mid |p| = 1\}$ is compact, we can assume

that $p_n \longrightarrow p_0$ for some $|p_0|$. By the convexity of f in p , it is continuous in particular. Hence it follows by (l, 1) that

$$\begin{aligned} f(x_0, p_n) - f(x_n, p_n) &= f(x_0, p_n) - f(x_0, p_0) + f(x_0, p_0) - f(x_n, p_n) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

for sufficiently large n and this contradicts the assumption.

$$(l, 2) \Rightarrow (l, 3)$$

Suppose that K is not l.s.c. at $x_0 \in X$. Then there exist a sequence $\{x_n\}$ with $x_n \longrightarrow x_0$, $q_0 \in K_{x_0}$ and $\varepsilon > 0$ such that

$$Kx_n \cap \varepsilon B(q_0) = \phi, \quad (1)$$

for every n , where $\varepsilon B(q_0) = \{q \in \mathbb{R}^d \mid d(q, q_0) \leq \varepsilon\}$. By the condition (l, 2), we have for sufficiently large n ,

$$f(x_0, p) - f(x_n, p) < \varepsilon \quad \text{for all } p \in \mathbb{R}^d \text{ with } |p| = 1. \quad (2)$$

We fix such n , and by the separation theorem and (1), there exists $p_0 \in \mathbb{R}^d$ with $|p_0| = 1$, such that

$$\sup_{q \in K_{x_n}} \langle q, p_0 \rangle \leq \inf_{q \in \varepsilon B(q_0)} \langle q, p_0 \rangle. \quad (3)$$

Now we take the supporting point $\bar{q}$ of $\varepsilon B(q_0)$ with respect to p_0 , that is, $\bar{q} \in \varepsilon B(q_0)$ and $\inf_{q \in \varepsilon B(q_0)} \langle q, p_0 \rangle = \langle \bar{q}, p_0 \rangle$. Then,

$$\begin{aligned} \inf_{q \in \varepsilon B(q_0)} \langle q, p_0 \rangle &= \langle q_0, p_0 \rangle - \langle q_0 - \bar{q}, p_0 \rangle \\ &= \langle q_0, p_0 \rangle - \varepsilon \\ &\leq \sup_{q \in K_{x_0}} \langle q, p_0 \rangle - \varepsilon \\ &= f(x_0, p_0) - \varepsilon. \end{aligned}$$

By (3), we obtain

$$f(x_n, p_0) \leq f(x_0, p_0) - \varepsilon.$$

Since p in (2) is arbitrary, this is a contradiction.

$$(l, 3) \Rightarrow (l, 1)$$

Suppose that $x_n \rightarrow x_0$ in X and $p_n \rightarrow p_0$ in $\mathbb{R}^d$. For every $\varepsilon > 0$, we take $q_0 \in K_{x_0}$ such that

$$\begin{aligned} \langle q_0, p_0 \rangle &\geq \sup_{q \in K_{x_0}} \langle q, p_0 \rangle - \varepsilon \\ &= f(x_0, p_0) - \varepsilon. \end{aligned}$$

By (l, 3), there exists a sequence $\{q_n\}$ such that each q_n belongs to K_{x_n} and $q_n \rightarrow q_0$. Since $\langle q_n, p_n \rangle \leq \sup_{q \in K_{x_n}} \langle q, p_n \rangle = f(x_n, p_n)$, we have

$$\begin{aligned} f(x_0, p_0) - f(x_n, p_n) &\leq \langle q_0, p_0 \rangle + \varepsilon - \langle q_n, p_n \rangle \\ &< 2\varepsilon \end{aligned}$$

for sufficiently large n . This implies that f is l.s.c. on $X \times \mathbb{R}^d$. ■

COROLLARY 4. Suppose that f satisfies one of three conditions in Theorem 3. Then for every $x_0 \in X$ and $p_0 \in \mathbb{R}^d$, there exists a continuous function L on $X \times \mathbb{R}^d$ satisfying

- (1) for every $x \in X$, $L(x, p)$ is linear in $p \in \mathbb{R}^d$,
- (2) $L(x, p) \leq f(x, p)$ for all $x \in X$ and $p \in \mathbb{R}^d$,
- (3) $L(x_0, p_0) = f(x_0, p_0)$.

PROOF: First we note that L is continuous on $X \times \mathbb{R}^d$ if it satisfies (1) and is continuous with respect to each variable. By the separation theorem or Hahn Banach theorem, there exists $q_0 \in \mathbb{R}^d$ such that $\langle q_0, p \rangle \leq f(x_0, p)$ for all $p \in \mathbb{R}^d$, and $\langle q_0, p_0 \rangle = f(x_0, p_0)$. Take a set valued mapping K' defined by

$$K'_x = \begin{cases} K_x & x \neq x_0 \\ \{q_0\} & x = x_0. \end{cases}$$

Since $q_0 \in K_{x_0}$, it is easy to see that K' is l.s.c., and hence we can take a continuous selection $q(x)$ of K'_x . Thus the function L defined by $L(x, p) = \langle q(x), p \rangle$ ($x \in X, p \in \mathbb{R}^d$) is what we want. ■

By an analogy, one can also prove the following.

COROLLARY 5. Suppose that f satisfies one of the three conditions in Theorem 3. Let E be a closed subset of X , and let L be a continuous function on $E \times \mathbb{R}^d$ satisfying

- (1) for every $x \in E$, $L(x, p)$ is linear in $p \in \mathbb{R}^d$,
- (2) $L(x, p) \leq f(x, p)$ for all $x \in E$ and $p \in \mathbb{R}^d$.

Then L can be continuously extended to $X \times \mathbb{R}^d$ such that (1) and (2) hold replacing E by X .

Next we consider the upper semicontinuity of K_x . We note that the following proposition and Lemma 3 have some symmetry but it is not perfect.

PROPOSITION 6. Under the hypotheses in Lemma 3, the following conditions are equivalent.

- (u, 0) For every $p \in \mathbb{R}^d$, $f(x, p)$ is u.s.c. in $x \in X$.
- (u, 1) f is u.s.c. on $X \times \mathbb{R}^d$.
- (u, 2) For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies

$$f(x, p) - f(x_0, p) < \varepsilon|p|, \quad \text{for all } p \in \mathbb{R}^d.$$

- (u, 3) $K : x \rightarrow K_x$ is u.s.c. on X .

REMARK: A set valued mapping K is said to be closed if for any sequence $\{x_n\}$ with $x_n \rightarrow x_0$, and $\{q_n\}$ with $q_n \in K_{x_n}$, $q_n \rightarrow q_0$ for some $q_0 \in \mathbb{R}^d$ implies $q_0 \in K_{x_0}$. This is also a notion of upper semicontinuity of set valued mappings. Since K_x is compact in our case, the upper semicontinuity of K implies the closedness. However, the converse is not true in general. The equivalence of (u, 0) and (u, 1) is still valid when f is only convex and not positively homogeneous in p .

PROOF: $(u, 0) \Rightarrow (u, 1)$

Suppose that $x_n \rightarrow x_0$ in X and $p_n \rightarrow p_0$ in $\mathbb{R}^d$. Since f is continuous in p , there exists $\bar{p}_1, \dots, \bar{p}_{d+1} \in \mathbb{R}^d$ such that

$$f(x_0, \bar{p}_i) \leq f(x_0, p_0) + \frac{\varepsilon}{2} \quad (i = 1, \dots, d+1)$$

and the convex hull $co\{\bar{p}_1, \dots, \bar{p}_{d+1}\}$ forms a neighborhood of p_0 . Moreover by the condition (u, 0),

$$f(x_n, \bar{p}_i) \leq f(x_0, \bar{p}_i) + \frac{\varepsilon}{2} \quad (i = 1, \dots, d+1)$$

holds for sufficiently large n . Since $p_n \in co\{\bar{p}_1, \dots, \bar{p}_{d+1}\}$ for sufficiently large n , we obtain by the convexity of $f(x, \cdot)$ that

$$\begin{aligned} f(x_n, p_n) &\leq \max_{1 \leq i \leq d+1} f(x_n, \bar{p}_i) \\ &\leq \max_{1 \leq i \leq d+1} f(x_0, \bar{p}_i) + \frac{\varepsilon}{2} \\ &\leq f(x_0, p_0) + \varepsilon. \end{aligned}$$

This proves that (u, 1) holds.

$$(u, 1) \Rightarrow (u, 2)$$

we can prove this by the same way as in $(l, 1) \Rightarrow (l, 2)$ in Lemma 3.

$$(u, 2) \Rightarrow (u, 3)$$

Take $x_0 \in X$ and $\varepsilon > 0$ arbitrarily, and Suppose that $x_n \rightarrow x_0$ in X . By (u, 2),

$$f(x_n, p) - f(x_0, p) \leq \varepsilon|p| \quad (p \in \mathbb{R}^d),$$

for sufficiently large n . Then $q \in K_{x_n}$ implies that

$$f(x_0, p) - \langle q, p \rangle \geq f(x_0, p) - f(x_n, p) > -\varepsilon|p| \text{ for all } p \in \mathbb{R}^d.$$

By the separation theorem, there exists $q_0 \in \mathbb{R}^d$ such that

$$-\varepsilon|p| \leq \langle q_0, p \rangle \leq f(x_0, p) - \langle q, p \rangle \quad (p \in \mathbb{R}^d).$$

This inequality implies that $|q_0| \leq \varepsilon$, and $q + q_0 \in K_{x_0}$. Hence we have $q \in K_{x_0} + \varepsilon B$ and this proves (u, 3).

$$(u, 3) \Rightarrow (u, 1)$$

For the reason stated in the remark of this theorem, we can assume that K is closed. Suppose that (u, 1) does not hold, then there exist sequences $\{x_n\}$ with $x_n \rightarrow x_0$ for some x_0 in X , and $\{p_n\}$ with $p_n \rightarrow p_0$ for some p_0 in $\mathbb{R}^d$, and $\varepsilon > 0$ such that $f(x_n, p_n) >$

$f(x_0, p_0) + \varepsilon$ for every n . Since $f(x_n, p_n) = \sup_{q \in K_{x_n}} \langle q, p_n \rangle$, we can choose a sequence $\{q_n\} \subset \mathbb{R}^d$ such that $q_n \in K_{x_n}$ and

$$|f(x_n, p_n) - \langle q_n, p_n \rangle| \longrightarrow 0 \quad (n \longrightarrow \infty).$$

By the definition of upper semicontinuity, K_{x_n} is uniformly bounded. Therefore the sequence $\{q_n\}$ is bounded, and we can take a convergent subsequence $\{q_m\}$ of $\{q_n\}$ with $q_m \longrightarrow q_0$ for some $q_0 \in \mathbb{R}^d$. Hence it follows that

$$\langle q_0, p_0 \rangle \geq f(x_0, p_0) + \varepsilon.$$

On the other hand, by the closedness of K , q_0 has to be an element of K_{x_0} , and this is a contradiction. ■

Combining Lemma 3 and Proposition 6, we also obtain the following theorem. To see the equivalence between (c, 0) and (c, 1), refer to Theorem 1.1 in [3].

PROPOSITION 7. *Under the hypotheses in Lemma 3, the following conditions are equivalent.*

- (c, 0) For every $p \in \mathbb{R}^d$, $f(x, p)$ is continuous in $x \in X$.
- (c, 1) f is continuous on $X \times \mathbb{R}^d$.
- (c, 2) For every $x_0 \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d(x_0, x) < \delta$ implies

$$|f(x, p) - f(x_0, p)| < \varepsilon |p|, \quad \text{for all } p \in \mathbb{R}^d.$$

- (c, 3) $K : x \longrightarrow K_x$ is continuous on X .

§3 PROOF OF THE DUALITY FORMULA

For a subset $U \subset \mathbb{R}^d$, we denote the inverse image of a set valued mapping K by

$$K^{-1}(U) = \{x \in X | K_x \cap U \neq \emptyset\}.$$

K is l.s.c. if and only if $K^{-1}(U)$ is open for every Open set $U \subset \mathbb{R}^d$. Moreover, we say K is $|\mu|$ -measurable if $K^{-1}(U)$ is $|\mu|$ -measurable for every open set $U \subset \mathbb{R}^d$. For the detail of the continuous selection theorem and the measurable selection theorem, we refer to [1], [5], and [7].

PROOF OF THEOREM 1: Note that ' $\geq$ ' part of the formulas are almost trivial and it suffices to prove the converse inequality. First we show a weaker version of the formula (F, 1) while the supremum is taken over $|\mu|$ -measurable function $w : X \rightarrow \mathbb{R}^d$ with $w(x) \in K_x$. For arbitrary $\varepsilon > 0$, and $x \in X$, put

$$\begin{aligned}\Gamma(x) &= \{p \in \mathbb{R}^d \mid \langle \overrightarrow{\mu(x)}, p \rangle \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\}, \\ \Gamma_0(x) &= \{p \in K_x \mid \langle \overrightarrow{\mu(x)}, p \rangle \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\}.\end{aligned}$$

Since $f_x(\overrightarrow{\mu(x)}) = \sup_{p \in K_x} \langle \overrightarrow{\mu(x)}, p \rangle$, $\Gamma(x)$ and $\Gamma_0(x)$ are nonempty closed convex sets in $\mathbb{R}^d$, and $\Gamma(x) = \Gamma_0(x) \cap K_x$. By the condition (1) and Lemma 3, K is l.s.c. as a set valued mapping, and also measurable in particular. Hence by [7, Theorem 1M], Γ is a $|\mu|$ -measurable set valued mapping provided that so is Γ_0 . Let U be an open set in $\mathbb{R}^d$. Since $\Gamma_0(x)$ is an affine half space, $\Gamma_0(x) \cap U \neq \emptyset$ if and only if $\Gamma_0(x) \cap D \neq \emptyset$ where D is an arbitrary countable dense subset of U . Hence we have

$$\begin{aligned}\Gamma_0^{-1}(U) &= \Gamma_0^{-1}(D) \\ &= \bigcup_{p \in D} A_p\end{aligned}$$

where $A_p = \{x \in X \mid \langle \overrightarrow{\mu(x)}, p \rangle \geq f_x(\overrightarrow{\mu(x)}) - \varepsilon\}$. We note that $f_x(\overrightarrow{\mu(x)})$ is $|\mu|$ -measurable because of the lower semicontinuity of f . Thus $\Gamma_0^{-1}(U)$ is $|\mu|$ -measurable, and by the measurable selection theorem we can take a measurable function w on X such that $w(x) \in \Gamma(x)$. In other words

$$\begin{aligned}\int_X \langle \overrightarrow{\mu(x)}, w(x) \rangle \varphi(x) d|\mu| &\geq \int_X (f_x(\overrightarrow{\mu(x)}) - \varepsilon) \varphi(x) d|\mu| \\ &= \int_X f(x, \mu) \varphi - \varepsilon \int_X \varphi d|\mu|\end{aligned}\tag{4}$$

Since $|\mu|$ is finite measure and φ is bounded, this yields the duality formula of weaker version.

We next construct a desired continuous function $v : X \longrightarrow \mathbb{R}^d$ from w which has been obtained above. By Lusin's theorem, for arbitrary $\delta > 0$ there exists a closed set $Y \subset X$ such that $|\mu|(Y^c) < \delta$ and w is continuous on Y . We define a set valued mapping K' by

$$K'_x = \begin{cases} \{w(x)\} & x \in Y \\ K_x & x \notin Y \end{cases}$$

for $x \in X$. We see by [1, Corollary 9.1.3] (the closedness of K is missing in the condition of this corollary) that K' is also l.s.c. and have a continuous selection. In other words, there exists a continuous function $v : X \longrightarrow \mathbb{R}^d$ such that $v(x) \in K_x$ on X and $v(x) = w(x)$ on Y . Hence we have

$$\begin{aligned} \int_X \langle \overrightarrow{\mu(x)}, w(x) \rangle \varphi d|\mu| &= \int_X \langle \overrightarrow{\mu(x)}, v(x) \rangle \varphi d|\mu| + \int_{Y^c} \langle \overrightarrow{\mu(x)}, w(x) \rangle \varphi d|\mu| \\ &\quad - \int_{Y^c} \langle \overrightarrow{\mu(x)}, v(x) \rangle \varphi d|\mu| \\ &\leq \int_X \langle \overrightarrow{\mu(x)}, w(x) \rangle \varphi d|\mu| + \int_{Y^c} f(x, \overrightarrow{\mu(x)}) \varphi d|\mu| \\ &\quad + \|v\| \int_{Y^c} \varphi d|\mu|. \end{aligned}$$

Since $f(x, p) \leq c$ for $x \in X$ and $|p| = 1$, we thus obtain from (4) that

$$\begin{aligned} \int_X f(x, \mu) \varphi &\leq \int_X \langle \overrightarrow{\mu(x)}, v(x) \rangle \varphi d|\mu| + (c + \|v\|) \|\varphi\| |\mu|(Y^c) \\ &\quad + \varepsilon \|\varphi\| |\mu|(X). \end{aligned}$$

We note that $v(x) \in K_x$ implies $\|v\| = \sup_{x \in X} |v(x)| \leq c$, which is independent of δ and ε . Since ε and δ are arbitrary, this yields the desired formula (F,1). ■

The formula (F,1) is still valid in the case when the effective domain of $f_x(\cdot)$ is a closed convex cone $C \subset \mathbb{R}^d$. The proof can be done by a similar way except some standard arguments. Moreover, the formula (F,1) of this case is used for the proof of Theorem 2. Indeed, under the conditions in Theorem 2, the homogenization $F(x, p_0, p)$ satisfies the conditions in Theorem 1 by replacing $\mathbb{R}^d$ by the cone $C = [0, \infty) \times \mathbb{R}^d$, and we can apply Theorem 1 for F . To end this note, we show this fact in the following proposition.

PROPOSITION 8. *If f satisfies (1),(2),(3) in Theorem 2, then the homogenization F satisfies (1),(2),(3) in Theorem 1 by replacing $\mathbb{R}^d$ by $C = [0, \infty) \times \mathbb{R}^d$.*

PROOF: It is stated in §1 that F satisfies (2). Moreover,

$$\begin{aligned} F(x, 0, p) &= \lim_{t \downarrow 0} f(x, \frac{p}{t})t \\ &\leq \lim_{t \downarrow 0} c(1 + |\frac{p}{t}|)t \\ &= c|p|, \\ F(x, p_0, p) &= f(x, \frac{p}{p_0})p_0 \\ &\leq c(1 + |\frac{p}{p_0}|)p_0 \\ &= c(|p_0| + |p|) \quad (p_0 \neq 0), \end{aligned}$$

and this proves (3). Hence it remains to prove (1). It is easy to see that F is l.s.c. in $(p_0, p) \in C = [0, \infty) \times \mathbb{R}^d$. Hence it follows from (1) in Theorem 2 that for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|(p_0, p) - (q_0, q)| < \delta, d(x_0, x) < \delta, q_0 \neq 0$ implies

$$\begin{aligned} F(x_0, p_0, p) - F(x, q_0, q) &= F(x_0, p_0, p) - F(x_0, q_0, q) + F(x_0, q_0, q) - F(x, q_0, q) \\ &< \varepsilon + (f(x_0, \frac{q}{q_0}) - f(x, \frac{q}{q_0}))q_0 \\ &< \varepsilon + \varepsilon(1 + |\frac{q}{q_0}|)q_0 \\ &= \varepsilon + \varepsilon(|q_0| + |q|). \end{aligned}$$

It is similar in the case of $q_0 = 0$. So F is l.s.c. on $X \times C$ and the proof is complete. ■

REFERENCES

1. J. P. Aubin , H. Frankowska, "Set-valued analysis," Birkhäuser, Boston-Basel-Berlin, 1990.
2. P. Aviles, Y. Giga, N. Komuro, *Duality formulas and variational integrals*, Advances in Math. Sci. Appl. **1** (1992), 207–227.

3. F. Demengel and R. Temam, *Convex functions of a measure and applications*, Indiana Univ. Math. J. **33** (1984), 673–709.
4. N. Komuro, *On basic properties of convex functions and convex integrands*, Hokkaido J. of Math. **18** (1989), 1–30.
5. E. Michael, *Continuous selections I*, Ann. of Math. **63** (1956), 361–382.
6. R. T. Rockafellar, *Integrals which are convex functional II*, Pacific. J. Math. **39** (1971), 439–469.
7. R. T. Rockafellar, “Integral functionals, normal integrands and measurable selections,” Springer Lecture Notes in Math. 543, 1976, pp. 157–207.
8. R. T. Rockafellar, “Convex Analysis,” Princeton University Press, 1970.