

Another Minimax Generalized Bayes Estimators of a Normal Variance

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1 Introduction

Let X and S be independent random variables where X has p -variate normal distribution $N_p(\theta, \sigma^2 I_p)$ and S/σ^2 has chi square distribution χ_n^2 with n degrees of freedom. We deal with the problem of estimating the unknown variance σ^2 by an estimator δ relative to the quadratic loss function $L_2(\delta, \sigma^2) = (\delta/\sigma^2 - 1)^2$.

Stein (1964) showed that the best affine equivariant minimax estimator is $\delta_0 = S/(n+2)$ and it can be improved by considering a class of scale equivariant estimators

$$\delta_\phi = \frac{1}{n+2}(1 - \phi(W))S, \quad (1)$$

for $W = \|X\|^2/S$. Explicitly, he found an improved estimator $\delta^S = \min\{S/(n+2), (\|X\|^2 + S)/(p+n+2)\} = (n+2)^{-1}(1 - \phi^S(W))S$, where $\phi^S(w) = \max\{0, (p - (n+2)w)/(p+n+2)\}$. Brewster and Zidek (1974) derived an improved generalized Bayes estimator $\delta^{BZ} = (n+2)^{-1}(1 - \phi^{BZ}(W))S$, where

$$\phi^{BZ}(w) = \frac{2(1+w)^{-n/2-1}}{n+p+2} \left(\int_0^1 t^{p/2-1} \{1 - wt/(w+1)\}^{n/2+1} dt \right)^{-1}. \quad (2)$$

They also gave the general sufficient condition for minimaxity, which we denote by the BZ-condition in this paper. They showed that δ_ϕ is minimax if

(BZ1) $\phi(w)$ is nonincreasing,

(BZ2) $0 \leq \phi(w) \leq \phi^{BZ}(w)$.

Clearly both $\phi^S(w)$ and $\phi^{BZ}(w)$ satisfy the BZ-condition.

On the other hand, Strawderman (1974) derived another sufficient condition for minimaxity, which we denote by the ST-condition in this paper. He showed that δ_ϕ is minimax if

(ST1) $(1+w)^\epsilon \phi(w)$ is nonincreasing,

(ST2) $0 \leq \phi(w) \leq B_2(p, n, \epsilon)$.

The upper bound $B_2(p, n, \epsilon)$ will be discussed detail in Section 2. Strawderman (1974) claimed that δ^{BZ} satisfies the ST-condition. But his claim is incorrect as pointed out in Ghosh (1994) and Pal *et al.* (1998). Ghosh (1994) proposed a generalized Bayes estimator $\delta^G = (n+2)^{-1}(1 - \phi^G(W))S$ where

$$\phi^G(w) = \frac{2(1+w)^{-n/2-1}}{n+p+2(a+2)} \left(\int_0^1 t^{p/2+a} \{1 - wt/(w+1)\}^{n/2+1} dt \right)^{-1} \quad (3)$$

and showed that $\phi^G(w)$ for $-p/2 - 1 < a \leq -1$ satisfies the BZ-condition for minimaxity. Pal *et al.* (1998) pointed out that $\phi^G(w)$ for some $a(< -1)$ also satisfies the ST-condition. As far as we know, however, generalized Bayes estimators which satisfy the ST-condition but do not satisfy the BZ-condition have not been found to date.

In this paper, we propose such estimators. We consider a generalized Bayes estimator $\delta^{GB} = (n+2)^{-1}(1 - \phi^{GB}(W))S$ where

$$\phi^{GB}(w) = \frac{2b(w+1)^{-1}}{p+n+2(a+2)} \frac{\int_0^1 t^{p/2+a+1} (1-t)^{b-1} \{1 - wt/(w+1)\}^{n/2-b} dt}{\int_0^1 t^{p/2+a} (1-t)^b \{1 - wt/(w+1)\}^{n/2-b+1} dt} \quad (4)$$

which, for $b > 0$ does not satisfy the BZ-condition. We show that $\phi^{GB}(w)$ for some a and $b > 0$ satisfies the ST-condition. We make two main contributions. The first is to enrich the class of minimax generalized Bayes estimators under L_2 . The second is to find within our class, a subclass of estimators of a particularly simple form $(n+2)^{-1}(1 + \alpha/(W+1))^{-1}S$. This second contribution is interesting since most known generalized Bayes minimax procedures such as (2) or (3) seem quite complicated while the empirical Bayes estimator δ^S is quite simple. Hence we produce generalized Bayes estimators with a form as simple as δ^S .

2 Strawderman's type sufficient condition for minimaxity

First we review Strawderman (1974)'s sufficient condition for minimaxity. Under L_2 , Strawderman (1974) proposed as a upper bound of ϕ

$$B_2^{ST}(p, n, \epsilon) = \min \left(\frac{2}{1+\epsilon}, 2 \frac{\Gamma(p/2 + n/2 + 2\epsilon + 2) \Gamma(n/2 + \epsilon + 1)}{\Gamma(p/2 + n/2 + \epsilon + 2) \Gamma(n/2 + 2\epsilon + 2)} \frac{p\epsilon}{p+n+2} \right). \quad (5)$$

He claimed that $B \leq 2/(1+\epsilon)$ is required to guarantee that the function $g(u)u^{\epsilon+1}$, where $g(u) = 2u - Bu^{\epsilon+1} - 2(n+2)/(p+n+2)$, changes sign only once from negative to positive on $u \in [0, 1]$. Pal *et al.* (1998) pointed out that $g(1) > 0$, that is, $B < 2p/(p+n+2)$, should be

also required and hence proposed $\min(B_2^{ST}, 2p/(p+n+2))$ as the upper bound. Here noting that the equation $g(u) = 0$ has at most two solutions on $[0, \infty]$, $g(0) < 0$ and $g(u)$ is negative for sufficiently large u , we see that $g(u)$ changes sign only once from negative to positive on $[0, 1]$ if and only if $g(1) > 0$. Therefore we propose the modified version of the ST-condition.

Theorem 2.1. *Estimators of the form (1) are minimax under L_2 provided $\epsilon > 0$, $(1+w)^\epsilon \phi(w)$ is nonincreasing and $0 \leq \phi < B_2(p, n, \epsilon)$ where*

$$B_2(p, n, \epsilon) = \min \left(\frac{2p}{p+n+2}, 2 \frac{\Gamma(p/2 + n/2 + 2\epsilon + 2) \Gamma(n/2 + \epsilon + 1)}{\Gamma(p/2 + n/2 + \epsilon + 2) \Gamma(n/2 + 2\epsilon + 2)} \frac{p\epsilon}{p+n+2} \right). \quad (6)$$

3 Minimax generalized Bayes estimators

In this section, we derive minimax generalized Bayes estimators satisfying the sufficient condition proposed in Theorem 2.1 for the loss L_2 . We consider a class of generalized Bayes estimators with respect to the following prior distribution. For $\eta = \sigma^{-2}$, let the conditional distribution of θ given λ and η , for $0 < \lambda < 1$, be normal with mean vector 0 and covariance matrix $\lambda^{-1}(1-\lambda)\eta^{-1}I_p$ and let the density functions of λ and η be proportional to $\lambda^a(1-\lambda)^b I_{(0,1)}(\lambda)$ and $\eta^c I_{(0,\infty)}(\eta)$, respectively. Then the joint distribution $g(\eta, x, s)$ of η, X, S is given by

$$\begin{aligned} g(\eta, x, s) &\propto \int \eta^{p/2} \exp\left(-\frac{\eta}{2}\|x - \theta\|^2\right) \left(\frac{\eta\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\frac{\lambda}{1-\lambda}\frac{\eta}{2}\|\theta\|^2\right) \\ &\quad \cdot \eta^c \lambda^a (1-\lambda)^b \eta^{n/2} \exp(-\eta s/2) d\theta d\lambda \\ &\propto \int \eta^{p/2} \left(\frac{\eta\lambda}{1-\lambda}\right)^{p/2} \exp\left(-\eta \frac{\|\theta - (1-\lambda)x\|^2}{2(1-\lambda)} - \frac{\eta\|x\|^2\lambda}{2}\right) \\ &\quad \cdot \eta^c \lambda^a (1-\lambda)^b \eta^{n/2} \exp(-\eta s/2) d\theta d\lambda \\ &\propto \eta^{(p+n)/2+c} \int_0^1 \lambda^{p/2+a} (1-\lambda)^b \exp\left(-\eta \frac{\|x\|^2\lambda + s}{2}\right) d\lambda. \end{aligned}$$

As the generalized Bayes estimator under L_2 loss is written as $E(\eta \mid X, S)/E(\eta^2 \mid X, S) = \int \eta g(\eta, x, s) d\eta / \int \eta^2 g(\eta, x, s) d\eta$, we have the estimator

$$\begin{aligned} \delta^{GB} &= \frac{\int_0^1 \lambda^{p/2+a} (1-\lambda)^b \int_0^\infty \eta^{(p+n)/2+c+1} \exp\left(-\eta \frac{\|X\|^2\lambda + S}{2}\right) d\eta d\lambda}{\int_0^1 \lambda^{p/2+a} (1-\lambda)^b \int_0^\infty \eta^{(p+n)/2+c+2} \exp\left(-\eta \frac{\|X\|^2\lambda + S}{2}\right) d\eta d\lambda} \\ &= \frac{1}{p+n+2(c+2)} \frac{\int_0^1 \lambda^{p/2+a} (1-\lambda)^b (1+\lambda W)^{-(n+p)/2-c-2} d\lambda}{\int_0^1 \lambda^{p/2+a} (1-\lambda)^b (1+\lambda W)^{-(n+p)/2-c-3} d\lambda} S \\ &= \frac{1}{p+n+2(c+2)} \frac{\int_0^1 t^{p/2+a} (1-t)^b \{1-tW/(W+1)\}^{n/2-a-b+c} dt}{\int_0^1 t^{p/2+a} (1-t)^b \{1-tW/(W+1)\}^{n/2-a-b+c+1} dt} S \\ &= \varphi^{GB}(W)S, \text{ (say)} \end{aligned} \quad (7)$$

which is well-defined if $a > -p/2 - 1$ and $b > -1$. In the following, as we have $\lim_{w \rightarrow \infty} \varphi^{GB}(w) = 1/\{n + 2 + 2(c - a)\}$ from (7), we only consider the case $a = c$, that is,

$$\varphi^{GB}(w) = \frac{1}{p + n + 2(a + 2)} \frac{\int_0^1 t^{p/2+a} (1-t)^b \{1 - tW/(W+1)\}^{n/2-b} dt}{\int_0^1 t^{p/2+a} (1-t)^b \{1 - tW/(W+1)\}^{n/2-b+1} dt}. \quad (8)$$

In particular, we have a simple form

$$\varphi^{GB}(w) = \left(n + 2 + \frac{p + 2a + 2}{w + 1} \right)^{-1} \quad (9)$$

by letting $b = n/2$ in (8). This is unexpected because generalized Bayes minimax shrinkage estimators such as (2) and (3) typically have a complicated form. The estimator (9) has a form which is comparable to Stein's estimator δ^S in its simplicity. We will show, in Proposition 3.5 below, that $\psi^{GB}(W)S$ given by (9) is minimax for certain a .

Some basic properties of behavior of φ^{GB} given by (9) are given in the following result.

Lemma 3.1. 1. $\varphi^{GB}(w)$ is increasing in w for $b \geq 0$.

2. φ^{GB} is decreasing in a for fixed $b \geq 0$ and w .

Since δ^{GB} for $a = -1$ and $b = 0$ corresponds to δ^{BZ} , the BZ-condition together with Lemma 3.1 implies that δ^{GB} for $-p/2 - 1 < a \leq -1$ and $b = 0$, which is equal to Ghosh's (1994) estimator, is minimax.

Proof. By the change of variables in (8), we have

$$\varphi^{GB}(w) = \frac{1}{p + n + 2(a + 2)} \frac{\int_0^v t^{p/2+a} (v-t)^b (1-t)^{n/2-b} dt}{\int_0^v t^{p/2+a} (v-t)^b (1-t)^{n/2-b+1} dt},$$

where $v = w/(w+1)$. For $v_1 > v_2$ and $b \geq 0$,

$$\begin{aligned} \frac{\int_0^{v_1} t^{p/2+a} (v_1-t)^b (1-t)^{n/2-b} dt}{\int_0^{v_1} t^{p/2+a} (v_1-t)^b (1-t)^{n/2-b+1} dt} &\geq \frac{\int_0^{v_2} t^{p/2+a} (v_1-t)^b (1-t)^{n/2-b} dt}{\int_0^{v_2} t^{p/2+a} (v_1-t)^b (1-t)^{n/2-b+1} dt} \\ &\geq \frac{\int_0^{v_2} t^{p/2+a} (v_2-t)^b (1-t)^{n/2-b} dt}{\int_0^{v_2} t^{p/2+a} (v_2-t)^b (1-t)^{n/2-b+1} dt}. \end{aligned}$$

The first inequality follows from the fact that the ratio of integrands of the numerator and the denominator is increasing, the second inequality from the fact that $\{(v_1 - t)/(v_2 - t)\}^b$ is increasing. This completes the proof of (i).

By the change of variables ($u = \eta s$, $t = u\lambda$), in the first equality in (7), we have

$$\varphi^{GB}(w) = \frac{\int_0^\infty u^{n/2} \exp(-u/2) h_w(u) du}{\int_0^\infty u^{n/2+1} \exp(-u/2) h_w(u) du},$$

where $h_w(u, a) = \int_0^u t^{p/2+a} (1-t/u)^b \exp(-wt/2) dt$. Hence, to prove (ii), it is sufficient to show that $h_w(u, a_1)/h_w(u, a_2)$ is increasing in u for $a_1 > a_2$. As in the proof of (i), we see that for

$u_1 > u_2$ and $b \geq 0$

$$\begin{aligned} \frac{\int_0^{u_1} t^{p/2+a_1}(u_1-t)^b \exp(-wt/2) dt}{\int_0^{u_1} t^{p/2+a_2}(u_1-t)^b \exp(-wt/2) dt} &\geq \frac{\int_0^{u_2} t^{p/2+a_1}(u_1-t)^b \exp(-wt/2) dt}{\int_0^{u_2} t^{p/2+a_2}(u_1-t)^b \exp(-wt/2) dt} \\ &\geq \frac{\int_0^{u_2} t^{p/2+a_1}(u_2-t)^b \exp(-wt/2) dt}{\int_0^{u_2} t^{p/2+a_2}(u_2-t)^b \exp(-wt/2) dt}, \end{aligned}$$

which completes the proof of (ii). $\square$

Note: Since the derivative of $\{(1-vt)/t\}^{n/2+1}$ is $(-n/2-1)\{(1-vt)/t\}^{n/2}t^{-2}$, an integration by parts in (8) gives $\phi^{GB}(w) = (n+2)^{-1}(1-\phi^{GB}(w))$ where

$$\begin{aligned} \phi^{GB}(w) &= \frac{2b(1-v)}{p+n+2(a+2)} \frac{\int_0^1 t^{p/2+a+1}(1-t)^{b-1}(1-vt)^{n/2-b} dt}{\int_0^1 t^{p/2+a}(1-t)^b(1-vt)^{n/2-b+1} dt} \text{ for } b > 0 \\ &= \frac{2(1-v)^{n/2+1}}{p+n+2(a+2)} \frac{1}{\int_0^1 t^{p/2+a}(1-vt)^{n/2+1} dt} \text{ for } b = 0. \end{aligned}$$

Since $v = w/(1+w)$ and all integrals above approach constant values, we have

$$\phi^{GB}(w) = \begin{cases} O\{(w+1)^{-n/2-1}\} & \text{for } b = 0 \\ O\{(w+1)^{-1}\} & \text{for } b > 0. \end{cases}$$

Since $\phi^{BZ}(w)$ is $\phi^{GB}(w)$ for $a = -1$ and $b = 0$, $\phi^{BZ}(w) = O\{(w+1)^{-n/2-1}\}$. This implies that $\phi^{GB}(w)$ for $b > 0$ is greater than $\phi^{BZ}(w)$ for sufficiently large w . Thus we have the following result.

Theorem 3.2. $\phi^{GB}(w)$ for $b > 0$ does not satisfy (BZ2) of the BZ-condition.

Next we investigate the properties of ϕ^{GB} in order to apply the ST-condition proposed in Theorem 2.1.

Theorem 3.3. 1. $\phi^{GB}(w) \leq (p+2a+2)/(p+n+2a+4)$.

2. $(1+w)^\epsilon \phi^{GB}(w)$ is monotone nonincreasing if

(a) $b = 0$ and $a < -p/2 - 2 + (n/2 + 1)/\epsilon$ or

(b) $0 < b \leq n/2 + 1$, $\epsilon \leq 1$ and $a < -p/2 - b - 2 + (n/2 + 1)/\epsilon$.

(c) $b > n/2 + 1$ and $a < -p/2 - b - 2 + b(b - n/2)/(\epsilon + b - n/2 - 1)$.

Proof. By Theorem 3.1 $\phi^{GB}(w)$ is decreasing in w and hence $\phi^{GB}(w) \leq \phi^{GB}(0) = (p+2a+2)/(p+n+2a+4)$, which completes the proof of (i).

For $b = 0$, The derivative of $(1+w)^\epsilon \phi(w)$ with respect to $v = w/(w+1)$ is written as

$$(1-v)^{-\epsilon-1} \phi(w) \left[-n/2 - 1 + \epsilon + (n/2 + 1)(1-v) \frac{\int_0^1 t^{p/2+a+1}(1-vt)^{n/2} dt}{\int_0^1 t^{p/2+a}(1-vt)^{n/2+1} dt} \right]. \quad (10)$$

Using the relation

$$\int_0^1 t^\alpha (1-t)^\beta (1-vt)^\gamma dt = (1-v)^{\beta+\gamma+1} \int_0^1 t^\beta (1-t)^\alpha (1-vt)^{-\alpha-\beta-\gamma-2} dt \quad (11)$$

for $\alpha > -1$ and $\beta > -1$, the term in bracket in (10) is written as

$$-n/2 - 1 + \epsilon + (n/2 + 1) \frac{\int_0^1 (1-t)^{p/2+a+1} (1-vt)^{-p/2-n/2-a-3} dt}{\int_0^1 (1-t)^{p/2+a} (1-vt)^{-p/2-n/2-a-3} dt}$$

which is less than $-n/2 - 1 + \epsilon + (n/2 + 1)(p/2 + a + 1)/(p/2 + a + 2) = \epsilon - (n/2 + 1)/(p/2 + a + 2)$ because $(1-vt)^{-1}$ is increasing in t . This completes the proof in the case $b = 0$.

For $b > 0$, the derivative of $(1+w)^\epsilon \phi(w)$ with respect to $v = w/(w+1)$, together with the relation (11) is written as

$$(1-v)^{-\epsilon-1} \phi(w) \left[\epsilon - 1 + (b - n/2) \frac{\int_0^1 t^{b-1} (1-t)^{p/2+a+2} (1-vt)^{-p/2-n/2-a-2} dt}{\int_0^1 t^{b-1} (1-t)^{p/2+a+1} (1-vt)^{-p/2-n/2-a-2} dt} + (n/2 + 1 - b) \frac{\int_0^1 t^b (1-t)^{p/2+a+1} (1-vt)^{-p/2-n/2-a-3} dt}{\int_0^1 t^b (1-t)^{p/2+a} (1-vt)^{-p/2-n/2-a-3} dt} \right]. \quad (12)$$

By applying a Maclaurin expansion to the integrals in (12), the term in bracket in (12) is written as

$$\epsilon - 1 + \frac{(b - n/2)(p/2 + a + 2)}{p/2 + a + b + 2} \frac{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 3, v)}{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 2, v)} - \frac{(b - n/2 - 1)(p/2 + a + 1)}{p/2 + a + b + 2} \frac{F(p/2 + n/2 + a + 3, b + 1, p/2 + a + b + 3, v)}{F(p/2 + n/2 + a + 3, b + 1, p/2 + a + b + 2, v)}, \quad (13)$$

where $F(a, b, c, x)$ is the hypergeometric function

$$F(a, b, c, x) = 1 + \sum_{i=1}^{\infty} \frac{(a)_i (b)_i}{(c)_i} \frac{x^i}{i!} \quad \text{for } (a)_i = a \cdot (a+1) \cdots (a+i-1).$$

From the inequality

$$\frac{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 3, v)}{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 2, v)} \geq \frac{F(p/2 + n/2 + a + 3, b + 1, p/2 + a + b + 3, v)}{F(p/2 + n/2 + a + 3, b + 1, p/2 + a + b + 2, v)},$$

(13) for $b \leq n/2 + 1$ is less than

$$\epsilon - 1 + \frac{p/2 + a + 1 - n/2 + b}{p/2 + a + b + 2} \frac{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 3, v)}{F(p/2 + n/2 + a + 2, b, p/2 + a + b + 2, v)} \leq \epsilon - \min \left(1, \frac{n/2 + 1}{p/2 + a + b + 2} \right),$$

which is nonpositive when $\epsilon \leq 1$ and $a < -p/2 - b - 2 + (n/2 + 1)/\epsilon$. This completes the proof in the case $0 < b \leq n/2 + 1$.

For $b > n/2 + 1$, (13) is less than

$$\epsilon - 1 + \frac{(b - n/2)(p/2 + a + 2)}{p/2 + a + b + 2},$$

which is nonpositive when $a < -p/2 - b - 2 + b(b - n/2)/(\epsilon + b - n/2 - 1)$. This completes the proof. $\square$

Combining Theorem 3.3 and Theorem 2.1, we have the following result.

Theorem 3.4. *The generalized Bayes estimator δ^{GB} given by (4) is minimax under L_2 loss if*

$$-p/2 - 1 < a < -p/2 - 1 + \max_{\epsilon} \min \left((n/2 + 1) \frac{B_2(p, n, \epsilon)}{1 - B_2(p, n, \epsilon)}, C_2(n, \epsilon, b) \right) \quad (14)$$

where

$$C_2(n, \epsilon, b) = \begin{cases} (n/2 + 1)/\epsilon - 1 & b = 0 \\ -b - 1 + \max(n/2 + 1, (n/2 + 1)/\epsilon) & 0 < b \leq n/2 + 1 \\ -b - 1 + b(b - n/2)/(\epsilon + b - n/2 - 1) & b > n/2 + 1. \end{cases}$$

Figure 1 reveals the upper bounds for minimaxity in the case $p = 10$. Note that the upper bound given by (14) is not always continuous in $b = 0$.

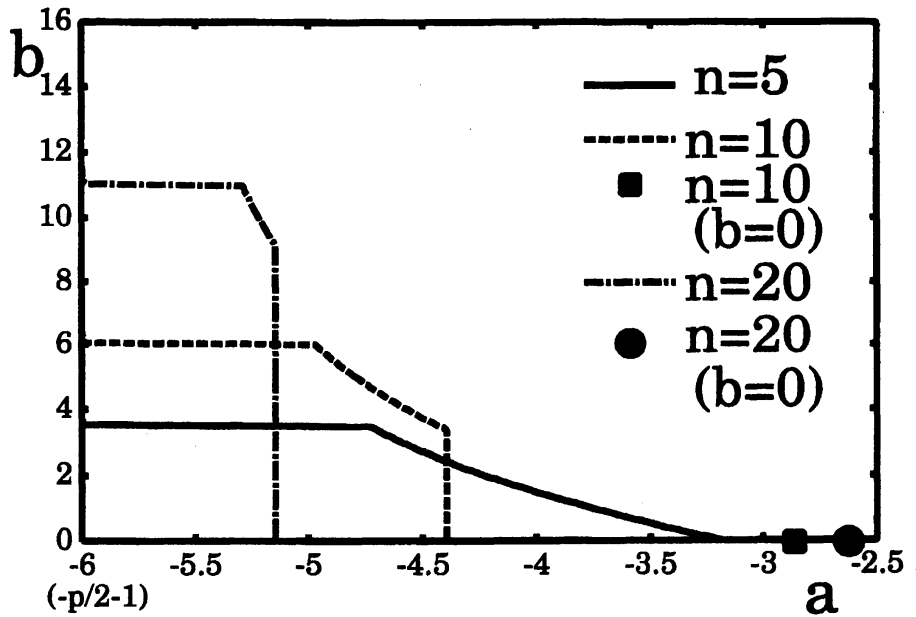


Figure 1 : Ranges of values for minimaxity in the case $p = 10$ under L_2 loss

As noted earlier, when $b = 0$ ϕ^{GB} satisfies the BZ-condition for $-p/2 - 1 < a \leq -1$ but does not satisfy the ST-condition when $a = -1$ (i.e. $\delta^{GB} = \delta^{BZ}$). Our contribution when $b = 0$ is to add an explicit upper bound on a so that the ST-condition is satisfied. This upper bound is of course less than -1 so that when $b = 0$. The class of estimators satisfying the BZ-condition contains the class satisfying the ST-condition. However when $b > 0$ the class of estimators δ^{GB} satisfying the BZ-condition is empty while the class satisfying the ST-condition is non-empty and is in fact quite rich.

Finally we highlight the following very simple case.

Proposition 3.5. *the generalized Bayes estimator*

$$\frac{1}{n+2} \left(1 + \frac{\alpha}{W+1} \right)^{-1} S$$

is minimax under L_2 if $0 < \alpha \leq \max_{\epsilon} \min (B_2(p, n, \epsilon)(1 - B_2(p, n, \epsilon))^{-1}, 1/\epsilon - 1)$.

Proof. Apply Theorem 3.4 to the estimator given in (9). □

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