

## A Localization of a Semigroup Ring, II

松田 隆輝 (Ryûki Matsuda)

茨城大学理学部 (Faculty of Science, Ibaraki University)

This is a continuation of our [M4]. Thus a submonoid  $S$  of a torsion-free abelian (additive) group is called a  $g$ -monoid. For a  $g$ -monoid  $S$ , the quotient group of  $S$  is denoted by  $q(S)$ , and for a commutative ring  $R$ , the total quotient ring of  $R$  is denoted by  $q(R)$ . Throughout the paper  $S$  denotes a  $g$ -monoid which is not  $\{0\}$ .

Let  $F(S)$  be the set of fractional ideals of the  $g$ -monoid  $S$ . A mapping  $I \mapsto I^*$  of  $F(S)$  to  $F(S)$  is called a star-operation on  $S$  if the following conditions hold for every element  $a \in q(S)$  and  $I, J \in F(S)$ :

$$(a)^* = (a); (a + I)^* = a + I^*; I \subset I^*;$$

$$\text{If } I \subset J, \text{ then } I^* \subset J^*; (I^*)^* = I^*.$$

Let  $*$  be a star-operation on  $S$ . If, for all finitely generated fractional ideals  $J_1, J_2$  and  $I$ ,  $(I + J_1)^* \subset (I + J_2)^*$  implies  $J_1^* \subset J_2^*$ , then  $*$  is called an e.a.b. star-operation on  $S$ .

Let  $F'(S)$  be the set of non-empty subsets of  $q(S)$  such that  $S + I \subset I$ . A mapping  $I \mapsto I^*$  of  $F'(S)$  to  $F'(S)$  is called a semistar-operation on  $S$  if the following conditions hold for every element  $a \in q(S)$  and  $I, J \in F'(S)$ :

$$(a + I)^* = a + I^*; I \subset I^*;$$

$$\text{If } I \subset J, \text{ then } I^* \subset J^*; (I^*)^* = I^*.$$

Let  $*$  be a semistar-operation on  $S$ . If, for all finitely generated fractional ideals  $J_1, J_2$  and  $I$ ,  $(I + J_1)^* \subset (I + J_2)^*$  implies  $J_1^* \subset J_2^*$ , then  $*$  is called an e.a.b. semistar-operation on  $S$ .

Let  $R$  be a commutative ring. A non-zero-divisor of  $R$  is called a regular element of  $R$ . If an ideal  $I$  of  $R$  contains at least one regular element, then  $I$  is called a regular ideal of  $R$ . If every regular ideal is generated by regular elements, then  $R$  is called a Marot ring. If, for every regular element  $f$  of the polynomial ring  $R[X]$ , the ideal of  $R$  generated by the coefficients of  $f$  is a regular ideal of  $R$ , then  $R$  is said to have property (A).

Let  $I$  be an  $R$ -submodule of  $q(R)$  such that  $rI \subset R$  for some regular  $r \in R$ . Then  $I$  is called a fractional ideal of  $R$ . Let  $F(R)$  be the set of non-zero fractional ideals of  $R$ . A mapping  $I \mapsto I^*$  of  $F(R)$  to  $F(R)$  is called a

star-operation on  $R$  if the following conditions hold for every regular element  $a \in q(R)$  and  $I, J \in F(R)$ :

- (1)  $(a)^* = (a)$ ; (2)  $(aI)^* = aI^*$ ; (3)  $I \subset I^*$ ;
- (4) If  $I \subset J$ , then  $I^* \subset J^*$ ; (5)  $(I^*)^* = I^*$ .

Let  $*$  be a star-operation on  $R$ . If, for all finitely generated non-zero fractional ideals  $J_1, J_2, I$  with  $I$  regular,  $(IJ_1)^* \subset (IJ_2)^*$  implies  $J_1^* \subset J_2^*$ , then  $*$  is called an e.a.b. star-operation on  $R$ .

Let  $F'(R)$  be the set of non-zero  $R$ -submodules of  $q(R)$ . A mapping  $I \longmapsto I^*$  of  $F'(R)$  to  $F'(R)$  is called a semistar-operation on  $R$  if the following conditions hold for every regular element  $a \in q(R)$  and  $I, J \in F'(R)$ :

- (1)  $(aI)^* = aI^*$ ; (2)  $I \subset I^*$ ;
- (3) If  $I \subset J$ , then  $I^* \subset J^*$ ; (4)  $(I^*)^* = I^*$ .

Let  $*$  be a semistar-operation on  $R$ . If, for all finitely generated non-zero fractional ideals  $J_1, J_2, I$  with  $I$  regular,  $(IJ_1)^* \subset (IJ_2)^*$  implies  $J_1^* \subset J_2^*$ , then  $*$  is called an e.a.b. semistar-operation on  $R$ .

Let  $f = \sum_1^n a_i X^{s_i} \in R[X; S]$ , where  $s_i \neq s_j$  for  $i \neq j$ , and  $a_i \neq 0$  for each  $i$ . Then the ideal  $(s_1, \dots, s_n)$  of  $S$  is denoted by  $e(f)$ , and the ideal  $(a_1, \dots, a_n)$  of  $R$  is denoted by  $c(f)$ .

**Proposition 1.** Let  $*$  be a star-operation on a domain  $D$ . The following conditions are equivalent:

- (1)  $*$  is e.a.b.
- (2) If  $f/g = f'/g'$ , where  $f, g, f', g' \in D[X]$  with  $g, g'$  non-zero, and if  $c(f)^* \subset c(g)^*$ , then  $c(f')^* \subset c(g')^*$ .

*Proof.* Assume that, if  $f/g = f'/g'$ , where  $f, g, f', g' \in D[X]$  with  $g, g'$  non-zero, and if  $c(f)^* \subset c(g)^*$ , then  $c(f')^* \subset c(g')^*$ . Let  $I, J_1, J_2$  be finitely generated non-zero fractional ideals of  $D$ , and assume that  $(IJ_1)^* \subset (IJ_2)^*$ . We may assume that  $I, J_1, J_2$  are ideals of  $D$ . Let  $I = (a_0, \dots, a_n)$ ,  $J_1 = (b_0, \dots, b_m)$  and  $J_2 = (c_0, \dots, c_l)$ . Put  $f = \sum a_i X^i$ ,  $g = \sum b_i X^{i(n+1)}$  and  $h = \sum c_i X^{i(n+1)}$ . Then  $c(fg) = IJ_1$ ,  $c(fh) = IJ_2$ . Since,  $(fg)/(fh) = g/h$  and  $c(fg)^* \subset c(fh)^*$ , we have  $c(g)^* \subset c(h)^*$ . That is,  $J_1^* \subset J_2^*$ . Hence  $*$  is e.a.b.

In the following, let  $D$  be a domain, and let  $A$  be a Marot ring with property

**Theorem 1.** Let  $*$  be a star-operation on  $A$ . The following conditions are equivalent:

- (i)  $*$  is e.a.b.
  - (ii) If  $f/g = f'/g'$ , where  $f, g, f', g' \in A[X; S]$  with  $g, g'$  regular, and if  $c(f)^* \subset c(g)^*$ , then  $c(f')^* \subset c(g')^*$ .
- (2) Let  $*$  be a star-operation on  $S$ . The following conditions are equivalent:
- (i)  $*$  is e.a.b.
  - (ii) If  $f/g = f'/g'$ , where  $f, g, f', g' \in D[X; S]$  with  $g, g'$  non-zero, and if  $e(f)^* \subset e(g)^*$ , then  $e(f')^* \subset e(g')^*$ .
- (3) Let  $*$  be a semistar-operation on  $A$ . The following conditions are equivalent:
- (i)  $*$  is e.a.b.
  - (ii) If  $f/g = f'/g'$ , where  $f, g, f', g' \in A[X; S]$  with  $g, g'$  regular, and if  $c(f)^* \subset c(g)^*$ , then  $c(f')^* \subset c(g')^*$ .
- (4) Let  $*$  be a semistar-operation on  $S$ . The following conditions are equivalent:
- (i)  $*$  is e.a.b.
  - (ii) If  $f/g = f'/g'$ , where  $f, g, f', g' \in D[X; S]$  with  $g, g'$  non-zero, and if  $e(f)^* \subset e(g)^*$ , then  $e(f')^* \subset e(g')^*$ .

The proof of Theorem 1 is similar to that of Proposition 1.

Let  $R$  be a commutative ring. If every finitely generated regular ideal of  $R$  is principal,  $R$  is called an r-Bézout ring. If every finitely generated regular ideal of  $R$  is invertible,  $R$  is called a Prüfer ring. A multiplicative system of  $R$  consisting of regular elements is called a regular multiplicative system of  $R$ , and a quotient ring of  $R$  with respect to a regular multiplicative system is called a regular quotient ring of  $R$ .

Let  $*$  be a star-operation (resp. semistar-operation) on a g-monoid  $S$ . If the set  $\{I^* \mid I \text{ is a finitely generated fractional ideal of } S\}$  is a group under the sum  $(I_1^*, I_2^*) \mapsto (I_1^* + I_2^*)^*$ , then  $S$  is called a Prüfer  $*$ -multiplication semigroup. Assume that  $*$  is an e.a.b. star-operation (resp. semistar-operation) on  $S$ , let  $D$  be a domain. Then the ring  $S_*^D = \{f/g \mid f, g \in D[X; S] - \{0\} \text{ with}$

$e(f)^* \subset e(g)^*\} \cup \{0\}$  is called the Kronecker function ring of  $S$  with respect to  $*$  and  $D$ .

Let  $*$  be a star-operation (resp. semistar-operation) on  $R$ . If the set  $\{I^* \mid I \text{ is a finitely generated regular fractional ideal of } R\}$  is a group under the product  $(I_1^*, I_2^*) \mapsto (I_1^* I_2^*)^*$ , then  $R$  is called a Prüfer  $*$ -multiplication ring. Assume that  $*$  is an e.a.b. star-operation (resp. semistar-operation) on  $A$ . Then the ring  $A_*^S = \{f/g \mid f, g \in A[X; S] - \{0\} \text{ with } g \text{ regular and } c(f)^* \subset c(g)^*\} \cup \{0\}$  is called the Kronecker function ring of  $A$  with respect to  $*$  and  $S$ .

Let  $P$  be a prime ideal of  $R$ . The overring  $\{x \in L \mid sx \in R \text{ for some } s \in R - P\}$  of  $R$  is denoted by  $R_{[P]}$ .

**Theorem 2.** Let  $*$  be an e.a.b. star-operation on  $A$ , and let  $T = \{g \mid g \text{ is a regular element of } A[X; S] \text{ with } c(g)^* = A\}$ . Then the following conditions are equivalent:

- (0)  $A[X; S]_T$  is a Prüfer ring.
- (1)  $A$  is a Prüfer  $*$ -multiplication ring.
- (2)  $A[X; S]_T = A_*^S$ .
- (3)  $A[X; S]_T$  is an r-Bezout ring.
- (4) Each regular prime ideal of  $A[X; S]_T$  is the extension of a prime ideal of  $A$ .
- (5)  $A_*^S$  is a regular quotient ring of  $A[X; S]$ .
- (6) Each prime ideal of  $A[X; S]_T$  is the contraction of a prime ideal of  $A_*^S$ .
- (7) Each regular prime ideal of  $A[X; S]_T$  is the contraction of a prime ideal of  $A_*^S$ .
- (8) Each valuation overring of  $A_*^S$  is of the form  $A[X; S]_{[PA[X; S]]}$ , where  $P$  is a prime ideal of  $A$  such that  $A_{[P]}$  is a valuation overring of  $A$ .
- (9)  $A_*^S$  is a flat  $A[X; S]$ -module.

Moreover, there exists a Prüfer Marot ring  $A$  with property (A) which satisfies the following conditions: Let  $*$  be any e.a.b.  $*$ -operation on  $A$ . Then there exists a prime ideal of  $A[X; \mathbb{Z}_0]_T$  which is not the extension of a prime ideal of  $A$ , where  $\mathbb{Z}_0$  is the g-monoid of non-negative integers.

For the proof of equivalence of (0)  $\sim$  (9) we confer [M3, Propositions 3.1 and 3.9 and Theorem 3.7]. Let  $k$  be a field, let  $X_1, X_2, \dots, Y_1, Y_2, \dots$  be indeter-

minates, and let  $D_0$  be a Prüfer domain. Let  $R = k[[X_1, X_2, \dots, Y_1, Y_2, \dots]]_1 / (X_i X_j, Y_i Y_j \mid i \neq j)$ , and let  $A = R \oplus D_0$ , where  $k[[X_1, X_2, \dots, Y_1, Y_2, \dots]]_1 = \bigcup_{n=1}^{\infty} k[[X_1, X_2, \dots, X_n, Y_1, Y_2, \dots, Y_n]]$ , and  $(X_i X_j, Y_i Y_j \mid i \neq j)$  is the ideal of  $k[[X_1, X_2, \dots, Y_1, Y_2, \dots]]_1$  generated by the subset  $\{X_i X_j, Y_i Y_j \mid i \neq j\}$ . Then  $A$  is such a ring (cf. [M1, Theorem (1.3)]).

A similar result to Theorem 2 holds for semistar-operations on the ring  $A$  as follows.

**Theorem 3.** Let  $*$  be an e.a.b. semistar-operation on  $A$ , and let  $W = \{g \mid g \text{ is a regular element of } A^*[S] \text{ such that } c(g)^* = A^*\}$ . Then the following conditions are equivalent:

- (0)  $A^*[X; S]_W$  is a Prüfer ring.
- (1)  $A$  is a Prüfer  $*$ -multiplication ring.
- (2)  $A^*[X; S]_W$  coincides with the Kronecker function ring  $A_*^S$  of  $A$  with respect to  $*$  and  $S$ .
- (3)  $A^*[X; S]_W$  is an r-Bezout ring.
- (4) Each regular prime ideal of  $A^*[X; S]_W$  is the extension of a prime ideal of  $A^*$ .
- (5)  $A_*^S$  is a regular quotient ring of  $A^*[X; S]$ .
- (6) Each prime ideal of  $A^*[X; S]_W$  is the contraction of a prime ideal of  $A_*^S$ .
- (7) Each regular prime ideal of  $A^*[X; S]_W$  is the contraction of a prime ideal of  $A_*^S$ .
- (8) Each valuation overring of  $A_*^S$  is of the form  $A^*[X; S]_{[Q]A^*[X; S]_W}$ , where  $Q$  is a prime ideal of  $A^*$  such that  $(A^*)_{[Q]}$  is a valuation overring of  $A^*$ .
- (9)  $A_*^S$  is a flat  $A^*[X; S]$ -module.

For the proof we confer [M3, Propositions 3.2, 3.8 and 3.9].

**Theorem 4.** Let  $D$  be a domain, and let  $*$  be an e.a.b. star-operation on a g-monoid  $S$ , and let  $T = \{g \mid g \text{ is a non-zero element of } D[X; S] \text{ with } e(g)^* = S\}$ . The following conditions are equivalent:

- (0)  $D[X; S]_T$  is a Prüfer ring.
- (1)  $S$  is a Prüfer  $*$ -multiplication semigroup.
- (2)  $D[X; S]_T$  coincides with the Kronecker function ring  $S_*^D$  of  $S$  with respect

to  $*$  and  $D$ .

- (3)  $D[X; S]_T$  is a Bezout ring.
- (4) Each prime ideal of  $D[X; S]_T$  is the extension of a prime ideal of  $S$ .
- (5)  $S_*^D$  is a quotient ring of  $D[X; S]$ .
- (6) Each prime ideal of  $D[X; S]_T$  is the contraction of a prime ideal of  $S_*^D$ .
- (7) Each valuation overring of  $S_*^D$  is of the form  $D[X; S]_{PD[X; S]}$ , where  $P$  is a prime ideal of  $S$  such that  $S_P$  is a valuation oversemigroup of  $S$ .
- (8)  $S_*^D$  is a flat  $D[X; S]$ -module.

For the proof we confer [MS, Theorems 8 and 25].

A similar result to Theorem 4 holds for semistar-operations on  $S$ .

**Theorem 5.** Let  $*$  be an e.a.b. semistar-operation on  $S$ , and let  $W = \{g \mid g \text{ is a non-zero element of } D[X; S^*] \text{ such that } e(g)^* = S^*\}$ . The following conditions are equivalent:

- (0)  $D[X; S^*]_W$  is a Prüfer ring.
- (1)  $S$  is a Prüfer  $*$ -multiplication semigroup.
- (2)  $D[X; S^*]_W = S_*^D$ .
- (3)  $D[X; S^*]_W$  is a Bezout ring.
- (4) Each prime ideal of  $D[X; S^*]_W$  is the extension of a prime ideal of  $S^*$ .
- (5)  $S_*^D$  is a quotient ring of  $D[X; S^*]$ .
- (6) Each prime ideal of  $D[X; S^*]_W$  is the contraction of a prime ideal of  $S_*^D$ .
- (7) Each valuation overring of  $S_*^D$  is of the form  $D[X; S^*]_{QD[X; S^*]}$ , where  $Q$  is a prime ideal of  $S^*$  such that  $(S^*)_Q$  is a valuation oversemigroup of  $S^*$ .
- (8)  $S_*^D$  is a flat  $D[X; S^*]$ -module.

For the proof we confer [M2, Proposition 4 and Theorem 23].

## Appendix

**Theorem.** Let  $S$  be a g-monoid, and let  $T$  be an extension semigroup. If  $T$  is a Noetherian semigroup, and if  $T$  is a finitely generated  $S$ -module, then  $S$  is a Noetherian semigroup.

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