UNIMODULAR FOURIER MULTIPLIERS ON MODULATION SPACES $M^{p,q}$ FOR $0 < p < 1$

NAOHITO TOMITA

1. Introduction

In this note, we consider the boundedness of the Fourier multiplier operator $e^{i|D|^\alpha}$ on modulation spaces, where $\alpha > 0$ and $e^{i|D|^\alpha}$ is defined by

$$e^{i|D|^\alpha} f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \, e^{i|\xi|^\alpha} \, \widehat{f}(\xi) \, d\xi.$$

In the case $\alpha = 2$, $u(t,x) = e^{it|D|^2} u_0(x)$ is the formal solution to the Schrödinger equation

$$\begin{cases} i\dfrac{\partial u}{\partial t}(t,x) = \Delta_x u(t,x) & (t > 0, \ x \in \mathbb{R}^n), \\ u(0,x) = u_0(x) & (x \in \mathbb{R}^n). \end{cases}$$

Modulation spaces $M_s^{p,q}$ were introduced by Feichtinger [3, 4] (see also Gröchenig [5]). We recall the definition of modulation spaces. Let $0 < p, q \le \infty$, $s \in \mathbb{R}$, and let $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that

$$\operatorname{supp} \psi \subset [-1,1]^n \qquad \text{and} \qquad \sum_{k\in\mathbb{Z}^n} \psi(\xi - k) = 1 \quad \text{for all } \xi \in \mathbb{R}^n. \tag{1.1}$$

Then the modulation space $M_s^{p,q}(\mathbb{R}^n)$ consists of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\|f\|_{M_s^{p,q}} = \left(\sum_{k\in\mathbb{Z}^n} (1+|k|)^{sq} \|\psi(D-k)f\|_{L^p}^q \right)^{1/q} < \infty,$$

where $\psi(D-k)f = \mathcal{F}^{-1}[\psi(\cdot - k)\,\widehat{f}]$. If $s = 0$, we simply write $M^{p,q}(\mathbb{R}^n)$ instead of $M_0^{p,q}(\mathbb{R}^n)$. We remark that $M_s^{2,2}$ coincides with the Sobolev space $W^{s,2}$.

It is known that $e^{i|D|^2}$ is bounded on L^p if and only if $p = 2$ (Hörmander [7]). However, $e^{i|D|^2}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \le p, q \le \infty$ (see Gröchenig-Heil [6], Toft [10], Wang-Zhao-Guo [11], Bényi-Gröchenig-Okoudjou-Rogers [1]). This is one of differences between L^p-spaces and modulation spaces. Bényi-Gröchenig-Okoudjou-Rogers ([1]) proved that if $0 \le \alpha \le 2$ then $e^{i|D|^\alpha}$ is bounded on $M^{p,q}(\mathbb{R}^n)$ for all $1 \le p, q \le \infty$. Furthermore, in the case $\alpha > 2$, Miyachi-Nicola-Rivetti-Tabacco-Tomita [9] showed that, for $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \ge (\alpha-2)n|1/p - 1/2|$ (see [9] for more general results). In particular, this says that if $\alpha > 2$ and $p \ne 2$ then $e^{i|D|^\alpha}$ is not bounded on modulation spaces $M^{p,q}$.

The purpose of this note is to consider the case $0 < p < 1$, and our main result is the following:

Theorem 1.1. *Let $0 < p < 1$, $0 < q \le \infty$, $\alpha > n(1/p - 1)$ and $s \in \mathbb{R}$. Then $e^{i|D|^\alpha}$ is bounded from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \ge \max\{0, \alpha - 2\} n(1/p - 1/2)$.*

We remark that Bényi-Okoudjou [2] considered the cases $0 \le \alpha \le 2$, $1 \le p \le \infty$ and $0 < q \le \infty$, and $\alpha \in \{1, 2\}$, $n/(n+1) < p \le \infty$ and $0 < q \le \infty$. In Remark 3.5, we also treat the case $\alpha \ge 0$, $1 \le p \le \infty$ and $0 < q \le \infty$.

We end this section by explaining the organization of this note. In Section 2, we give the relation between L^p-boundedness and $M^{p,q}$-boundedness. In Section 3, we give the proof of Theorem 1.1.

2. Relation between L^p-boundedness and $M^{p,q}$-boundedness

Let $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ be the Schwartz spaces of all rapidly decreasing smooth functions and tempered distributions, respectively. We define the Fourier transform $\mathcal{F}f$ and the inverse Fourier transform $\mathcal{F}^{-1}f$ of $f \in \mathcal{S}(\mathbb{R}^n)$ by

$$\mathcal{F}f(\xi) = \widehat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\, f(x)\, dx \quad \text{and} \quad \mathcal{F}^{-1}f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\, f(\xi)\, d\xi.$$

For $m \in \mathcal{S}'(\mathbb{R}^n)$, we define the Fourier multiplier operator $m(D)$ by

$$m(D)f = \mathcal{F}^{-1}[m\,\widehat{f}] = [\mathcal{F}^{-1}m] * f \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n).$$

To avoid the fact that $\mathcal{S}(\mathbb{R}^n)$ is not dense in $M^{p,q}_s(\mathbb{R}^n)$ if $p = \infty$ or $q = \infty$, we use the following definition of the boundedness of Fourier multiplier operators on modulation spaces: We say that $m(D)$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if there exists a constant $C > 0$ such that $\|m(D)f\|_{M^{p,q}} \le C\|f\|_{M^{p,q}_s}$ for all $f \in \mathcal{S}(\mathbb{R}^n)$, and set

$$\|m(D)\|_{\mathcal{L}(M^{p,q}_s, M^{p,q})} = \sup\{\|m(D)f\|_{M^{p,q}} \mid f \in \mathcal{S}(\mathbb{R}^n),\ \|f\|_{M^{p,q}_s} = 1\}.$$

Similarly, we set

$$\|m(D)\|_{\mathcal{L}(L^p, L^q)} = \sup\{\|m(D)f\|_{L^q} \mid f \in \mathcal{S}(\mathbb{R}^n),\ \|f\|_{L^p} = 1\},$$

and simply write $\|m(D)\|_{\mathcal{L}(L^p)} = \|m(D)\|_{\mathcal{L}(L^p,L^p)}$ if $p = q$.

The notation $A \asymp B$ stands for $C^{-1}A \le B \le CA$ for some positive constant C independent of A and B. For $1 \le p \le \infty$, p' is the conjugate exponent of p (that is, $1/p + 1/p' = 1$). Throughout the rest of this note, $\psi \in \mathcal{S}(\mathbb{R}^n)$ is the same as in (1.1).

Lemma 2.1 ([8, Lemma 2.6])**.** *Let $0 < p < 1$, and let Γ be a compact subset of $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that*

$$\|f * g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^p}$$

for all $f, g \in L^p(\mathbb{R}^n)$ with $\operatorname{supp}\widehat{f} \subset \xi + \Gamma$ and $\operatorname{supp}\widehat{g} \subset \xi' + \Gamma$, where $C > 0$ is independent of $\xi, \xi' \in \mathbb{R}^n$.

The following is on the relation between L^p-boundedness and $M^{p,q}$-boundedness which is a slight modification of [9, Lemma 2.2]:

Lemma 2.2. *Let $0 < p, q \le \infty$, $s \in \mathbb{R}$ and $m \in \mathcal{S}'(\mathbb{R}^n)$. Then $m(D)$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if there exists a constant $C > 0$ such that*

$$\|\psi(D-k)m(D)f\|_{L^p} \le C(1+|k|)^s\|\psi(D-k)f\|_{L^p} \tag{2.1}$$

for all $k \in \mathbb{Z}^n$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. We assume that (2.1) holds for some constant $C > 0$. Then, by our assumption,

$$\begin{aligned}\|m(D)f\|_{M^{p,q}} &= \left(\sum_{k\in\mathbb{Z}^n}\|\psi(D-k)m(D)f\|_{L^p}^q\right)^{1/q} \\ &\le C\left(\sum_{k\in\mathbb{Z}^n}(1+|k|)^{sq}\|\psi(D-k)f\|_{L^p}^q\right)^{1/q} = C\|f\|_{M^{p,q}_s}\end{aligned}$$

for all $f \in \mathcal{S}$, and we obtain the boundedness of $m(D)$ from $M^{p,q}_s$ to $M^{p,q}$.

We next assume that $0<p<1$ and $m(D)$ is bounded from $M^{p,q}_s$ to $M^{p,q}$. Let $\varphi \in \mathcal{S}$ be such that $\varphi = 1$ on $\operatorname{supp}\psi$, $\operatorname{supp}\varphi \subset [-2,2]^n$ and $|\sum_{k\in\mathbb{Z}^n}\varphi(\xi-k)| \ge C > 0$ for all $\xi \in \mathbb{R}^n$. Note that $\|f\|_{M^{p,q}_s} \asymp \big(\sum_{k\in\mathbb{Z}^n}(1+|k|)^{sq}\|\varphi(D-k)f\|_{L^p}^q\big)^{1/q}$. Since $\psi = \varphi\,\psi$, $\operatorname{supp}\psi(\cdot-(k+\ell)) \subset (k+\ell)+[-1,1]^n$ and $\operatorname{supp}\psi(\cdot-k)\widehat{f} \subset k+[-1,1]^n$ for all $k,\ell\in\mathbb{Z}^n$, we have by Lemma 2.1 and the boundedness of $m(D)$ from $M^{p,q}_s$ to $M^{p,q}$

$$\begin{aligned}
&\|\psi(D-k)m(D)f\|_{L^p} = \|\varphi(D-k)(m(D)\psi(D-k)f)\|_{L^p}\\
&\le \left(\sum_{\ell\in\mathbb{Z}^n}\|\varphi(D-\ell)(m(D)\psi(D-k)f)\|_{L^p}^q\right)^{1/q}\\
&\le C\|m(D)(\psi(D-k)f)\|_{M^{p,q}} \le C\|m(D)\|_{\mathcal{L}(M^{p,q}_s,M^{p,q})}\|\psi(D-k)f\|_{M^{p,q}_s}\\
&= C\|m(D)\|_{\mathcal{L}(M^{p,q}_s,M^{p,q})}\left(\sum_{|\ell|\le 2\sqrt{n}}(1+|k+\ell|)^{sq}\|\psi(D-(k+\ell))\psi(D-k)f\|_{L^p}^q\right)^{1/q}\\
&\le C\|m(D)\|_{\mathcal{L}(M^{p,q}_s,M^{p,q})}\left(\sum_{|\ell|\le 2\sqrt{n}}(1+|k+\ell|)^{sq}\|\mathcal{F}^{-1}[\psi(\cdot-(k+\ell))]\|_{L^p}^q\|\psi(D-k)f\|_{L^p}^q\right)^{1/q}\\
&\le C(1+|k|)^s\|m(D)\|_{\mathcal{L}(M^{p,q}_s,M^{p,q})}\|\mathcal{F}^{-1}\psi\|_{L^p}\|\psi(D-k)f\|_{L^p}
\end{aligned}$$

for all $k\in\mathbb{Z}^n$ and $f\in\mathcal{S}$, where we have used $(1+|k+\ell|)^s \le (1+|k|)^s(1+|\ell|)^{|s|}$. Hence, we obtain (2.1) with $0<p<1$. For $1\le p\le\infty$, by using Young's inequality ($\|f*g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$) instead of Lemma 2.1, we can prove (2.1) in the same way. $\square$

3. Proof of Theorem 1.1

The proof of the following lemma is based on that of [9, Lemma 3.1]:

Lemma 3.1. *Let $0<p<1$, $N=[n(1/p-1/2)]+1$ and $\alpha > n(1/p-1)$, where $[n(1/p-1/2)]$ stands for the largest integer $\le n(1/p-1/2)$. If m is a $C^N(\mathbb{R}^n\setminus\{0\})$-function with compact support satisfying*

$$|\partial^\beta m(\xi)| \le C_\beta|\xi|^{\alpha-|\beta|} \qquad \text{for all } \xi\ne 0 \text{ and } |\beta|\le N,$$

then $\mathcal{F}^{-1}m \in L^p(\mathbb{R}^n)$.

Proof. Assume that $\operatorname{supp} m \subset \{|\xi|\le 2^{j_0}\}$, where $j_0\in\mathbb{Z}$. Let $\varphi\in\mathcal{S}$ be such that $\operatorname{supp}\varphi \subset \{1/2\le|\xi|\le 2\}$ and $\sum_{j\in\mathbb{Z}}\varphi(\xi/2^j) = 1$ for all $\xi\ne 0$. Since $\operatorname{supp}\varphi(\cdot/2^j) \subset \{2^{j-1}\le|\xi|\le 2^{j+1}\}$, we see that

$$m(\xi) = \sum_{j=-\infty}^{j_0}\varphi(\xi/2^j)\,m(\xi) = \sum_{j=-\infty}^{j_0} m_j(\xi/2^j),$$

where $m_j(\xi) = \varphi(\xi)\,m(2^j\xi)$. By using $p<1$, we have

$$\|\mathcal{F}^{-1}m\|_{L^p}^p \le \sum_{j=-\infty}^{j_0}\|2^{jn}(\mathcal{F}^{-1}m_j)(2^j\cdot)\|_{L^p}^p = \sum_{j=-\infty}^{j_0}2^{jn(p-1)}\|\mathcal{F}^{-1}m_j\|_{L^p}^p. \tag{3.1}$$

Let r be the conjugate exponent of $2/p$, and set $N=[n/(pr)]+1$. Then $N=[n(1/p-1/2)]+1$. By Hölder's inequality and Plancherel's theorem,

$$\begin{aligned}
\|\mathcal{F}^{-1}m_j\|_{L^p} &= \|(1+|\xi|)^{-N}(1+|\xi|)^N\mathcal{F}^{-1}m_j\|_{L^p}\\
&\le \|(1+|\xi|)^{-pN}\|_{L^r}^{1/p}\|(1+|\xi|)^N\mathcal{F}^{-1}m_j\|_{L^2} \le C\sum_{|\beta|\le N}\|\partial^\beta m_j\|_{L^2}
\end{aligned} \tag{3.2}$$

for all $j \in \mathbb{Z}$, where we have used the fact $prN > n$. Since $\operatorname{supp} \varphi \subset \{2^{-1} \le |\xi| \le 2\}$, we have by our assumption

$$
\begin{aligned}
|\partial^\beta m_j(\xi)| &= \left| \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} (\partial^{\beta_1}\varphi)(\xi)\, 2^{j|\beta_2|} (\partial^{\beta_2} m)(2^j\xi) \right| \\
&\le \sum_{\beta_1+\beta_2=\beta} C_{\beta_1,\beta_2} |(\partial^{\beta_1}\varphi)(\xi)|\, 2^{j|\beta_2|} (C_{\beta_2} |2^j\xi|^{\alpha-|\beta_2|}) \le C_\beta 2^{j\alpha}
\end{aligned}
\tag{3.3}
$$

for all $j \in \mathbb{Z}$ and $|\beta| \le N$. On the other hand, $\operatorname{supp} m_j \subset \{2^{-1} \le |\xi| \le 2\}$ for all $j \in \mathbb{Z}$. Therefore, by (3.1)-(3.3),

$$
\begin{aligned}
\|\mathcal{F}^{-1} m\|_{L^p}^p &\le \sum_{j=-\infty}^{j_0} 2^{jn(p-1)} \|\mathcal{F}^{-1} m_j\|_{L^p}^p \\
&\le C \sum_{j=-\infty}^{j_0} 2^{-jn(1/p-1)p} \sum_{|\beta|\le N} \|\partial^\beta m_j\|_{L^2}^p \le C \sum_{j=-\infty}^{j_0} 2^{j(\alpha-n(1/p-1))p} < \infty.
\end{aligned}
$$

The proof is complete. $\square$

For $\alpha > 0$ and $k \in \mathbb{Z}^n$, we set

$$
\sigma_\alpha(\xi) = |\xi|^\alpha \qquad \text{and} \qquad \tau_{\alpha,k}(\xi) = \sigma_\alpha(\xi + k) - \sigma_\alpha(k) - (\nabla\sigma_\alpha)(k)\cdot\xi. \tag{3.4}
$$

Lemma 3.2. *Let $0 < p < 1$ and $\alpha > n(1/p-1)$. Then there exists a constant $C > 0$ such that*

$$
\|\psi(D-k) e^{i\sigma_\alpha(D)} f\|_{L^p} \le C \|\psi(D-k) f\|_{L^p}
$$

for all $|k| < 4\sqrt{n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Let η be a Schwartz function with compact support. Then

$$
|\partial_\xi^\beta [\eta(\xi)(e^{i\sigma_\alpha(\xi)} - 1)]| \le C_\beta |\xi|^{\alpha-|\beta|}
$$

for all $\xi \ne 0$ and β. Hence, it follows from Lemma 3.1 that

$$
\mathcal{F}^{-1}[\eta\, e^{i\sigma_\alpha}] = \mathcal{F}^{-1}[\eta(e^{i\sigma_\alpha} - 1)] + \mathcal{F}^{-1}\eta \in L^p.
$$

Take $\varphi \in \mathcal{S}$ such that $\operatorname{supp}\varphi$ is compact and $\varphi = 1$ on $\operatorname{supp}\psi$. Then, by Lemma 2.1 and the first part of this proof with $\eta = \varphi(\cdot - k)$, for $|k| < 4\sqrt{n}$,

$$
\begin{aligned}
\|\psi(D-k) e^{i\sigma_\alpha(D)} f\|_{L^p} &= \|\varphi(D-k) e^{i\sigma_\alpha(D)} \psi(D-k) f\|_{L^p} \\
&\le C \|\mathcal{F}^{-1}[\varphi(\cdot - k) e^{i\sigma_\alpha}]\|_{L^p} \|\psi(D-k) f\|_{L^p} \\
&\le C \|\psi(D-k) f\|_{L^p}
\end{aligned}
\tag{3.5}
$$

for all $f \in \mathcal{S}$. This completes the proof. $\square$

Lemma 3.3. *Let $0 < p < 1$. Then there exists a constant $C > 0$ such that*

$$
\|\psi(D-k) e^{i\sigma_\alpha(D)} f\|_{L^p} \le C |k|^{\max\{0,\alpha-2\} n(1/p-1/2)} \|\psi(D-k) f\|_{L^p}
$$

for all $|k| \ge 4\sqrt{n}$ and $f \in \mathcal{S}(\mathbb{R}^n)$.

Proof. Throughout this proof, we assume that $|k| \ge 4\sqrt{n}$ and $f \in \mathcal{S}$. Let $\varphi \in \mathcal{S}$ be such that $\operatorname{supp}\varphi \subset [-2,2]^n$ and $\varphi = 1$ on $\operatorname{supp}\psi$. Then, by Lemma 2.1,

$$
\begin{aligned}
\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} &= \|\varphi(D-k)e^{i\sigma_\alpha(D)}\psi(D-k)f\|_{L^p} \\
&= \|(\mathcal{F}^{-1}[\varphi(\cdot - k)\, e^{i\sigma_\alpha}]) * \psi(D-k)f\|_{L^p} \\
&\le C\|\mathcal{F}^{-1}[\varphi(\cdot - k)\, e^{i\sigma_\alpha}]\|_{L^p}\|\psi(D-k)f\|_{L^p} \\
&= C\|\mathcal{F}^{-1}[\varphi\, e^{i\sigma_\alpha(\cdot + k)}]\|_{L^p}\|\psi(D-k)f\|_{L^p}.
\end{aligned}
\tag{3.6}
$$

Let us estimate $\|\mathcal{F}^{-1}[\varphi\, e^{i\sigma_\alpha(\cdot+k)}]\|_{L^p}$. Since

$$
\mathcal{F}^{-1}[\varphi\, e^{i\sigma_\alpha(\cdot+k)}](x) = e^{i\sigma_\alpha(k)}\mathcal{F}^{-1}[\varphi\, e^{i\tau_{\alpha,k}}](x + (\nabla\sigma_\alpha)(k)),
$$

where $\tau_{\alpha,k}$ is defined by (3.4), we see that

$$
\|\mathcal{F}^{-1}[\varphi\, e^{i\sigma_\alpha(\cdot+k)}]\|_{L^p} = \|\mathcal{F}^{-1}[\varphi\, e^{i\tau_{\alpha,k}}]\|_{L^p}. \tag{3.7}
$$

By Taylor's formula,

$$
\tau_{\alpha,k}(\xi) = 2\sum_{|\beta|=2}\frac{\xi^\beta}{\beta!}\int_0^1 (1-t)\,(\partial^\beta\sigma_\alpha)(k+t\xi)\,dt. \tag{3.8}
$$

If $\xi \in [-2,2]^n$, then $|k+t\xi| \asymp |k|$ for all $0 \le t \le 1$. Since $|\partial^\gamma\sigma_\alpha(\eta)| \le C_\gamma|\eta|^{\alpha-|\gamma|}$, we have by (3.8)

$$
\begin{aligned}
|\partial^\gamma\tau_{\alpha,k}(\xi)| &= \left|\sum_{|\beta|=2}\sum_{\gamma_1+\gamma_2=\gamma} C_{\beta,\gamma_1,\gamma_2}\partial^{\gamma_1}(\xi^\beta)\int_0^1(1-t)\,t^{|\gamma_2|}\,(\partial^{\beta+\gamma_2}\sigma_\alpha)(k+t\xi)\,dt\right| \\
&\le C_\beta\sum_{|\beta|=2}\sum_{\gamma_1+\gamma_2=\gamma}|\partial^{\gamma_1}(\xi^\beta)|\int_0^1|k+t\xi|^{\alpha-|\beta|-|\gamma_2|}\,dt \le C_\beta|k|^{\alpha-2}
\end{aligned}
$$

for all multi-indices γ. Hence, by noting $|k|^{\max\{0,\alpha-2\}} \ge 1$, we have

$$
\begin{aligned}
&|\partial^\gamma(\varphi(\xi)\, e^{i\tau_{\alpha,k}(\xi)})| \\
&= \left|\sum_{N=0}^{|\gamma|}\sum_{\mu+\nu_1+\dots+\nu_N=\gamma} C_{\mu,\nu_1,\dots,\nu_N}(\partial^\mu\varphi(\xi))\,(\partial^{\nu_1}\tau_{\alpha,k}(\xi))\dots(\partial^{\nu_N}\tau_{\alpha,k}(\xi))\,e^{i\tau_{\alpha,k}(\xi)}\right| \\
&\le C_\gamma\sum_{N=0}^{|\gamma|}\sum_{\mu+\nu_1+\dots+\nu_N=\gamma}\|\partial^\mu\varphi\|_{L^\infty}(C_{\nu_1}|k|^{\alpha-2})\dots(C_{\nu_N}|k|^{\alpha-2}) \\
&\le C_\gamma|k|^{\max\{0,\alpha-2\}|\gamma|}.
\end{aligned}
$$

Then, setting $\varphi_{\alpha,k}(\xi) = \varphi(\xi)\, e^{i\tau_{\alpha,k}(\xi)}$, we have

$$
|\partial_\xi^\gamma[\varphi_{\alpha,k}(\xi/|k|^{\max\{0,\alpha-2\}})]| \le C_\gamma\chi_{[-2|k|^{\max\{0,\alpha-2\}},2|k|^{\max\{0,\alpha-2\}}]^n}(\xi) \tag{3.9}
$$

for all multi-indices γ, where χ_A denote the characteristic function of A. Therefore, by (3.2), (3.6), (3.7) and (3.9),

$$
\begin{aligned}
&\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} \le C\|\mathcal{F}^{-1}\varphi_{\alpha,k}\|_{L^p}\|\psi(D-k)f\|_{L^p} \\
&= C|k|^{\max\{0,\alpha-2\}n(1/p-1)}\|\mathcal{F}^{-1}[\varphi_{\alpha,k}(\cdot/|k|^{\max\{0,\alpha-2\}})]\|_{L^p}\|\psi(D-k)f\|_{L^p} \\
&\le C|k|^{\max\{0,\alpha-2\}n(1/p-1)}\left(\sum_{|\gamma|\le N}\|\partial^\gamma[\varphi_{\alpha,k}(\cdot/|k|^{\max\{0,\alpha-2\}})]\|_{L^2}\right)\|\psi(D-k)f\|_{L^p}
\end{aligned}
$$

$$\leq C|k|^{\max\{0,\alpha-2\}n(1/p-1/2)}\|\psi(D-k)f\|_{L^p},$$

where $N = [n(1/p-1/2)]+1$ and $C>0$ is independent of k satisfying $|k| \geq 4\sqrt{n}$. The proof is complete. $\square$

Before proving Theorem 1.1, we give the following remark on the case $0 \leq \alpha \leq 2$:

Remark 3.4. Let $0 \leq \alpha \leq 2$ and $1 \leq p \leq \infty$. In this case, $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ only if $s \geq 0$.

We first consider the case $p=2$. By Plancherel's theorem,

$$\begin{aligned}\|e^{i\sigma_\alpha(D)}f\|_{M^{2,q}} &= \left(\sum_{k\in\mathbb{Z}^n}\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^2}^q\right)^{1/q} \\ &= \left\{\sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^{n/2}}\|\psi(\cdot-k)e^{i\sigma_\alpha}\widehat{f}\|_{L^2}\right)^q\right\}^{1/q} \\ &= \left\{\sum_{k\in\mathbb{Z}^n}\left(\frac{1}{(2\pi)^{n/2}}\|\psi(\cdot-k)\widehat{f}\|_{L^2}\right)^q\right\}^{1/q} = \|f\|_{M^{2,q}}.\end{aligned}$$

Hence, the boundedness of $e^{i\sigma_\alpha(D)}$ from $M_s^{2,q}$ to $M^{2,q}$ implies the embedding $M_s^{2,q} \hookrightarrow M^{2,q}$. Therefore, $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{2,q}$ to $M^{2,q}$ only if $s \geq 0$.

We next consider the case $1 \leq p \leq \infty$ and $p \neq 2$. If $m(D)$ is bounded from $M_s^{p,q}$ to $M^{p,q}$, then $m(D)$ is also bounded from $M_s^{p',q}$ to $M^{p',q}$. This follows from the facts that $m(D)$ is bounded from $M_s^{p,q}$ to $M^{p,q}$ if and only if $\sup_{k\in\mathbb{Z}^n}(1+|k|)^{-s}\|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)} < \infty$ ([9, Lemma 2.2]) and $\|\psi(D-k)m(D)\|_{\mathcal{L}(L^p)} = \|\psi(D-k)m(D)\|_{\mathcal{L}(L^{p'})}$. Then, by interpolation, if $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}$ to $M^{p,q}$ for some $s<0$, then $e^{i\sigma_\alpha(D)}$ is also bounded from $M_s^{2,q}$ to $M^{2,q}$. Therefore, $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}$ to $M^{p,q}$ only if $s \geq 0$.

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Let $0<p<1$, $0<q\leq\infty$, $\alpha > n(1/p-1)$ and $s \in \mathbb{R}$.

We first assume that $s \geq \max\{0,\alpha-2\}n(1/p-1/2)$. By Lemmas 3.2 and 3.3,

$$\begin{aligned}\|\psi(D-k)e^{i\sigma_\alpha(D)}f\|_{L^p} &\leq C(1+|k|)^{\max\{0,\alpha-2\}n(1/p-1/2)}\|\psi(D-k)f\|_{L^p} \\ &\leq C(1+|k|)^s\|\psi(D-k)f\|_{L^p}\end{aligned}$$

for all $k \in \mathbb{Z}^n$ and $f \in \mathcal{S}$. Hence, by Lemma 2.2, we have the boundedness of $e^{i\sigma_\alpha(D)}$ from $M_s^{p,q}$ to $M^{p,q}$.

We next assume that $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}$ to $M^{p,q}$. By Lemma 2.2, we may assume $q \geq 1$. We note that $e^{i\sigma_\alpha(D)}$ is bounded on $M^{2,q}$ (see Remark 3.4). Hence, it follows from interpolation with the boundedness on $M^{2,q}$ that, if $s < \max\{0,\alpha-2\}n(1/p-1/2)$, then $e^{i\sigma_\alpha(D)}$ is bounded from $M_{\widetilde{s}}^{\widetilde{p},q}$ to $M^{\widetilde{p},q}$, where $1<\widetilde{p}<2$ and $\widetilde{s} < \max\{0,\alpha-2\}n(1/\widetilde{p}-1/2)$. However, in the case $1<\widetilde{p}<2$, $e^{i\sigma_\alpha(D)}$ is bounded from $M_{\widetilde{s}}^{\widetilde{p},q}$ to $M^{\widetilde{p},q}$ only if $\widetilde{s} \geq \max\{0,\alpha-2\}n(1/\widetilde{p}-1/2)$ (see Remark 3.4 and [9]). Therefore, s must satisfy $s \geq \max\{0,\alpha-2\}n(1/p-1/2)$. $\square$

We end this note by giving the following remark on the case $1 \leq p \leq \infty$ and $0<q<1$:

Remark 3.5. Let $\alpha \geq 0$, $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. Lemma 2.2 says that $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ for some $0<q\leq\infty$ if and only if $e^{i\sigma_\alpha(D)}$ is bounded from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ for all $0<q\leq\infty$. In particular, the boundedness of $e^{i\sigma_\alpha(D)}$ from $M_s^{p,q}(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ with $0<q<1$ is equivalent to that with $1\leq q\leq\infty$. On the other hand, by [1, 9] and Remark 3.4, $e^{i\sigma_\alpha(D)}$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha-2\}n|1/p-1/2|$, where $1 \leq q \leq \infty$. Combining these facts, we see that $e^{i\sigma_\alpha(D)}$ is bounded from $M^{p,q}_s(\mathbb{R}^n)$ to $M^{p,q}(\mathbb{R}^n)$ if and only if $s \geq \max\{0, \alpha-2\}n|1/p-1/2|$, where $0 < q < 1$.

Acknowledgement

The author gratefully acknowledges helpful discussions with Professor Akihiko Miyachi. He also thanks the referee for his/her valuable comments.

References

[1] Á. Bényi, K. Gröchenig, K. Okoudjou and L.G. Rogers, Unimodular Fourier multipliers for modulation spaces, J. Funct. Anal. 246 (2007), 366-384.

[2] Á. Bényi and K. Okoudjou, Local well-posedness of nonlinear dispersive equations on modulation spaces, Bull London Math. Soc. 41 (2009), 549-558.

[3] H.G. Feichtinger, Modulation spaces on locally compact abelian groups, in: M. Krishna, R. Radha and S. Thangavelu (Eds.), Wavelets and their Applications, Chennai, India, Allied Publishers, New Delhi, 2003, pp. 99-140, Updated version of a technical report, University of Vienna, 1983.

[4] H.G. Feichtinger, Modulation spaces: looking back and ahead, Sampl. Theory Signal Image Process. 5 (2006), 109-140.

[5] K. Gröchenig, Foundations of Time-Frequency Analysis, Birkhäuser, Boston, 2001.

[6] K. Gröchenig and C. Heil, Modulation spaces and pseudodifferential operators, Integral Equations Operator Theory 34 (1999), 439-457.

[7] L. Hörmander, Estimates for translation invariant operators in L^p-spaces, Acta Math. 104 (1960), 93-140.

[8] M. Kobayashi, Modulation spaces $M^{p,q}$ for $0 < p, q \leq \infty$, J. Funct. Spaces Appl. 4 (2006), 329-341.

[9] A. Miyachi, F. Nicola, S. Rivetti, A. Tabacco and N. Tomita, Estimates for unimodular Fourier multipliers on modulation spaces, Proc. Amer. Math. Soc. 137 (2009), 3869-3883.

[10] J. Toft, Continuity properties for modulation spaces, with applications to pseudo-differential calculus-I, J. Funct. Anal. 207 (2004), 399-429.

[11] B. Wang, L. Zhao and B. Guo, Isometric decomposition operators, function spaces $E^\lambda_{p,q}$ and applications to nonlinear equation, J. Funct. Anal. 233 (2006), 1-39.

Naohito Tomita, Department of Mathematics, Graduate School of Science, Osaka University, Toyonaka, Osaka 560-0043, Japan

E-mail address: tomita@math.sci.osaka-u.ac.jp