

A MICROLOCAL VERSION OF THE RIEMANN-HILBERT CORRESPONDANCE

Emmanuel Andronikof[†] (Univ. Paris 13)

1. - Introduction

Let X be a complex n -dimensional manifold. Recall that the “Riemann-Hilbert correspondance” consists of the two following commutative diagrams, together with the assertion that all the arrows are equivalences of categories :

$$(1.1) \quad \begin{array}{ccccc} & & \text{Rhom}(\cdot, \mathcal{O}_X) & & \\ & \swarrow & & \searrow & \\ D_{\mathbf{C}-c}^b(X)^\circ & \xrightleftharpoons[\text{Sol}]{RH} & D_{r-h}^b(\mathcal{D}_X) & \xrightarrow{\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} (\cdot)} & D_h^b(\mathcal{D}_X^\infty) \\ & \nwarrow & \text{Sol} & \nearrow & \end{array}$$

$$(1.2) \quad \begin{array}{ccccc} & & \text{Rhom}(\cdot, \mathcal{O}_X) & & \\ & \swarrow & & \searrow & \\ \text{Perv}(X)^\circ & \xrightleftharpoons[\text{Sol}]{RH} & \text{RegHol}(\mathcal{D}_X) & \xrightarrow{\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} (\cdot)} & \text{Hol}(\mathcal{D}_X^\infty) \\ & \nwarrow & \text{Sol} & \nearrow & \end{array}$$

We make use of the following notations :

$D_{\mathbf{C}-c}^b(X)$ is the derived category of bounded complexes of sheaves of $\mathbf{C}$ -vector spaces on X with $\mathbf{C}$ -constructible cohomology.

$\text{RegHol}(\mathcal{D}_X)$ is the abelian category of regular holonomic (left) $\mathcal{D}_X$ -modules.

$\text{Hol}(\mathcal{D}_X^\infty)$ is the category of modules of the form $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} \mathcal{M}$ where $\mathcal{M}$ is a holonomic $\mathcal{D}$ -module.

$D_{r-h}^b(\mathcal{D}_X)$ is the derived category of bounded complexes of $\mathcal{D}_X$ -modules with regular holonomic cohomology. $D_h^b(\mathcal{D}_X^\infty)$ is the derived category of bounded complexes of admissible $\mathcal{D}_X^\infty$ -modules (in the sense of [S-K-K]) with cohomology in $\text{Hol}(\mathcal{D}_X^\infty)$.

$\text{Perv}(X)$ is the full abelian subcategory of “perverse sheaves” of $D_{\mathbf{C}-c}^b(X)$, where we adopt for our purpose a definition shifted by $n = \dim_{\mathbf{C}} X$ from the usual one, i.e. given $F \in \text{Ob } D_{\mathbf{C}-c}^b(X)$, we say F is an object of $\text{Perv}(X)$ iff $F[n]$ is perverse in the usual sense of [BBD] (e.g. if $Y \subset X$ is a purely d -codimensional complex set then we say that $\mathbf{C}_Y[-d]$ is perverse; see § 4).

Recall that one sets $\text{Sol}(\mathcal{M}) = R\text{Hom}_{\mathcal{D}}(\mathcal{M}, \mathcal{O})$ or $R\text{Hom}_{\mathcal{D}^\infty}(\mathcal{M}, \mathcal{O})$ accordingly, and that the arrows bearing that name in (1.1) and (1.2) were constructed in [K 1], that the equivalence under $\mathcal{D}_X^\infty \otimes_{\mathcal{D}_X} (\cdot)$ was proven in [K-K] and that the construction of the temperate $R\text{Hom}(\cdot, \mathcal{O})$ -functor RH and proof that RH is an equivalence was performed in [K 1, 2].

[†] Département de Mathématiques - Laboratoire C.N.R.S. URA 742 - Université Paris XIII
93430 Villetaneuse - France

An independent proof that Sol is an equivalence is performed in [M 1 and 2]. See [B] for a review of these results.

The point of interest here is to give a microlocal version of (1.2). Namely, if $\pi : T^*X \rightarrow X$ is the cotangent bundle of X , and $p \in \overset{\circ}{T^*}X = \underset{\text{def}}{T^*X \setminus T_X^*X}$, one has the abelian category $\text{Reghol}(\mathcal{E}_{X,p})$ of germs of regular holonomic modules over the ring of microdifferential operators $\mathcal{E}_{X,p}$ of [S-K-K] which should be equivalent to a category defined by a suitable microlocalization of $\text{Perv}(X)$. The precise statement goes as follows.

We set $\mathbb{C}^\times := \mathbb{C} \setminus \{0\}$ and $\gamma : T^*X \rightarrow T^*X/\mathbb{C}^\times$.

THEOREM 1. — *One has the following commutative diagram (1.9) and all the horizontal arrows are equivalences of categories.*

$$(1.9) \quad \begin{array}{ccccc} & & \gamma^{-1}R\gamma_*\mu\text{hom}(\cdot, \mathcal{O}_X) & & \\ & \swarrow & \downarrow & \searrow & \\ \text{Perv}(X; \mathbb{C}^\times p)^\circ & \xrightarrow{\mu RH} & \text{Reghol}(\mathcal{E}_{X,p}) & \xrightarrow{\mathcal{E}_{X,p}^\infty \otimes_{\mathcal{E}_{X,p}} (\cdot)} & \mathcal{H}ol(\mathcal{E}_{X,p}^\infty) \\ & \downarrow & \downarrow \mathcal{E}_{X,p}^{\mathbb{R},f} \otimes_{\mathcal{E}_{X,p}} (\cdot) & & \downarrow \mathcal{E}_{X,p}^{\mathbb{R}} \otimes_{\mathcal{E}_{X,p}} (\cdot) \\ & \text{Perv}(X; p)^\circ & \xleftarrow{T-\mu\text{hom}(\cdot, \mathcal{O}_X)} & \text{Reghol}(\mathcal{E}_{X,p}^{\mathbb{R},f}) & \xrightarrow{\mathcal{E}_{X,p}^{\mathbb{R}} \otimes_{\mathcal{E}_{X,p}} (\cdot)} & \mathcal{H}ol(\mathcal{E}_{X,p}^{\mathbb{R}}) \\ & & \xleftarrow{\text{Sol}_p} & \xrightarrow{\mu\text{hom}(\cdot, \mathcal{O}_X)_p} & \\ & & & & \text{Sol}_p \end{array}$$

Here :

$\mathcal{E}_X^\infty$ is the sheaf of infinite order microdifferential operators of [S-K-K],

$\mathcal{E}_X^{\mathbb{R}}$ is the sheaf of holomorphic microlocal operators of [S-K-K] and $\mathcal{E}_X^{\mathbb{R},f}$ is its temperate analogue of [A].

An object of $\text{Reghol} \mathcal{E}_{X,p}^{\mathbb{R},f}$ is by definition of the form $\mathcal{E}_{X,p}^{\mathbb{R},f} \otimes_{\mathcal{E}_X} \mathcal{M}$ with $\mathcal{M} \in \text{Ob Reghol } \mathcal{E}_{X,p}$, with a similar definition for $\mathcal{H}ol(\mathcal{E}_{X,p}^\infty)$ and $\mathcal{H}ol(\mathcal{E}_{X,p}^{\mathbb{R}})$.

The categories $\text{Perv}(X; \mathbb{C}^\times p)$ and $\text{Perv}(X; p)$ are defined below.

$\mu\text{hom}(\cdot, \cdot)$ is Kashiwara and Schapira's functor of [K-S 2], and $T-\mu\text{hom}(\cdot, \mathcal{O}_X)$ is the temperate version of $\mu\text{hom}(\cdot, \mathcal{O}_X)$ of [A], while $\mu RH := \gamma^{-1}R\gamma_* T-\mu\text{hom}(\cdot, \mathcal{O}_X)$.

Assuming the definition of $\text{Perv}(X; p)$, the construction of Sol_p is implicit in [K-S 1].

The various microlocalizations of $\text{Perv}(X)$ are performed by essential use of the microlocal theory of sheaves of Kashiwara and Schapira [K-S 2] and by using the microlocal characterisation of perverse sheaves of loc.cit.

We stress the point that these microlocalizations rely on necessary *real* (subanalytic) geometry.

The main tool in the proof is the invariance by canonical transformations which allows one to make use of the generic position theorem of [K-K] which reduces the situation to that of (regular holonomic) $\mathcal{D}$ -modules.

2. – The category $D_{\mathbb{R}-c}^b(X; \Omega)$

Let X be a real analytic manifold, $D^b(X)$ the derived category of the category of bounded complexes of sheaves on X and $D_{\mathbb{R}-c}^b(X)$ its full triangulated subcategory of complexes with $\mathbb{R}$ -constructible cohomology. The following is detailed in [A, Appendix].

If $\Omega \subset T^*X$ is any subset of the cotangent bundle of X the fundamental category occurring in [K-S 2] is

$$D^b(X; \Omega) \stackrel{\text{def}}{=} D^b(X) / \mathcal{N}_\Omega$$

where $\mathcal{N}_\Omega$ is the null-system of objects F whose micro-support $SS(F)$ does not meet Ω (cf. loc.cit).

We set here

$$D_{\mathbb{R}-c}^b(X; \Omega) = D_{\mathbb{R}-c}^b(X) / \mathcal{N}_\Omega \cap \text{Ob } D_{\mathbb{R}-c}^b(X).$$

Note that if $\Omega' \subset \Omega$ there is a canonical functor $D_{\mathbb{R}-c}^b(X; \Omega) \longrightarrow D_{\mathbb{R}-c}^b(X; \Omega')$.

If $\Omega = \{p\}$ is a point we write $D^b(X; p)$ instead of $D^b(X; \{p\})$ and so forth.

By the results of [K-S 2] it is easy to see that

LEMMA 2.1. — $D_{\mathbb{R}-c}^b(X; p)$ is a full triangulated subcategory of $D^b(X; p)$.

An adaptation of the microlocal kernel operations of [K-S 2] yields also the invariance under “extended canonical transformations” of loc.cit.

More precisely, let Y be another real manifold and denote by q_j the j -th projection of $X \times Y$ and by $(\cdot)^a$ the antipodal map of T^*Y .

Let $p_X \in T^*X$, $p_Y \in T^*Y$ and $K \in \text{Ob } D_{\mathbb{R}-c}^b(X \times Y)$ satisfying the following condition :

$$(2.1) \quad SS(K) \cap (\{p_X\} \times T^*Y) \subset \{(p_X, p_Y^a)\} \text{ in the neighborhood of that point.}$$

For $F \in \text{Ob } D_{\mathbb{R}-c}^b(Y)$ one defines a pro-object of $D_{\mathbb{R}-c}^b(X; p_X)$ by setting

$$(2.2) \quad \Phi_K^\mu(F) = \varprojlim Rq_{1!}(K_{X \times V} \otimes q_2^{-1}F)$$

where V runs over the set of relatively compact open subanalytic neighborhoods of $x = \pi(p)$. Actually one has the

LEMMA 2.2. — For $K \in \text{Ob } D_{\mathbb{R}-c}^b(X \times Y)$ satisfying (2.1), this pro-object is an object of $D_{\mathbb{R}-c}^b(X; p_X)$ and the functor $\Phi_K^\mu : D_{\mathbb{R}-c}^b(Y; p_Y) \rightarrow D_{\mathbb{R}-c}^b(X; p_X)$ is well defined.

Note that the functor $\Phi_K(\cdot) = Rq_{1!}(K \otimes q_2^{-1}(\cdot))$ would not be defined here in general.

PROPOSITION 2.3. — Let $\varphi : (T^*Y)_{p_Y} \rightarrow (T^*X)_{p_X}$ be a germ of canonical transformation and Λ its associated germ of Lagrangian manifold in $T^*(X \times Y)$. One may find $K \in \text{Ob } D_{\mathbb{R}-c}^b(X \times Y)$ with $SS(K) \subset \Lambda$ in the neighborhood of (p_X, p_Y^a) , such that $\Phi_K^\mu : D_{\mathbb{R}-c}^b(Y; p_Y) \rightarrow D_{\mathbb{R}-c}^b(X; p_X)$ is an equivalence of categories.

3. – The category $D_{\mathbf{C}-c}^b(X; \Omega)$

Let now X be a complex n -dimensional manifold, and $X_{\mathbb{R}}$ the underlying real manifold. Recall that for $F \in \text{Ob } D_{\mathbb{R}-c}^b(X)$ one has the following characterisation (cf. [K-S 2]) :

$$(3.1) \quad (F \in \text{Ob } D_{\mathbf{C}-c}^b(X)) \iff (SS(F) \text{ is } \mathbf{C}^\times\text{-conical}) \iff (SS(F) \text{ is } \mathbf{C}\text{-Lagrangian}),$$

thus we may define for any subset $\Omega \subset T^*X$ a full triangulated subcategory of $D_{\mathbb{R}-c}^b(X; \Omega)$ by setting

$$(3.2) \quad \begin{cases} D_{\mathbf{C}-c}^b(X; \Omega) \stackrel{\text{def}}{=} \text{the full subcategory of } D_{\mathbb{R}-c}^b(X; \Omega) \text{ of the objects} \\ F \in \text{Ob } D_{\mathbb{R}-c}^b(X) \text{ such that } SS(F) \text{ is } \mathbf{C}^\times\text{-conic in a neighborhood of } \Omega. \end{cases}$$

PROPOSITION 3.1 (See [A, Appendix]). — *Let Y be another copy of X , $\varphi : (T^*Y)_{p_Y} \rightarrow (T^*X)_{p_X}$ be a germ of complex canonical transformation and $\Lambda \subset T^*(X \times Y)$ its associated complex Lagrangian submanifold. Then*

- (i) *there exists $K \in \text{Ob}(D_{\mathbf{C}-c}^b(X \times Y; (p_X, p_Y^a)))$ with $SS(K) \subset \Lambda$ in a neighborhood of (p_X, p_Y^a) such that the functor of proposition 2.3 induces an equivalence of categories*

$$\Phi_K^\mu : D_{\mathbf{C}-c}^b(Y; p_Y) \rightarrow D_{\mathbf{C}-c}^b(X; p_X),$$

- (ii) *if moreover φ is globally defined on the orbit $\mathbf{C}^\times p_Y$ then there is $K \in \text{Ob}(D_{\mathbf{C}-c}^b(X \times Y; \mathbf{C}^\times(p_X, p_Y^a)))$, with $SS(K) \subset \Lambda = \mathbf{C}^\times \Lambda$ in a neighborhood of $\mathbf{C}^\times(p_X, p_Y^a)$ such that Φ_K^μ induces an equivalence of categories*

$$\Phi_K^\mu : D_{\mathbf{C}-c}^b(Y; \mathbf{C}^\times p_Y) \rightarrow D_{\mathbf{C}-c}^b(X; \mathbf{C}^\times p_X).$$

Point (i) follows easily from proposition 2.3 by (3.1) because Φ_K^μ preserves local $\mathbf{C}^\times$ -conicity, then (ii) stems from (i) and formula (2.2) that shows that Φ_K^μ is defined at any point in the fiber of π over $\pi(p)$.

For example one has $D_{\mathbf{C}-c}^b(X; T^*X) = D_{\mathbf{C}-c}^b(X)$ and if $x \in X \cong T_X^*X$ one has the equivalence $(F \in \text{Ob } D_{\mathbf{C}-c}^b(X; x)) \iff (F \in \text{Ob } D_{\mathbb{R}-c}^b(X) \text{ and } F|_V \in \text{Ob } D_{\mathbf{C}-c}^b(V) \text{ for some open neighborhood } V \text{ of } x)$.

Note that, in general the objects of $D_{\mathbf{C}-c}^b(X; p)$ do not have $\mathbf{C}$ -constructible cohomologies and the natural functor $D_{\mathbf{C}-c}^b(X)/\mathcal{N}_p \cap D_{\mathbf{C}-c}^b(X) \rightarrow D_{\mathbf{C}-c}^b(X; p)$ is not an equivalence.

On the other hand, one has the following geometrical version of the generic position theorem. Recall (cf. [K-K]) that a complex Lagrangian subset $\Lambda \subset T^*X$ is said to have a *generic position* at $p \in \overset{\circ}{T^*}X$ iff

$$(3.3) \quad \Lambda \cap \pi^{-1}\pi(p) = \mathbf{C}^\times p \text{ in a neighborhood of } p.$$

PROPOSITION 3.2. — *Let $F \in \text{Ob } D_{\mathbf{C}-c}^b(X; p)$ such that $SS(F)$ is in a generic position at p . Then there exists $F' \in \text{Ob } D_{\mathbf{C}-c}^b(X; \pi(p))$ such that $F' \simeq F$ in $D^b(X; p)$.*

The proof goes by showing that one may “cut-off” the non $\mathbf{C}$ -Lagrangian part of $SS(F)$ in $\pi^{-1}\pi(p)$, i.e. one finds kernels K, K^* in $D_{\mathbf{C}-c}^b(X \times X; (p, p^a))$ and an open subanalytic neighborhood U of x in X such that K, K^* satisfy the conditions of proposition 3.1 (i), $\Phi_{K^*}^\mu$ is a quasi-inverse of Φ_K^μ and $F' := \Phi_{K^*}^\mu((\Phi_K^\mu F)_U)$ is such that $SS(F')$ is $\mathbf{C}^\times$ -invariant in $\pi^{-1}(U)$. Thus $F' \in \text{Ob } D_{\mathbf{C}-c}^b(X; \pi(p))$ by (3.1) and $F' \simeq F$ in $D^b(X; p)$ by proposition 3.1.

One may get a quicker proof by using a refined version, obtained in [D'A-S], of a microlocal cut-off lemma of [K-S 2] where one is allowed non-convex sets.

4. – Microlocalization of Perverse Sheaves

In [K-S 2] one finds the following microlocal characterisation of perverse sheaves :

On object $F \in \text{Ob } D_{\mathbb{C}-c}^b(X)$ is a perverse sheaf iff it satisfies the following condition

$$(4.1) \quad \left\{ \begin{array}{l} \text{For any non-singular point } p \in SS(F) \text{ such that } \pi : SS(F) \rightarrow X \\ \text{has constant rank in a neighborhood of } p, \text{ there exists a complex } d\text{-codimensional} \\ \text{submanifold } Y \subset X \text{ such that } F \simeq \mathbb{C}_Y[-d] \text{ in } D^b(X; p) \text{ (cf. [K-S 2, (10.3.7)])}. \end{array} \right.$$

Thus for any subset $\Omega \subset T^*X$ we may define a full subcategory $\text{Perv}(X; \Omega)$ of $D_{\mathbb{C}-c}^b(X; \Omega)$ in the following manner.

DEFINITION 4.1. — $\text{Ob } \text{Perv}(X; \Omega) \stackrel{\text{def}}{=} \{F \in \text{Ob } D_{\mathbb{C}-c}^b(X; \Omega); F \text{ satisfies condition (4.1) at any } p \text{ in a neighborhood of } \Omega\}$.

Then the following results from §3 and the characterisation (4.1).

PROPOSITION 4.2. — *Let $\Omega = \{p\}$ (resp. $\Omega = \mathbb{C}^\times p$).*

- (i) $\text{Perv}(X; \Omega)$ is invariant by extended canonical transformation in the sense of proposition 3.1 (i) (resp. proposition 3.1 (ii)).
- (ii) Let $F \in \text{Perv}(X; p)$ (resp. $\text{Perv}(X; \mathbb{C}^\times p)$) such that $SS(F)$ is in a generic position at p . Then there is $F' \in \text{Perv}(X; \pi(p))$ such that $F \simeq F'$ in $D^b(X; p)$.
- (iii) $\text{Perv}(X; \Omega)$ is a full abelian subcategory of $D_{\mathbb{C}-c}^b(X; \Omega)$.

5. – The equivalence $\text{Perv}(X; \mathbb{C}^\times p)^\circ \xrightarrow{\mu RH} \text{Reghol } \mathcal{E}_{X,p}$

Recall that Kashiwara's functor RH of cohomology with bounds of [K 2, 3] is defined on $\mathbb{R}$ -constructible complexes, more precisely

$$RH : D_{\mathbb{R}-c}^b(X)^\circ \rightarrow D^b(\mathcal{D}_X),$$

(where $D^b(\mathcal{D}_X)$ stands for $D^b(\text{Mod } \mathcal{D}_X)$), and it is microlocalized in [A] as a functor

$$T\text{-}\mu\text{hom}(\cdot, \mathcal{O}_X) : D_{\mathbb{R}-c}^b(X)^\circ \rightarrow D_{\mathbb{R}_{>0}}^b(\pi^{-1} \mathcal{D}_X),$$

where the latter category is the full subcategory subcategory of the complexes of $D^b(\pi^{-1} \mathcal{D}_X) := D^b(\text{Mod}(\pi^{-1} \mathcal{D}_X))$ with $\mathbb{R}_{>0}$ -homogenous cohomology. Since one has

$$\text{supp}(T\text{-}\mu\text{hom}(F, \mathcal{O}_X)) \subset SS(F),$$

then for any subset $\Omega \subset T^*X$, the functor of triangulated categories

$$T\text{-}\mu\text{hom}(\cdot, \mathcal{O}_X) : D_{\mathbb{R}-c}^b(X; \Omega)^\circ \rightarrow D_{\mathbb{R}_{>0}}^b(\pi_\Omega^{-1} \mathcal{D}_X)$$

is well-defined, where $\pi_\Omega := \pi|_\Omega : \Omega \rightarrow X$. If moreover $\Omega = \mathbb{C}^\times \Omega$ is a $\mathbb{C}^\times$ -invariant subset we set for $F \in \text{Ob } D_{\mathbb{R}-c}^b(X)$:

$$(5.1) \quad \mu RH(F) \stackrel{\text{def}}{=} \gamma^{-1} R\gamma_* T\text{-}\mu\text{hom}(F, \mathcal{O}_X) \in \text{Ob } D_{\mathbb{R}_{>0}}^b(\pi_\Omega^{-1} \mathcal{D}_X).$$

Recall also the following facts

For any $F \in \text{Ob } D_{\mathbb{R}-c}^b(X)$ and any $j \in \mathbb{Z}$, $H^j T\text{-}\mu\text{hom}(F, \mathcal{O}_X)$ is an $\mathcal{E}_X^{\mathbb{R},f}$ -module,

$$\mathcal{E}_X^{\mathbb{R},f} \text{ is faithfully flat on } \mathcal{E}_X \text{ and } \gamma^{-1} R\gamma_* \mathcal{E}_X^{\mathbb{R},f} \cong \mathcal{E}_X,$$

and we have invariance by canonical transformations, that is, with the hypotheses of proposition 3.1 (i), one may find a section

$$s \in H^0(T\text{-}\mu\text{hom}(K, \Omega_{X \times Y/X}))_{(p_X, p_Y^*)},$$

(where $\Omega_{X \times Y/X}$ means the sheaf of maximum degree forms relative to $X \times Y \rightarrow X$) such that

the correspondance $P \in \mathcal{E}_{X, p_X}^{\mathbb{R},f} \mapsto Q \in \mathcal{E}_{Y, p_Y}^{\mathbb{R},f}$ such that $Ps = sQ$ is a ring isomorphism compatible with a natural isomorphism $T\text{-}\mu\text{hom}(F, \mathcal{O}_Y)_{p_Y} \xrightarrow{\sim} T\text{-}\mu\text{hom}(\Phi_{K[n]}^\mu F, \mathcal{O}_X)_{p_X}$.

Finally we have a basic formula :

$$T\text{-}\mu\text{hom}(F, \mathcal{O}_X) \simeq \mathcal{E}_X^{\mathbb{R},f} \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1} RH(F), \quad \text{for } F \in \text{Ob } D_{\mathbb{C}-c}^b(X),$$

from which we get

$$(5.2) \quad \mu RH(F) = \mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1} RH(F) \quad \text{for } F \in \text{Ob } D_{\mathbb{C}-c}^b(X).$$

The key point is then the

LEMMA 5.1. — *Formula (5.1) actually defines a functor*

$$\mu RH : \text{Perv}(X; \mathbb{C}^\times p)^\circ \rightarrow \text{RegHol}(\mathcal{E}_{X,p}).$$

Proof : Let $F \in \text{Ob } \text{Perv}(X; \mathbb{C}^\times p)$. By the invariance by extended (resp. quantized) canonical transformations, we may assume that $SS(F)$ has generic position at p , thus, by proposition 4.2 (iii) we may find $F' \in \text{Perv}(X; \pi(p))$ such that $F \simeq F'$ in $D^b(X; p)$, thus

$$\mu RH(F)_p \simeq \mu RH(F')_p \simeq (\mathcal{E}_X \otimes_{\pi^{-1}\mathcal{D}_X} \pi^{-1} RH(F'))_p,$$

by (5.2), and the latter is an object concentrated in degree zero, which coincides with the germ at p of a regular holonomic $\mathcal{E}_X$ -module. ♥

That $\mu RH : \text{Perv}(X; \mathbb{C}^\times p)^\circ \rightarrow \text{RegHol}(\mathcal{E}_{X,p})$ is an equivalence is then readily deduced, by using again invariance by canonical transformations, from Kashiwara and Kawai's generic position theorem of [K-K].

Details will appear elsewhere.

References

- [A] **Andronikof, E.** : Microlocalisation tempérée. To appear in Mémoires de la S.M.F. in 1994. See also : C.R. Acad. Sci. t.303, 347-350 (1986) and t.304, n°17, 511-514 (1987).
- [B-B-D] **Beilinson, A.A - Bernstein, J. - Deligne, P.** : Faisceaux pervers. Astérisque 100 (1982).
- [B] **Björk, J-E.** : Analytic $\mathcal{D}$ -Modules. Kluwer (1993).

- [D'A-S] **D'Agnolo, A. - Schapira, P.** : in preparation.
- [K 1] **Kashiwara, M.** : On the maximally overdetermined systems of linear differential equations. Publ. R.I.M.S. Kyoto Univ. 10, 563-579 (1975).
- [K 2] **Kashiwara, M.** : Faisceaux constructibles et systèmes holonomes à points singuliers réguliers. Sémin. Goulaouic-Schwartz. Ecole Polytechnique (1979-80).
- [K 3] **Kashiwara, M.** : The Riemann-Hilbert problem for holonomic systems. Publ. R.I.M.S., Kyoto Univ., t. 20, 319-365 (1984).
- [K-K] **Kashiwara, M. - Kawai, T.** : On the holonomic systems of microdifferential equations, III. Systems with regular singularities. Publ. R.I.M.S., Kyoto, Univ., t. 17, 813-979 (1981).
- [K-S 1] **Kashiwara, M. - Schapira, P.** : Micro-hyperbolic systems. Acta Math., 142, 1-55 (1979).
- [K-S 2] **Kashiwara, M. - Schapira, P.** : Sheaves on manifolds. Grundlehren der math. Wiss. 292, Springer (1990).
- [M 1 and 2] **Mebkhout, Z.** : Une équivalence de catégories Comp. Math. 51, 51-62 (1984), et : Une autre équivalence de catégories. Comp. Math. 51, 63-88 (1984).
- [S-K-K] **Sato, M. - Kawai, T. - Kashiwara, M.** : Hyperfunctions and pseudo-differential equations - in Proceedings Katata 1971, Komatsu H. (ed.), Lecture Notes in Math. 287, Springer-Verlag, 265-529 (1973).