

Varieties of modules and p -blocks of finite groups

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1 Introduction

Let G be a finite group and k an algebraically closed field of characteristic $p > 0$. For a finitely generated kG -module M , we denote the closed subvariety of $V_G(k)$ defined by the annihilator of $\text{Ext}_{kG}^*(M, M)$ in $H^*(G, k)$ by $V_G(M)$, where $V_G(k)$ is the maximal ideal spectrum of the cohomology ring $H^*(G, k)$. In this note, we consider varieties of indecomposable modules in a p -block of kG . It is known that for any homogeneous closed subvariety V of $V_G(k)$, there is a kG -module M such that $V_G(M) = V$. On the other hand, if M is an indecomposable kG -module, then the variety $V_G(M)$ is connected (as a projective variety) ([C]). Our main result is the following.

Theorem *Let B be a block of kG with defect group D . Let V be a connected homogeneous closed subvariety of $V_G(k)$. Then $V = V_G(L)$ for some indecomposable kG -module L in B if and only if $V = \text{res}_{G,D}^*(W)$ for some connected homogeneous closed subvariety W of $V_D(k)$.*

Here, we say V is connected if it is connected as a projective variety.

In section 4, we study extensions of graded modules over a graded algebra A , which is a twisted algebra (in the sense of [Z]) of the group algebra of an elementary abelian 2-group. We state some results on the complexity $c(M)$ and the rate of growth of $\text{Ext}_A^*(M, M)$ for a graded A -module M . It is known that these are equal if A is a group algebra.

2 Varieties of modules in B

Let B be a block of kG and D a defect group of B . We denote by V_B the union of varieties of all finitely generated kG -modules in B . It is easy to see

that $V_B = \text{res}_{G,D}^*(V_D(k))$, where $\text{res}_{G,D}^*$ is the map induced by the restriction, $H^*(G, k) \rightarrow H^*(D, k)$. Moreover, there is a finitely generated kG -module M in B such that $V_G(M) = V_B$. If V is a connected homogeneous closed subvariety of V_B , then V is not necessarily a variety of some indecomposable kG -module in B . The problem is that, in general, V does not come from a *connected* homogeneous closed subvariety of $V_D(k)$ (see Example 2.4).

Theorem 2.1 *Let H be a subgroup of G , M a kG -module and V a connected homogeneous closed subvariety of $V_G(k)$. Suppose that the trivial kH -module is a direct summand of M as a kH -module. If $V = \text{res}_{G,H}^*(W)$ for some connected homogeneous closed subvariety W of $V_H(k)$, then there exists an indecomposable kG -module L such that $V_G(L) = V$ and $\text{Hom}_{kG}(M, L) \neq 0$.*

Now, we consider the varieties of kG -modules in a block B . Since the varieties of kG -modules in B is contained in V_B , we consider only such a variety.

Corollary 2.2 *Let B be a block of kG with defect group D . Let V be a connected homogeneous closed subvariety of V_B . Then $V = V_G(L)$ for some indecomposable kG -module L in B if and only if $V = \text{res}_{G,D}^*(W)$ for some connected homogeneous closed subvariety W of $V_D(k)$.*

Proof. Suppose that there exists an indecomposable kG -module L in B such that $V_G(L) = V$. Then there exists an indecomposable kD -module N such that $L|N \uparrow^G$ and $N|L \downarrow_D$. So we have that $\text{res}_{G,D}^*(V_D(N)) = V$. Conversely, suppose that there exists a connected homogeneous closed subvariety W of $V_D(k)$ such that $\text{res}_{G,D}^*(W) = V$. Note that there exists a kG -module M in B such that $k|M \downarrow_D$. By Theorem 2.1, there exists an indecomposable kG -module L such that $V_G(L) = V$ and $\text{Hom}_{kG}(M, L) \neq 0$. In particular, L belongs to B .

Let V be a connected homogeneous closed subvariety of $V_G(k)$. If H is a Sylow p -subgroup of G , then it is easy to see that $V = \text{res}_{G,H}^*(W)$ for some connected homogeneous closed subvariety W of $V_H(k)$. So we have,

Corollary 2.3 ([H1]) *Let V be a connected homogeneous closed subvariety of $V_G(k)$. Then there exists an indecomposable kG -module L such that $V_G(L) = V$ and $\text{Hom}_{kG}(k, L) \neq 0$.*

Example 2.4 Let $p = 2$. Let G be a 2-nilpotent group generated by

$$x_i, y_i, z_i, u, v \quad (i = 1, 2)$$

with relations,

$$x_i^2 = y_i^3 = z_i^2 = u^2 = v^2 = 1, \quad y_i^{x_i} = y_i^{-1},$$

$$\begin{aligned}
x_i^v &= x_j, \quad y_i^v = y_j, \quad z_i^v = z_j, \\
[a_i, b_j] &= [a_i, z_i] = [a_i, u] = [v, u] = 1, \\
(a, b &= x, y, z, \quad 1 \leq i \neq j \leq 2).
\end{aligned}$$

So $G \cong ((S_3 \times C_2) \wr C_2) \times C_2$, where we denote the symmetric group of degree 3 by S_3 and a cyclic group of order 2 by C_2 . We set

$$D = \langle x_2, z_1, z_2, u \rangle, \quad E = \langle x_2 u, z_1 \rangle, \quad F = \langle x_2, z_2 \rangle,$$

$$V = \text{res}_{G,E}^*(V_E(k)) \cup \text{res}_{G,F}^*(V_F(k)).$$

Then G has a 2-block B with defect group D . Moreover, V is a connected homogeneous closed subvariety of V_B . But there exists no connected homogeneous closed subvariety W of $V_D(k)$ such that $V = \text{res}_{G,D}^*(W)$. So there exists no indecomposable kG -module L in B such that $V_G(L) = V$.

3 Some associated primes in $H^*(G, k)$

Let G be a p -group. The complexity $c(M)$ of a finitely generated kG -module M is the smallest nonnegative integer c such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \Omega^n(M)}{n^c} = 0.$$

It is known that $c(M) = \dim V_G(M) = \dim H^*(G, k)/I(M)$ where $I(M)$ is the annihilator of $H^*(G, M)$ in $H^*(G, k)$ ([B, Chapter 5]). So there exists a minimal associated prime P of $H^*(G, M)$ such that $\dim H^*(G, k)/P = c(M)$ ([M, Theorem 6.5]). Since P is an associated prime ideal, there exists a homogeneous element $x \in H^*(G, M)$ such that $P = \text{ann } x$. In particular, $\dim H^*(G, M)/\text{ann } x = c(M)$.

Definition Let G be a p -group and M a finitely generated kG -module. Suppose that $1 \leq i \leq c(M)$. We define $m_i(M)$ to be the smallest integer $m \geq 0$ such that $\dim H^*(G, k)/\text{ann } x \geq i$ for some $x \in H^m(G, M)$. Then we have

$$m_1(M) \leq m_2(M) \leq \cdots \leq m_{c(M)}(M) < \infty$$

by the above argument.

Example 3.1 (1) Let $p = 2$ and $G = C_2 \times C_2$. If M is a nonprojective indecomposable kG -module, then M is either periodic or isomorphic to $\Omega^n(k)$ for some $n \in \mathbb{Z}$. If M is periodic, then $m_1(M) = 0$. On the other hand, we

have $m_2(\Omega^n(k)) = \max\{n, 0\}$.

(2) Let $p = 2$ and $G = C_2 \times C_2 \times C_2$. Fix any positive integer n . Then,

$$\sup\{m_2(M) | M : \text{f.g. } kG\text{-module}, c(M) = 3, \dim_k M \leq n\} < \infty.$$

Question 3.2 Let G be a p -group. Fix $n, i > 0$. Then,

$$\sup\{m_i(M) | M : \text{f.g. } kG\text{-module}, i \leq c(M), \dim_k M \leq n\} < \infty ?$$

4 Extensions of modules over some graded algebras

In this section, we assume that $p = 2$. Let

$$A = k \langle x_1, \dots, x_r \rangle / (x_i^2, a_i x_i x_j + a_j x_j x_i, 1 \leq i, j \leq r)$$

for $a_i \in k, a_i \neq 0$. Then A is a finite dimensional local selfinjective graded k -algebra with $\deg x_i = 1$ (see [H2], [Z] for more details). For a finitely generated A -module M , we define the rate of growth $\gamma(\text{Ext}_A^*(M, M))$ of $\text{Ext}_A^*(M, M)$ to be the smallest nonnegative integer s such that

$$\lim_{n \rightarrow \infty} \frac{\dim_k \text{Ext}_A^n(M, M)}{n^s} = 0.$$

Then, $0 \leq \gamma(\text{Ext}_A^*(M, M)) \leq c(M) \leq r$.

Theorem 4.1 Let M be a finitely generated graded A -module. If $\gamma(\text{Ext}_A^*(M, M)) = 0$, then $c(M) \leq r/2$.

Remark (1) If $a_i = 1$ for every i , then A is a group algebra of an elementary abelian 2-groups. In this case, $\gamma(\text{Ext}_A^*(M, M)) = c(M)$.

(2) ([H2]) If we take $a_1, \dots, a_r$ suitably, then A satisfies the following.

(*) For any $1 \leq s \leq r/2$, there exists a graded A -module M such that $c(M) = s$ and $\gamma(\text{Ext}_A^*(M, M)) = 0$.

Suppose that $r = 3$. If M is a graded A -module and $c(M) = 3$, then $\gamma(\text{Ext}_A^*(M, M)) \geq 1$ by Theorem 4.1. Using Example 3.1(2), we can prove the following.

Proposition 4.2 Suppose that $r = 3$. If M is a graded A -module with

$c(M) = 3$, then $\gamma(\text{Ext}_A^*(M, M)) \geq 2$.

Question 4.3 Suppose that $r = 3$. If M is a graded A -module with $c(M) = 3$, then $\gamma(\text{Ext}_A^*(M, M)) = 3$?

Suppose that Example 3.1(2) is true if we replace $m_2(M)$ by $m_3(M)$. Then the equality in Question 4.3 holds.

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