

ジョセフソン接合の電気力学を記述する モデル方程式の周期解

Periodic solutions of the model equation
describing electrodynamics of the Josephson junction

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Abstract

A novel method is developed for constructing periodic solutions of a model equation describing nonlocal Josephson electrodynamics. This method consists of reducing the equation to a system of linear ordinary differential equations through a sequence of nonlinear transformations. The periodic solutions are obtained in the form of parametric representation. It is found that the large time asymptotic of the solution exhibits a steady profile which does not depend on initial conditions. Last, the exact method is applied to the sine-Hilbert equation to obtain periodic solutions. The detail of this report has been published in J. Phys. A: Math. Theor. **42** (2009) 025401.

1. Model equation

1.1 Nonlocal model equation

Consider a Josephson junction with a thin layer between two superconductors. The phase difference $\phi(x, t)$ across the Josephson junction is described by the following model equation:

$$\omega_J^{-2} \phi_{tt} + \omega_J^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} K_0 \left(\frac{|x - x'|}{\lambda_L} \right) \phi_{x'x'}(x', t) dx' + \gamma. \quad (1)$$

K_0 : modified Bessel function of order zero, ω_J : Josephson plasma frequency,
 λ_L : London penetration depth, λ_J : Josephson penetration depth,
 γ : bias current density across the junction, η : positive parameter characterizing the resistance of a unit area of the tunneling junction.

Let l be the characteristic space scale of ϕ . When $\lambda_L \ll l$, then $K_0(x) \sim \pi \delta(x)$ and Eq. (1) reduces to the perturbed sine-Gordon equation

$$\omega_J^{-2} \phi_{tt} + \omega_J^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\lambda_L} \phi_{xx} + \gamma. \quad (2)$$

If $l \ll \lambda_L$, then $K_0(|x|) \sim -\ln|x|$ and Eq. (1) becomes

$$\omega_J^{-2} \phi_{tt} + \omega_J^{-2} \eta \phi_t = -\sin \phi + \frac{\lambda_J^2}{\pi \lambda_L} \int_{-\infty}^{\infty} \frac{\phi_{x'}(x', t)}{x' - x} dx' + \gamma. \quad (3)$$

In the following, we consider the overdamped case $\eta \gg 1$ and the zero bias current $\gamma = 0$. Eq. (3) can then be written in an appropriate dimensionless form as

$$\phi_t = -\sin \phi + H \phi_x, \quad H \phi_x = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\phi_{x'}(x', t)}{x' - x} dx'. \quad (4)$$

1.2 Remarks

- Equation (1) is derived from Maxwell's equations combined with the London equation and the Josephson equation:

Yu. Alief et al, Superconductivity **5** (1992) 230

A. Gurevich, Phys. Rev. **B46** (1992) 3187.

- Equation (4) has been proposed for the first time in a purely mathematical context:

Y. Matsuno, J. Math. Phys. **33** (1992) 3039.

- As for a review on nonlocal Josephson electrodynamics:

A.A. Abdumalikov et al, Superconductor Science and Technology, **22** (2009) 023001

R.G. Mints, J. Low Temp. Phys. **106** (1997) 183.

2. Exact method of solution

2.1 A nonlinear dynamical system

- Dependent variable transformation

We seek periodic solution of (4) of the form

$$\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^N \frac{1}{\beta} \sin \beta(x - x_j), \quad (5)$$

where $x_j = x_j(t)$ are complex functions of t with $\text{Im } x_j(t) > 0$, β is a positive parameter, N is an arbitrary positive integer and f^* denotes the complex conjugate expression of f . Using a formula for the Hilbert transform, one has $H \phi_x = -(\ln f^* f)_x$. Substitution of this expression and (5) into (4) gives the following bilinear equation for f and f^*

$$i(f_t^* f - f^* f_t) = \frac{i}{2}(f^2 - f^{*2}) - f_x^* f - f^* f_x. \quad (6)$$

- A system of nonlinear ODEs for x_j

We divide (6) by f^*f , substitute f from (5) and then evaluate the residue at $x = x_j$ on both sides. This gives a system of nonlinear ODEs for x_j

$$\dot{x}_j = -\frac{1}{2\beta} \frac{\prod_{l=1}^N \sin \beta(x_j - x_l^*)}{\prod_{\substack{l=1 \\ (l \neq j)}}^N \sin \beta(x_j - x_l)} + i, \quad j = 1, 2, \dots, N, \quad (7)$$

where an overdot denotes differentiation with respect to t .

We introduce the following notations:

$$z = e^{2i\beta x}, \quad \xi_j = e^{2i\beta x_j}, \quad \eta_j = e^{2i\beta x_j^*}, \quad j = 1, 2, \dots, N, \quad (8a)$$

$$s_1 = \sum_{j=1}^N x_j, \quad s_2 = \sum_{j<l}^N x_j x_l, \quad \dots, \quad s_N = \prod_{j=1}^N x_j, \quad (8b)$$

$$u_1 = \sum_{j=1}^N \xi_j, \quad u_2 = \sum_{j<l}^N \xi_j \xi_l, \quad \dots, \quad u_N = \prod_{j=1}^N \xi_j, \quad (8c)$$

$$v_1 = \sum_{j=1}^N \eta_j, \quad v_2 = \sum_{j<l}^N \eta_j \eta_l, \quad \dots, \quad v_N = \prod_{j=1}^N \eta_j, \quad (8d)$$

$$t_j = \sum_{l=1}^N \xi_l^j, \quad j = 1, 2, \dots, N. \quad (8e)$$

In terms of $u_j (j = 1, 2, \dots, N)$ and s_1 , f can be written as

$$f = \frac{e^{-i\beta(Nx-s_1)}}{(2\beta i)^N} (z^N - u_1 z^{N-1} + u_2 z^{N-2} + \dots + (-1)^N u_N). \quad (9)$$

Thus, $u_j (j = 1, 2, \dots, N)$ and s_1 determine the function f completely.

Let us derive a system of equations for u_j . To this end, We rewrite (7) in terms of ξ_j and η_j as

$$\dot{\xi}_j = -\frac{1}{2} \alpha u_N \frac{\prod_{l=1}^N (\xi_j - \eta_l)}{\prod_{\substack{l=1 \\ (l \neq j)}}^N (\xi_j - \xi_l)} - 2\beta \xi_j, \quad j = 1, 2, \dots, N, \quad (10a)$$

where

$$\alpha = \prod_{j=1}^N (\xi_j \eta_j)^{-1/2} = e^{-i\beta(s_1 + s_1^*)}, \quad u_N = \prod_{j=1}^N \xi_j = e^{2i\beta s_1}. \quad (10b)$$

Later, we show that α is a constant independent of t and u_N obeys a single nonlinear ODE.

2.2 Linearization

The system of nonlinear ODEs (10) can be linearized in terms of the variables u_j defined by (8c). We multiply ξ_j^{n-1} on both sides of (10a) and sum up with respect to j from 1 to N to obtain

$$\frac{1}{n} \dot{t}_n = -\frac{\alpha}{2} u_N \sum_{s=0}^n (-1)^s v_s I_{n-s} - 2\beta t_n, \quad n = 1, 2, \dots, N, \quad (11a)$$

where I_{n-s} is defined by

$$I_{n-s} = \sum_{j=1}^N \frac{\xi_j^{N+n-s-1}}{\prod_{\substack{l=1 \\ (l \neq j)}}^N (\xi_j - \xi_l)}. \quad (11b)$$

In deriving (11), we have used the identity

$$I_n = 0, \quad -N+1 \leq n \leq -1. \quad (11c)$$

- Time evolution of u_j

The time evolution of u_n follows from (11a) with the help of the formulas

$$u_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j u_j t_{n-j}, \quad 1 \leq n \leq N, \quad \sum_{j=0}^n (-1)^j u_j I_{n-j} = 0, \quad n \geq 1, \quad (12)$$

where $u_0 = 1$ and $I_0 = 1$. In fact, differentiating the first formula in (12) by t and substituting (11a) for $\dot{t}_{n-j}$, we can show that the quantity h_n defined by

$$h_n = \dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{N-n}^* + 2\beta n u_n, \quad n = 1, 2, \dots, N, \quad (13)$$

satisfies the relation

$$h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j} + \frac{(-1)^{n+1} r_n}{2n\alpha}, \quad (14a)$$

where

$$r_n = \sum_{j=1}^n u_{N-j+n}^* \left[-\sum_{s=1}^j (-1)^{n-s} s I_{j-s} + (-1)^{n-j} t_j \right]. \quad (14b)$$

The quantity in the brackets on the right-hand side of (14b) can be shown to vanish identically so that $r_n \equiv 0$. It follows from this and (14a) that

$$h_n = \frac{(-1)^{n-1}}{n} \sum_{j=0}^{n-1} (-1)^j h_j t_{n-j}, \quad n = 1, 2, \dots, N. \quad (15)$$

Solving (15) with the initial condition $h_0 = \alpha u_N/2 - u_N^*/(2\alpha) = 0$, we obtain the relations $h_n \equiv 0$ ($n = 1, 2, \dots, N$). Thus, we see that u_n evolves according to the following system of ODEs

$$\dot{u}_n + \frac{\alpha}{2} u_N u_n - \frac{\alpha^{-1}}{2} u_{N-n}^* + 2\beta n u_n = 0, \quad n = 1, 2, \dots, N. \quad (16)$$

It is remarkable that u_N obeys a single nonlinear ODE of the form

$$\dot{u}_N + \frac{\alpha}{2} u_N^2 - \frac{\alpha^{-1}}{2} + 2\beta N u_N = 0, \quad u_N = e^{2i\beta s_1}, \quad \alpha = \sqrt{\frac{u_N^*}{u_N}}, \quad (17)$$

and other $N - 1$ variables $u_1, u_2, \dots, u_{N-1}$ constitute a system of *linear* ODEs. Rewriting (17) in terms of s_1 , we can put it into a nonlinear ODE for s_1

$$\dot{s}_1 = \frac{1}{2i\beta} \sinh(2\beta \operatorname{Im} s_1) + iN, \quad (18)$$

where $\operatorname{Im} s_1$ implies the imaginary part of s_1 .

3. Periodic solutions

3.1 Construction of periodic solutions

The first step for constructing periodic solutions is to integrate (18). It follows from the real and imaginary parts of (18) that

$$\operatorname{Re} \dot{s}_1 = 0, \quad \operatorname{Im} \dot{s}_1 = -\frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1) + N. \quad (19)$$

Thus, the real part of s_1 becomes a constant $\operatorname{Re} s_1(t) = \operatorname{Re} s_1(0) \equiv b$ whereas integration of the equation for $\operatorname{Im} s_1$ yields an explicit expression. In terms of a new variable $y = 2\beta \operatorname{Im} s_1$, it is given by

$$e^{-y} = \frac{2\nu_N \left(-\tanh \frac{y_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N + 1) \tanh \frac{y_0}{2} - 2\beta N + 1 \right\} \sinh \nu_N t}{2\nu_N \left(\tanh \frac{y_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1 \right\} \sinh \nu_N t}, \quad (20)$$

where $\nu_N = \sqrt{(\beta N)^2 + (1/4)}$ and $y_0 = y(0) = 2\beta \operatorname{Im} s_1(0)$, For $n = 1, 2, \dots, N - 1$, on the other hand, (16) can be written in the form

$$\dot{u}_n = - \left(\frac{1}{2} e^{-2\beta \operatorname{Im} s_1} + 2\beta n \right) u_n + \frac{\alpha^{-1}}{2} u_{N-n}^*. \quad (21)$$

Note from (10b) and $\operatorname{Re} s_1 = b$ that $\alpha = e^{-2i\beta b}$ becomes a constant. The solution of the initial value problem for (21) can be put into the form of a rational function

$$u_n(t) = \frac{G_n}{F}, \quad n = 1, 2, \dots, N - 1, \quad (22a)$$

with

$$F = 2\nu_N \left(\tanh \frac{y_0}{2} + 1 \right) \cosh \nu_N t + \left\{ (2\beta N - 1) \tanh \frac{y_0}{2} + 2\beta N + 1 \right\} \sinh \nu_N t, \quad (22b)$$

$$G_n = 2\nu_N \left(\tanh \frac{y_0}{2} + 1 \right) \left[u_n(0) \cosh \nu_n t + \frac{1}{\nu_n} \left\{ \beta(N - 2n)u_n(0) + \frac{\alpha^{-1}}{2} u_{N-n}^*(0) \right\} \sinh \nu_n t \right], \quad (22c)$$

where $\nu_n = \sqrt{\beta^2(N - 2n)^2 + (1/4)}$. We see that the expression (22) with $n = N$ produces (20) and hence it can be used for all u_n .

3.2 Properties of solutions

- Asymptotic form of the solution as $t \rightarrow \infty$

$$u_n \rightarrow 0, \quad n = 1, 2, \dots, N - 1, \quad u_N \rightarrow e^{2i\beta b} (\sqrt{4(\beta N)^2 + 1} - 2\beta N), \quad (23)$$

$$\phi \sim 2 \tan^{-1} \left[\frac{\sqrt{4(\beta N)^2 + 1} - 1}{2\beta N} \tan \beta \left(Nx - b - \frac{N\pi}{2\beta} \right) \right], \quad (24)$$

$$u \equiv \phi_x \sim \frac{4(\beta N)^2}{\sqrt{4(\beta N)^2 + 1} + (-1)^N \cos 2\beta(Nx - b)}. \quad (25)$$

- Novel features of solutions

1) The asymptotic form of u does not depend on initial conditions except for a phase constant b . It represents a train of nonlinear periodic *standing* waves.

2) The initial profile of u with a spatial period π/β evolves into a periodic wave with a period $\pi/N\beta$.

3) The amplitude of the wave $A(= u_{max} - u_{min})$ is a constant independent of the wavenumber. Indeed, $u_{max} = \sqrt{4(\beta N)^2 + 1} + 1$, $u_{min} = \sqrt{4(\beta N)^2 + 1} - 1$ and hence $A = 2$.

4) The steady profile (25) satisfies the Peierls equation $H\phi_x = \sin \phi$ in the theory of dislocation

R. Peierls, Proc. Phys. Soc. **52** (1940) 256.

Example 1: $N = 1$, $x_1(0) = 3i$, $\beta = 0.2$

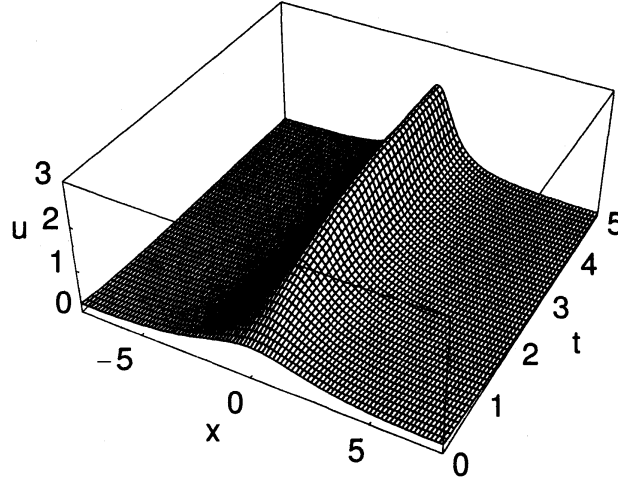


Figure 1. Time evolution of u for $N = 1$ (periodic case).

Example 2: $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$, $\beta = 0.4$

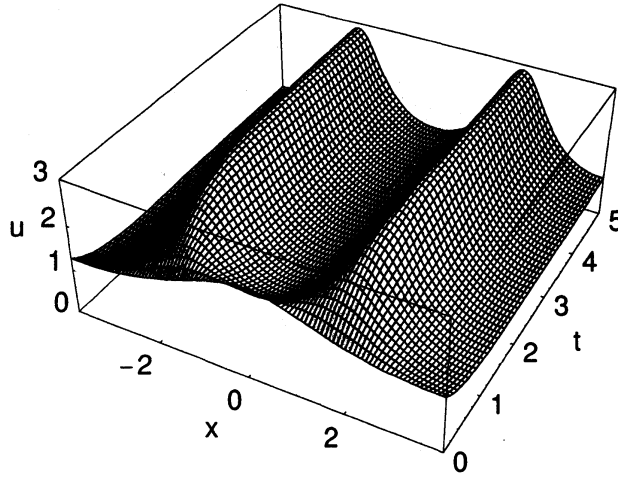


Figure 2. Time evolution of u for $N = 2$ (periodic case).

3.3 Long-wave limit $\beta \rightarrow 0$

The long-wave limit $\beta \rightarrow 0$ of the periodic solutions can be derived easily. We quote the results:

$$\phi = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^N (x - x_j) = \sum_{j=0}^N s_j(t) x^{N-j}, \quad (s_0 = 1), \quad (26)$$

$$\dot{s}_j = -i \operatorname{Im} s_j + i(N - j + 1)s_{j-1}, \quad j = 1, 2, \dots, N. \quad (27)$$

For $N = 2$, the solution reads as follows:

$$f = x^2 - s_1 x + s_2, \quad (28a)$$

$$s_1 = b_1 + i[-(a_1 - 2)(1 - e^{-t}) + a_1], \quad (28b)$$

$$s_2 = -2t - (a_1 - 2)(1 - e^{-t}) + b_2 + i[-(a_2 - b_1)(1 - e^{-t}) + a_2]. \quad (28c)$$

The large time asymptotic of the solution $u \equiv \phi_x$ is given by a superposition of N Lorentzian pulses

$$u \sim \sum_{j=1}^N \frac{2}{(x - \sqrt{2t} x_{j,N})^2 + 1}, \quad (29)$$

where $x_{j,N}$ is the n th root of the Hermite polynomial of order N . These results have been detailed in Matsuno (1992).

Example 1: Nonperiodic case $N = 2$, $x_1(0) = 4i$, $x_2(0) = 2i$

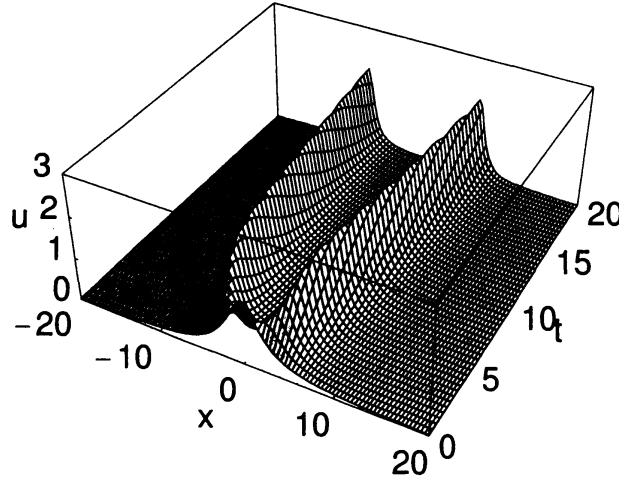


Figure 3. Time evolution of u for $N = 2$ (nonperiodic case).

4. Application

The exact method of solution developed so far can be applied to obtain periodic solutoins of the sine-Hilbert (sH) equation

$$H\theta_t = -\sin \theta, \quad \theta = \theta(x, t). \quad (30)$$

4.1 Remark

- The sH equation was introduced by Degasperis and Santini in a purely mathematical context:

Phys. Lett. A **98** (1983) 240.

- The reduction to a Riemann-Hilbert scattering problem was given by Degasperis et al:

J. Math. Phys. **26** (1985) 2469.

- An exact method of solution by means of bilinear transformation method was developed by Matsuno:

Phys. Lett. **A119** (1986) 229; Phys. Lett. **A120** 187(1987); J. Phys. A: Math. Gen. **20**(1987) 3587.

4.2 Periodic solutions

Here, we summarize the procedure for constructing periodic solutions of the sH equation. We seek periodic solutions of the form (5)

$$\theta = i \ln \frac{f^*}{f}, \quad f = \prod_{j=1}^N \frac{1}{\beta} \sin \beta(x - x_j), \quad (31)$$

The corresponding bilinear equation for f is given by

$$(f^* f)_t = \frac{1}{2}(f^2 - f^{*2}). \quad (32)$$

The system of equations for x_j becomes

$$\dot{x}_j = \frac{1}{2i\beta} \frac{\prod_{l=1}^N \sin \beta(x_j - x_l^*)}{\prod_{\substack{l=1 \\ (l \neq j)}}^N \sin \beta(x_j - x_l)}, \quad j = 1, 2, \dots, N, \quad (33)$$

and u_j satisfies the system of equations

$$\dot{u}_j = i \left(-\frac{c}{2} u_N u_j + \frac{1}{2c} u_{N-j}^* \right), \quad c = \sqrt{\frac{u_N^*}{u_N}}, \quad j = 1, 2, \dots, N. \quad (34)$$

The above system can be solved analytically and solutions are given explicitly.

Example: $N = 1$

Substituting $u_1 = e^{2i\beta s_1}$ into (34)

$$\dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1), \quad (35a)$$

$$\operatorname{Re} \dot{s}_1 = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1), \quad \operatorname{Im} \dot{s}_1 = 0, \quad (35b)$$

$$x_1 = s_1 = at + b + i \frac{1}{2\beta} \sinh^{-1}(2\beta a), \quad a = \frac{1}{2\beta} \sinh(2\beta \operatorname{Im} s_1) \quad b = \operatorname{Res}_1(0), \quad (35c)$$

$$u \equiv \theta_x = \frac{4\beta^2 a}{\sqrt{1 + 4\beta^2 a^2} - \cos 2\beta(x - at - b)}. \quad (36)$$

Note that the solution is not a standing wave but a traveling wave.

5. Summary

- We have constructed periodic solutions of a resistive model describing Josephson electrodynamics by means of a novel linearization procedure.
- The large time asymptotic of the periodic solution has a steady profile which is formed by a balance between nonlinearity and dissipation. This feature is in striking contrast to periodic solutions of nonlinear dispersive wave equations.
- The exact method of solution developed here was applied to the sine-Hilbert equation to obtain periodic solutions.