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DOCTORAL THESIS

**Variational Analysis and Homogenization
of Elastoplastic Materials in the
Rate-Independent Systems Framework**

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Abstract

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In this thesis, we develop a variational homogenization for finite-strain elastic-plastic composite materials based on the framework of rate-independent systems (RIS). We study materials with periodic layered structures with very small periodic lengths ($\varepsilon > 0$), consisting of alternating rigid and soft layers. The soft layers possess two crystallographic slip systems but are constrained to allow only one slip direction to be active at a time at each material point. In contrast, the rigid layers permit only local rotations. In the first half of the paper, we outline the fundamental components of elastic-plastic deformation theory—namely, additive strain decomposition in small-strain theory, multiplicative strain decomposition in finite-strain theory, free energy, and dissipation distance, and then describe the fundamental theory of RIS. We then formulate a time-discrete minimization problem that serves as the basis for the time-evolution solution of RIS.

In a time-discretized formulation, we analyze an ε -dependent incremental minimization problem at each time step and derive the homogenized energy $\mathcal{E}_0$ in the limit $\varepsilon \rightarrow 0$ using static Γ -convergence. This relaxation is necessary because nonconvex constraints and rapid microstructural oscillations may make it impossible to prove the existence of a minimizer at the ε -level. As a main result, we provide an explicit characterization of $\mathcal{E}_0$ when the two slip directions are orthogonal, and partial results are obtained for general angular relationships. The construction extends the classical lamination method to incorporate two slip systems with local single-slip activation, consistent with the rotational constraints imposed on the rigid layers.

The resulting homogenized energy $\mathcal{E}_0$ is intended to function as the energetic component of the evolutionary RIS. Under the standard assumptions of static Γ -convergence of energy functionals and Mosco convergence of the dissipation $\mathcal{D}_\varepsilon$ to $\mathcal{D}_0$, the limit system of the RIS is characterized by $(\mathcal{E}_0, \mathcal{D}_0)$. This analysis clarifies the energetic mechanisms enabling kink-band deformation in layered crystalline composites such as LPSO-type magnesium alloys, providing a rigorous foundation for future time-evolution studies. The derivation of the effective dissipation $\mathcal{D}_0$ remains an open problem for future work.

Furthermore, as a supplementary result of this work, we develop a small-strain, rate-independent elastoplastic model that does not assume a periodic structure. In this setting, we consider a single-slip plasticity model incorporating an inertial (kinetic energy) term and establish the existence of an energy solution (weak equation of motion, local stability, and energy inequality). This part is logically independent from the homogenization analysis and serves as a baseline model for future finite-strain dynamic analyses. The RIS framework including inertia is expected to represent more realistic behavior and provide a more natural setting for homogenization problems.

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List of Abbreviations

RIS	R ate I ndependent S ystem
ERIS	E volutionary R ate I ndependent S ystem
LPSO	L ong P eriod S tacking O rded

List of Symbols

$v^\perp$	vector obtained by rotating vector $v \in \mathbb{R}^2$ by $\pi/2$ counterclockwise
I	2×2 identity matrix
$SO(n)$	the rotation group in $\mathbb{R}^n$, i.e., the group consisting of all orthogonal $n \times n$ matrices with determinant 1
$GL(n)$	the general linear group of all invertible real $n \times n$ matrices
$GL^+(n)$	the subgroup of $GL(n)$ with positive determinant, representing orientation-preserving deformations in $\mathbb{R}^n$
$\mathfrak{gl}(n)$	the space of all real $n \times n$ matrices, i.e., the Lie algebra of $GL(n)$
$\mathbb{R}_{\text{sym}}^{d \times d}$	the space of all symmetric real $d \times d$ matrices, i.e., $\{A \in \mathbb{R}^{d \times d} \mid A = A^\top\}$
$\mathbb{1}_X$	indicator function of a set X
$(a)_+^s$	means $(\max\{0, a\})^s$, $a \in \mathbb{R}$, $s > 0$
c	a generic positive constant, independent of the involved parameters, unless stated otherwise
e_i	the unit vector along the positive direction of coordinate axis x_i in $\mathbb{R}^2$
∂_i	the first-order partial differential operator with respect to x_i
$L^p(D)$	the Lebesgue space of functions with integrable p -powers for $1 \leq p < \infty$
$L^\infty(D)$	the Lebesgue space of essentially bounded functions
$L_{\text{loc}}^p(D)$	the Lebesgue space of functions with locally integrable p -powers
$L_0^p(D)$	the set of functions in $L^p(D)$ with vanishing mean
$W^{k,p}(D)$	the Sobolev space of functions with all k -th derivatives in $L^p(D)$
$\rightarrow$	strong convergence in a normed linear space
$\rightharpoonup$	weak convergence in a normed linear space
$\xrightarrow{\Gamma}$	Γ -convergence of functionals
$\xrightarrow{\text{ev-}\Gamma}$	evolutionary Γ -convergence
$\xrightarrow{\text{ev-}\Gamma_{\text{app}}}$	approximate evolutionary Γ -convergence
$AC([0, T]; X)$	the space of absolutely continuous functions from $[0, T]$ into X
$BV_d([0, T]; Q)$	the space of Q -valued functions of bounded variation on $[0, T]$ with respect to the metric d
$\mathcal{M}([0, T]; X)$	the Banach space of X -valued finite Radon measures on $[0, T]$, endowed with the usual norm defined by duality

Chapter 1

Introduction

A central task in modern materials science is to engineer materials whose performance exceeds that of existing ones. One effective route is to combine distinct constituents and to control their spatial arrangement at the microscale. With a judicious choice of components and layering patterns, composite systems can exhibit marked improvements in strength, ductility, stiffness, corrosion resistance, and other properties. While experimental, engineering, and numerical studies have extensively explored how microstructural architecture influences macroscopic response (see, e.g., Low and Dong, 2021; Mantic, 2023; Milton, 2002), mathematical tools have evolved in parallel. The main variational approach is homogenization: by letting the microstructural length scale tend to zero, one derives an effective (averaged) model that retains the essential physics yet is more amenable to analysis and computation. For variational formulations based on energy minimization, the framework of Γ -convergence (originating with De Giorgi and coauthors) provides a rigorous and physically consistent notion of limit for sequences of energy functionals (De Giorgi, 1975).

Recent interest has been fueled by the discovery of Long-Period Stacking Ordered (LPSO) magnesium alloys and the associated strengthening mechanism linked to kink-band formation in their periodic layered millefeuille architecture (Drozdenko et al., 2022; Hagihara et al., 2019; Plummer et al., 2021; Wadee, Hunt, and Peletier, 2004). LPSO alloys display alternating soft, nearly pure Mg layers and solute-rich, relatively rigid layers. Building on the first homogenization-based study of bilayered materials by Christowiak and Kreisbeck—who analyzed planar structures with band-shaped rigid layers and softer layers constrained to one available slip direction (Christowiak and Kreisbeck, 2017) (see also Braides and Garroni, 1995; Cherdantsev and Cherednichenko, 2012; Francfort and Giacomini, 2014 for related results)—a compression model in the single-slip setting has shown good agreement with experimentally observed kinking (Drozdenko et al., 2022). Since actual LPSO alloys have multiple slip systems in the soft layers, which are believed to contribute to strengthening, it is natural to move beyond the one-slip idealization. In this work we therefore address a two-slip scenario in the soft layers, under a single-slip constraint (only one slip-plane normal can be activated at each material point), coupled with rigid layers that allow only local rotations.

Against this backdrop, this thesis develops a variational homogenization theory for finite-strain elastoplasticity within the Rate-Independent Systems (RIS) framework, with a view toward time-evolving problems. We work at the level of the incremental minimization problems that underlie energetic (rate-independent) evolutions, and we derive an explicit homogenized energy in a setting where, due to nonconvex constraints and rapid microscopic oscillations, one generally cannot guarantee the existence of minimizers for the original static ε -problems. This homogenized energy

is designed to serve as the energy component of an effective RIS and thus as a first step toward an evolutionary limit for the full time-dependent problem.

Chapter 2 formulates constitutive modeling for elastoplastic deformation from a variational perspective. In the small-strain regime we recall the classical additive decomposition of strain, and in the finite-strain regime we adopt the Kröner–Lee multiplicative decomposition to handle large rotations and deformations. On this basis we introduce the free energy and dissipation structure needed for RIS: a time-dependent internal energy $\mathcal{E}$ and a dissipation distance $\mathcal{D}$ on a suitable state space. Chapter 3 then places these ingredients in the abstract architecture of RIS, following (Mielke and Roubíček, 2015). Instead of differential evolution equations, RIS uses the pair of variational principles—energetic stability and energy balance—and represents time evolution through an intrinsic incremental scheme

$$(y_{n+1}, z_{n+1}) \in \arg \min_{(y,z)} \{ \mathcal{E}(t_{n+1}, y, z) + \mathcal{D}(z_n, z) \},$$

whose compactness and lower-semicontinuity properties imply the existence of energetic solutions in continuous time. Each increment can be treated by standard finite-element or nonlinear optimization techniques, thereby providing a bridge between analysis and computation.

Direct RIS analysis for composites with ε -periodic microstructure is hindered already at the static incremental level by (i) nonconvex stored-energy densities and admissible sets, and (ii) fine-scale oscillations as $\varepsilon \downarrow 0$, which can prevent attainment of minima. Chapter 4 introduces static Γ -convergence as a principled relaxation tool: the Γ -limit replaces a possibly nonattaining problem by a well-posed one without lowering the variational infimum; with equicoercivity/compactness, minimizers of the relaxed functional exist. In parallel, homogenization reduces the two-scale ($O(\varepsilon)$ and $O(1)$) description to an effective macroscopic model whose energy is the Γ -limit of the ε -energies, eliminating the need to resolve the microstructure while preserving the correct coarse-scale behavior.

The link to time evolution is provided by evolutionary Γ -convergence: under standard assumptions—uniform-in-time Γ -convergence of the energies and Mosco convergence of the convex dissipation potentials—the discrete RIS solutions converge to energetic solutions of the limit system $(\mathcal{E}_0, \mathcal{D}_0)$ (Mielke, Roubíček, and Stefanelli, 2008; Mielke, 2014; Mielke, Rossi, and Savaré, 2012; Mielke, 2016). In this prototypical setting one may employ the static Γ -limit $\mathcal{E}_0$ directly within the incremental scheme while preserving the evolutionary limit (cf. (4.2)). Establishing the static Γ -limit is therefore the necessary first step toward an effective macroscopic RIS for composites where a direct evolutionary analysis is infeasible.

We study a static homogenization problem for a layered crystalline composite consisting of alternating rigid and soft layers. The soft layers admit two crystallographic slip systems; the rigid layers permit only local rotations. The admissible deformation modes and the resulting energy density enforce nonconvex constraints, so the static incremental problems cannot be treated by standard convex variational methods. We therefore carry out a periodic homogenization via static Γ -convergence (Section 4.1) to obtain the macroscopic energy that represents the effective incremental cost in the limit $\varepsilon \downarrow 0$. The principal theorem establishes the complete homogenized energy when the two slip directions are orthogonal; for a general angle between slip directions we provide partial results. The construction builds on (Christowiak and Kreisbeck, 2017), which covers the single-slip case, and it requires a modification of the classical laminate method to accommodate simultaneous activation of multiple

slip systems together with the rotational constraints of the rigid layers.

The goal is not merely to approximate static minimization problems. The homogenized energy derived here is intended to serve as the energy component $\mathcal{E}_0$ of an effective RIS for the layered composite. In view of the evolutionary Γ -convergence framework summarized in Chapter 4, when the static Γ -convergence of the energies and the Mosco convergence of the dissipations $\mathcal{D}_\varepsilon$ are both available, one expects that the limiting system can be described precisely by the pair $(\mathcal{E}_0, \mathcal{D}_0)$, where $\mathcal{E}_0$ is the static Γ -limit of the energies and $\mathcal{D}_0$ is the Mosco limit of the dissipations (cf. (4.2)). In that favorable case, the homogenized system provides an ideal approximation of the original (pre-homogenized) RIS for small ε . Even if the convergence does not occur in this precise form, however, the static Γ -limit still retains a clear significance, as it approximates the time-discrete solutions of the RIS model. The present work establishes the energy part of this program; deriving an effective dissipation $\mathcal{D}_0$ is left for future work.

As a complement to the homogenization program, we include a self-contained chapter 6 that does not address periodic microstructures. In the infinitesimal-strain setting we formulate a rate-independent elastoplastic model with single-slip plasticity augmented by inertia (kinetic energy). Within an energetic framework (state space, free energy, dissipation, and kinetic term) we prove the existence of energetic solutions satisfying weak momentum balance, local stability, and an energy inequality. This chapter is intended as a physically closer counterpart to quasi-static RIS and as a methodological stepping stone toward future inertial analyses at finite strain. Once static or evolutionary Γ -limits for the composite energies and dissipations are available, the inertial RIS developed here provides a natural macroscopic framework in which to investigate dynamic phenomena; the chapter is logically independent of the homogenization results presented in this thesis.

Chapter 2

Foundations of Elastoplasticity and the Rate-Independent Variational Framework

In this chapter, we clarify the foundational framework of constitutive modeling for elastoplastic deformation. We begin with the classical formulation based on the additive decomposition of strain in the small-strain regime and then proceed to the finite-strain theory, where large rotations and deformations are handled via the Kröner–Lee multiplicative decomposition. Building on these formulations, we introduce a variational structure involving energy functionals and dissipation, thereby preparing the theoretical background for the Rate-Independent System (RIS) framework developed in the subsequent chapter.

2.1 Structure and Basic Elements of Small-Strain Elastoplasticity

Elastoplasticity provides a mathematical model to describe the coexistence of elastic and plastic deformations in materials. It forms a fundamental theoretical basis for understanding nonlinear material behavior, especially in mechanical structures and metal forming processes. In this section, we focus on small-strain elastoplasticity under the assumption of infinitesimal deformations, and introduce its principal structural components step by step. The presentation given here follows standard formulations found in the literature, in particular the comprehensive treatments by Simo and Hughes, 1998, Lubliner, 2008, and Hill, 1950, to which we refer for further background and physical interpretation.

Additive Strain Decomposition In the small-strain setting, the symmetric strain tensor ε is obtained by linearizing the displacement gradient $u: \Omega \rightarrow \mathbb{R}^3$ as follows:

$$\varepsilon := \frac{1}{2} \left(\nabla u + \nabla u^\top \right),$$

where ∇u is the displacement gradient tensor (Jacobian), and $\nabla u^\top$ denotes its transpose. The strain tensor $\varepsilon \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ is thus symmetric and represents the local infinitesimal deformation at each material point.

In the theory of small-strain elastoplasticity, the total strain is additively decomposed into a reversible elastic strain ε_e and an irreversible plastic strain ε_p :

$$\varepsilon = \varepsilon_e + \varepsilon_p.$$

Here, both $\varepsilon_e, \varepsilon_p \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ are symmetric tensors, with ε_e capturing the recoverable elastic response, and ε_p representing the accumulated plastic deformation.

Elastic Energy and Stress Response The elastic response is modeled via Hooke's law, which posits a linear relationship between elastic strain and stress. The elastic energy density is given by

$$\psi(\varepsilon_e) = \frac{1}{2} \varepsilon_e : \mathbb{C} \varepsilon_e,$$

where $\mathbb{C}$ is the fourth-order elasticity tensor, symmetric and positive definite on symmetric tensors. The colon $:$ denotes the standard inner product on second-order tensors:

$$A : B = \sum_{i,j=1}^3 A_{ij} B_{ij}.$$

The Cauchy stress tensor $\sigma \in \mathbb{R}_{\text{sym}}^{3 \times 3}$ is then obtained by differentiating ψ with respect to ε_e :

$$\sigma := \frac{\partial \psi}{\partial \varepsilon_e} = \mathbb{C} \varepsilon_e = \mathbb{C}(\varepsilon - \varepsilon_p).$$

Momentum Balance and External Loadings. The balance equation couples the stress response with external loading. In the dynamic case, the equation of motion reads:

$$\rho \ddot{u} - \operatorname{div} \sigma = b \quad \text{in } \Omega,$$

where ρ is the mass density, b the body force, and σ the Cauchy stress.

The boundary conditions are given by:

$$\sigma n = \bar{t} \quad \text{on } \Gamma_N, \quad u = u_D \quad \text{on } \Gamma_D,$$

where Γ_N denotes the Neumann boundary on which the surface traction $\bar{t}$ is prescribed, and Γ_D denotes the Dirichlet boundary on which the displacement u_D is prescribed.

In the quasistatic case, the inertial term vanishes, yielding:

$$-\operatorname{div} \sigma = b.$$

Yield Condition Elastic materials respond linearly up to a certain stress level, beyond which plastic flow begins. This threshold is described by the yield condition, and the corresponding level set in stress space is called the yield surface. A widely used example is the von Mises yield criterion:

$$f(\sigma) := \|\sigma^{\text{dev}}\| - \sigma_y \leq 0, \quad \text{where} \quad \sigma^{\text{dev}} := \sigma - \frac{1}{3} \operatorname{tr}(\sigma) \mathbb{1},$$

with $\sigma_y > 0$ denoting the yield stress, $\operatorname{tr}(\sigma)$ the trace of the stress tensor, and $\mathbb{1}$ the identity tensor. Here $\|A\|$ denotes the Frobenius norm of a tensor A .

Plastic Flow Rule and Kuhn–Tucker Conditions Once the yield surface is reached, plastic strain begins to evolve over time. This evolution is governed by the flow rule.

In classical settings, the associated flow rule takes the form:

$$\dot{\varepsilon}_p \in \lambda \frac{\partial f}{\partial \sigma}, \quad \lambda \geq 0,$$

where λ is a non-negative scalar representing the plastic multiplier. Together with the yield condition $f(\sigma) \leq 0$, this leads to the Kuhn–Tucker conditions:

$$\lambda \geq 0, \quad f(\sigma) \leq 0, \quad \lambda f(\sigma) = 0,$$

ensuring that plastic flow occurs only when the stress lies on the yield surface.

Hardening Mechanisms In many materials, the yield stress itself evolves with plastic deformation. This is captured by the hardening law, of which two classical forms are:

- Isotropic hardening: the yield surface expands uniformly, corresponding to an increase in σ_y ;
- Kinematic hardening: the yield surface translates in stress space, modeling Bauschinger-type effects.

These hardening effects are typically modeled by introducing internal variables z , such as the accumulated plastic strain or backstress tensors, and extending the energy and yield function to depend on them, i.e., $f(\sigma, z)$.

In summary, small-strain elastoplasticity is structured around the additive strain decomposition, the elastic energy–stress relationship, the yield condition and flow rule for plastic evolution, and hardening laws encoding history dependence.

2.2 Structure and Basic Elements of Finite-Strain Elastoplasticity

In the previous section, we discussed the framework of small-strain elastoplasticity under the assumption of infinitesimal deformations. There, we introduced the key structural components such as the additive decomposition of strain, the energy–stress relation, the yield condition, the flow rule governing plastic evolution, and hardening mechanisms. The small-strain theory is valid when deformations are sufficiently small, and is widely used in classical structural analysis and linearized modeling of metallic materials.

However, in many practical engineering applications and material behaviors, it is often necessary to consider large deformations, including finite rotations and nonlinear geometric effects. In such cases, the additive strain decomposition used in small-strain theory becomes geometrically inconsistent, and a more general formulation is required.

To address this issue, finite-strain elastoplasticity introduces the deformation gradient $F = \nabla y$ and adopts the multiplicative decomposition proposed by Kröner and Lee (Lee, 1969; Simo and Hughes, 1998). This framework allows for a geometrically consistent separation of elastic and plastic effects, even under large deformations.

Kinematics and the Multiplicative Decomposition In the finite-strain setting, the deformation is described by a map

$$y: \Omega \subset \mathbb{R}^3 \rightarrow \mathbb{R}^3,$$

which maps a material point $x \in \Omega$ in the reference configuration to its current position $y(x)$. The gradient of this map defines the deformation gradient tensor:

$$F(x) := \nabla y(x) \in \text{GL}^+(3),$$

where $\text{GL}^+(3)$ denotes the set of invertible 3×3 matrices with positive determinant, ensuring orientation preservation and local invertibility.

In the Kröner–Lee framework, the deformation gradient is multiplicatively decomposed into elastic and plastic parts:

$$F = F_e F_p,$$

where $F_p \in \text{GL}^+(3)$ represents the plastic deformation from the reference to an intermediate configuration, and $F_e \in \text{GL}^+(3)$ denotes the elastic deformation from the intermediate to the current configuration. This decomposition enables a clear and geometrically consistent separation of reversible and irreversible deformations.

Relation to the Small-Strain Limit A natural question arises regarding the relationship between the additive strain decomposition in small-strain theory and the multiplicative decomposition used in finite-strain models. In fact, it is well known that the multiplicative decomposition reduces to the additive form in the limit of small deformations, as we now demonstrate.

Let the elastic and plastic distortions be small:

$$F_e = I + \beta_e, \quad F_p = I + \beta_p, \quad \|\beta_e\|, \|\beta_p\| \ll 1.$$

Expanding $F = F_e F_p$ to first order gives

$$F = (I + \beta_e)(I + \beta_p) = I + \beta_e + \beta_p + \beta_e \beta_p.$$

Since the total deformation is a gradient, $F = I + \nabla_x u$, by neglecting quadratic terms we obtain

$$\nabla_x u \approx \beta_e + \beta_p.$$

Taking the symmetric part gives the additive decomposition

$$\varepsilon := \frac{1}{2}(\nabla u + \nabla u^\top) \approx \underbrace{\frac{1}{2}(\beta_e + \beta_e^\top)}_{\varepsilon_e} + \underbrace{\frac{1}{2}(\beta_p + \beta_p^\top)}_{\varepsilon_p} = \varepsilon_e + \varepsilon_p.$$

In the dislocation-free case on a simply connected domain (i.e., $\text{Curl}_x F_p = 0$ so that $F_p = \nabla_x \varphi_p$), one may write $\beta_p = \nabla_x u_p$ and, with the elastic map defined on the intermediate configuration, $\beta_e = \nabla_{x_p} u_e$. Consequently, to first order,

$$\nabla_x u \approx \nabla_{x_p} u_e + \nabla_x u_p,$$

where $\nabla_{x_p} u_e$ is evaluated at $x_p = \varphi_p(x)$; replacing this evaluation point by x changes only higher-order terms.

Therefore, the multiplicative decomposition reduces to the additive strain decomposition in the small-deformation limit. This confirms that classical small-strain theory arises as the linearized limit of the finite-strain framework, and the Kröner–Lee model can be regarded as a natural and geometrically consistent extension of the small-strain theory.

Free Energy and Frame Indifference The elastic response is governed by a free energy density function

$$W: GL^+(3) \rightarrow [0, \infty),$$

which depends only on the elastic part $F_e := FF_p^{-1}$. The total stored energy is given by

$$\int_{\Omega} W(FF_p^{-1}) dx.$$

To ensure frame indifference, W must satisfy

$$W(RF_e) = W(F_e) \quad \text{for all } R \in SO(3),$$

i.e., invariance under superposed rigid body rotations.

Stress and Driving Forces Let $F = \nabla y$ and $F_e := FF_p^{-1}$. Define the elastic Piola stress in the intermediate configuration by

$$P_e := \partial_{F_e} W(F_e).$$

The first Piola–Kirchhoff stress conjugate to F is then

$$P := \partial_F W(FF_p^{-1}) = P_e F_p^{-\top}.$$

The plastic driving force (density-level derivative with respect to F_p) reads

$$\partial_{F_p} W(FF_p^{-1}) = -F_e^{\top} P_e F_p^{-\top} = -M F_p^{-\top}, \quad M := F_e^{\top} P_e \text{ (Mandel stress)}.$$

Momentum Balance and External Loadings Let P denote the first Piola stress as before. The (reference) momentum balance is

$$\rho \ddot{y} - \text{Div } P = b \quad \text{in } \Omega, \quad Pn = \bar{t} \text{ on } \Gamma_N, \quad y = y_D \text{ on } \Gamma_D,$$

with b the body force density and $\bar{t}$ the prescribed traction on the Neumann part Γ_N of the boundary. This dynamic equation is obtained by Hamilton's principle for the action

$$\mathcal{S}[y] = \int_{t_0}^{t_1} \left(\int_{\Omega} \frac{1}{2} \rho |\dot{y}|^2 dx - \int_{\Omega} W(F_e) dx + \mathcal{W}_{\text{ext}}(t, y) \right) dt.$$

Here $\mathcal{W}_{\text{ext}}(t, y)$ collects the potential of external loadings, typically given by

$$\mathcal{W}_{\text{ext}}(t, y) = \int_{\Omega} b(t, x) \cdot y(x) dx + \int_{\Gamma_N} \bar{t}(t, x) \cdot y(x) dS.$$

In the quasistatic case, the inertial term vanishes and the equilibrium reduces to

$$-\text{Div } P = b \text{ in } \Omega, \quad Pn = \bar{t} \text{ on } \Gamma_N, \quad y = y_D \text{ on } \Gamma_D.$$

This equilibrium is equivalently characterized by the stationarity of the total potential energy

$$\Pi(t, y; F_p) = \int_{\Omega} W(F_e) dx - \mathcal{W}_{\text{ext}}(t, y),$$

namely the y -Euler–Lagrange condition $\delta_y \Pi(t, y; F_p) = 0$ with the above natural boundary terms.

Yield Condition and Plastic Incompressibility Plastic flow is activated when the stress state reaches a yield threshold. We define the Mandel stress in the intermediate configuration by

$$M := F_e^\top P_e$$

A von Mises-type yield criterion is then imposed on its deviatoric part:

$$\|M^{\text{dev}}\| \leq \sigma_y,$$

where $\sigma_y > 0$ denotes the yield stress. Here M^{dev} is the deviatoric part of M .

Dissipation Potential and Flow Rule The evolution of the plastic variable F_p is governed by a dissipation potential

$$\mathcal{R}(\xi) = \begin{cases} \sigma_y \|\xi^{\text{dev}}\|, & \text{if } \text{tr}(\xi) = 0, \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.1)$$

which is convex and positively one-homogeneous. These properties ensure that the model is rate-independent, i.e., the dissipation depends only on the total amount of plastic strain, not on the rate at which it occurs.

The plastic flow is then described by a Biot-type differential inclusion:

$$0 \in \partial \mathcal{R}(\dot{F}_p F_p^{-1}) + \frac{\partial \mathcal{E}}{\partial F_p}(F, F_p),$$

or, equivalently,

$$\partial \mathcal{R}(\xi) \ni M^{\text{dev}}.$$

This inclusion simultaneously encodes the yield condition and the flow rule in a unified variational framework.

Extensions: Hardening Models As in the small-strain setting, hardening effects can be included by allowing the yield stress σ_y to depend on internal variables such as accumulated plastic strain or a backstress tensor. The free energy W and yield condition $f(M, z)$ must then be extended accordingly to include these additional variables (Simo and Miehe, 1992).

Remark 1 (Energetic prototype). *Following Kružík and Roubíček, 2019, we explicitly state here a representative form of the total energy. This model specializes the generally introduced stored energy into a purely elastic form, characterized by its dependence on the elastic deformation gradient F_e obtained via multiplicative decomposition, together with the internal variables. The total energy functional is then defined as*

$$\mathcal{E}(t, F_e, z) = \int_{\Omega} \psi(F_e(x), z(x)) \, dx - \mathcal{W}_{\text{ext}}(t, y),$$

where

- ψ is the elastic energy density function, satisfying frame indifference. For isotropic materials, ψ depends only on the right Cauchy–Green tensor $C_e = F_e^\top F_e$ and on $\det F_e$.
- $\mathcal{W}_{\text{ext}}(t, y)$ denotes the work of external forces, which depends on the time t . Its time variation directly influences the evolution of the system.

This formulation allows both the internal state and the action of external forces to be represented within a unified energetic framework, forming the basis for the evolution laws and variational formulations.

In summary, the Kröner–Lee multiplicative decomposition provides a geometrically consistent foundation for finite-strain elastoplasticity. Combined with energy storage and rate-independent dissipation, it yields a unified variational framework for complex material behavior.

Within this framework, the momentum balance transmits the external loads $b, \bar{t}$ into the internal stress P , thereby raising the stress level in the body. Plastic flow, however, is not activated immediately; it occurs only once the stress state reaches the yield threshold introduced above, namely $\|M_{\text{dev}}\| = \sigma_y$ involving the Mandel stress. This condition represents the finite-strain analogue of the small-strain yield criterion discussed in §1.1. Thus, external forces act through the equilibrium equations to drive the system towards the yield surface, beyond which plastic evolution is triggered.

2.3 Finite-Strain Crystal Plasticity Model

In the previous section, we established the general framework of finite-strain elastoplasticity based on the Kröner–Lee multiplicative decomposition $F = F_e F_p$. This formulation provides a geometrically consistent separation of elastic and plastic effects, and is applicable to materials undergoing large deformations, finite rotations, and nonlinear geometry.

A particularly important class of such materials is crystalline solids, whose plastic deformation is primarily driven by crystallographic slip on specific slip systems. The resulting anisotropy, coupled with the multiplicative decomposition, requires a constitutive description that resolves the kinematics of slip at the crystal level and its interaction with the elastic lattice distortion. Finite-strain crystal plasticity theory offers such a framework, extending the general multiplicative decomposition to incorporate slip-system level flow rules, hardening laws, and lattice rotation effects (Asaro, 1983; Roters et al., 2010; Simo and Miehe, 1992; Miehe, 1998).

In the following, we outline the basic kinematic relations, energetic structure, and dissipative mechanisms of a finite-strain crystal plasticity model, which will serve as the constitutive setting for the main results of this work.

Kinematics and Plastic Incompressibility We consider N_s crystallographic slip systems (s_α, m_α) in the intermediate configuration, with unit slip directions s_α and unit slip-plane normals m_α ($s_\alpha \perp m_\alpha$). The plastic velocity gradient is

$$L_p := \dot{F}_p F_p^{-1} = \sum_{\alpha=1}^{N_s} \dot{\gamma}_\alpha s_\alpha \otimes m_\alpha,$$

where $\dot{\gamma}_\alpha$ denotes the slip rate on system α .

Consistent with the finite-strain framework, we impose plastic incompressibility. Using $\text{tr}(s_\alpha \otimes m_\alpha) = s_\alpha \cdot m_\alpha = 0$ for each Schmid tensor, we have

$$\text{tr } L_p = \sum_{\alpha=1}^{N_s} \dot{\gamma}_\alpha (s_\alpha \cdot m_\alpha) = 0, \quad \frac{d}{dt} \log \det F_p = \text{tr } L_p = 0.$$

Hence $\det F_p(t) = \det F_p(0)$ for all t ; prescribing $\det F_p(0) = 1$ yields $\det F_p(t) \equiv 1$, reflecting that crystallographic slip in metals does not produce volumetric change (see (2.1)).

Single-Slip Activation (assumption for the main result) While the above kinematic description allows for multiple slip systems to operate, the analysis in this thesis adopts the single-slip activation assumption: for almost every $X \in \Omega$ and $t \in (0, T)$,

$$\#\{\alpha \in \{1, \dots, N_s\} : \dot{\gamma}_\alpha \neq 0\} \leq 1,$$

This assumption does not restrict the number of available slip systems N_s ; it only forbids simultaneous activation of more than one system. This setting is physically justified (Franciosi, Berveiller, and Zaoui, 1980; Franciosi, 1985).

Exponential Map and First-Order Approximation In continuous time, the evolution equation $\dot{F}_p = L_p F_p$ can be formally integrated to obtain

$$F_p(t_{k+1}) = \exp\left(\int_{t_k}^{t_{k+1}} L_p(\tau) d\tau\right) F_p(t_k),$$

where the exponential map represents the exact time integration of L_p . In numerical implementations, this continuous evolution must be discretized. While the exponential map provides a geometrically exact update, it is computationally more demanding. In this work, we adopt a first-order approximation (valid for small slip increments)

$$F_p^{k+1} \approx \left(I + \sum_{\alpha=1}^{N_s} \Delta\gamma_\alpha s_\alpha \otimes m_\alpha\right) F_p^k,$$

where $\Delta\gamma_\alpha$ denotes the slip increment in step $k \rightarrow k+1$. This scheme is widely used not only in static incremental analysis but also in dynamic crystal plasticity and Rate-Independent formulations discussed later.

Energetic Structure The total stored energy is decomposed into an elastic part and a hardening part as

$$\mathcal{E}(t, y, F_p, \gamma) = \underbrace{\int_{\Omega} W(F F_p^{-1}) dx}_{\text{elastic energy}} + \underbrace{\int_{\Omega} H(\gamma) dx}_{\text{hardening energy}} - \mathcal{W}_{\text{ext}}(t, y),$$

where $\gamma = (\gamma_\alpha)_{\alpha=1}^{N_s}$ denotes the accumulated slip on each system. We explicitly include a hardening potential $H(\gamma)$ in the energy functional; this choice highlights the energetic-dissipative structure. In classical crystal plasticity, hardening is often introduced directly through the yield function without an explicit energy term.

An example of isotropic hardening is the quadratic form

$$H(\gamma) = \frac{1}{2} \sum_{\alpha=1}^{N_s} h_\alpha \gamma_\alpha^2,$$

with positive hardening moduli h_α .

Dissipation Potential The irreversible character of plastic deformation is modeled via a convex, positively one-homogeneous dissipation potential

$$\mathcal{R}(\dot{\gamma}) = \int_{\Omega} \sum_{\alpha=1}^{N_s} \tau_{c,\alpha}(\gamma) |\dot{\gamma}_{\alpha}| dx,$$

where $\tau_{c,\alpha}(\gamma)$ denotes the current critical resolved shear stress on slip system α , possibly depending on the accumulated slip through a hardening law.

Evolution Law The evolution of the internal variables γ follows from the balance between the thermodynamic driving forces (e.g., resolved shear stresses τ_{α} derived from the Mandel stress) and the subdifferential of the dissipation potential:

$$0 \in \partial \mathcal{R}(\dot{\gamma}) + \frac{\partial \mathcal{E}}{\partial \gamma}.$$

This variational inclusion simultaneously encodes the yield condition, the flow rule, and the hardening behavior.

Remark 2. In the main results of this work, we specialize to the case where the elastic part F_e is a pure rotation, so that $W \equiv 0$ and there is no elastic energy. The hardening potential is taken as

$$H(\gamma) = \gamma_1^2 + \gamma_2^2,$$

a simple quadratic form. This setting is a particular case of the general crystal plasticity model.

2.4 Rate-Independent Structure of Elastoplasticity

We show that both small- and finite-strain elastoplasticity admit a unified rate-independent formulation driven by two ingredients: (i) a time-dependent stored energy $\mathcal{E}(t, \cdot)$ and (ii) a convex, lower-semicontinuous, positively one-homogeneous dissipation potential $\mathcal{R}$. Precise definitions and the energetic solution concept are deferred to the next chapter; here we record the minimal structure and explain how the classical models fit into it. This RIS viewpoint is adopted because it is mathematically tractable: it is invariant under time reparameterizations, supports time-incremental minimization, and will be the backbone for compactness and Γ -convergence arguments used later.

Let the state be (y, z) , where y denotes the placement (displacement in the small-strain regime) and z collects internal variables (e.g. ε_p in small strain or F_p in finite strain). We write the total energy as

$$\mathcal{E}(t, y, z) := \mathcal{E}_{\text{int}}(y, z) - \mathcal{W}_{\text{ext}}(t, y),$$

and assume that the dissipation potential $\mathcal{R}: Z \rightarrow [0, \infty]$ is convex, lower semicontinuous, positively one-homogeneous, and independent of y (the elastic variable is non-dissipative). The positive one-homogeneity

$$\mathcal{R}(\lambda \dot{z}) = \lambda \mathcal{R}(\dot{z}) \quad \text{for all } \lambda \geq 0$$

implies invariance with respect to time reparameterizations and underlies the description of hysteresis, path dependence, and strain localization (Mielke, 2003; Mielke and Roubíček, 2015).

In the quasi-static RIS setting, the evolution is governed by the pair

(quasi-static balance)	$0 = \partial_y \mathcal{E}(t, y, z),$
(flow rule)	$0 \in \partial \mathcal{R}(\dot{z}) + \partial_z \mathcal{E}(t, y, z).$

Remark 3 (Quasi-static vs. inertia). *The “true” balance including inertia reads $\rho \ddot{y} + \partial_y \mathcal{E}(t, y, z) = 0$. Retaining inertia breaks time-reparameterization invariance (only the constitutive dissipation remains rate-independent) and leads to a different framework, often termed a “rate-independent system with inertia.” In this thesis we primarily adopt the quasi-static RIS; an analysis of inertial effects is presented in Chapter 6.*

The finite-strain formulation developed earlier fits the RIS template with $z = F_p$. For the small-strain model, let the internal variable be $z = \varepsilon_p$ and, for the displacement u , set $\varepsilon(u)$ and ε_e as above. We take the total energy

$$\mathcal{E}(t, u, \varepsilon_p) := \int_{\Omega} \psi(\varepsilon_e) \, dx - \langle \ell(t), u \rangle, \quad \varepsilon_e = \varepsilon(u) - \varepsilon_p,$$

where ψ is the elastic stored-energy density and $\ell(t)$ represents the external load. In the classical von Mises case with yield function $f(\sigma) = \|\sigma^{\text{dev}}\| - \sigma_y \leq 0$, a natural choice of dissipation potential is

$$\mathcal{R}(\dot{\varepsilon}_p) = \begin{cases} \sigma_y \|\dot{\varepsilon}_p^{\text{dev}}\|, & \text{if } \text{tr}(\dot{\varepsilon}_p) = 0, \\ +\infty, & \text{otherwise,} \end{cases}$$

which is convex, lower semicontinuous, positively one-homogeneous, and enforces plastic incompressibility. The RIS inclusion becomes

$$0 \in \partial \mathcal{R}(\dot{\varepsilon}_p) + \partial_{\varepsilon_p} \mathcal{E}(t, u, \varepsilon_p), \tag{2.2}$$

with stress defined by the energetic relation $\sigma := \partial_{\varepsilon_e} \psi(\varepsilon_e)$. Since $\varepsilon_e = \varepsilon(u) - \varepsilon_p$, (2.2) yields the subdifferential flow rule

$$\sigma \in \partial \mathcal{R}(\dot{\varepsilon}_p) \iff \dot{\varepsilon}_p \in \partial I_K(\sigma), \quad K := \{\tau \in \mathbb{R}_{\text{sym}}^{3 \times 3} \mid f(\tau) \leq 0\},$$

which, at smooth points of the yield surface ($f(\sigma) = 0$), is equivalent to the normality rule $\dot{\varepsilon}_p = \lambda \frac{\partial f}{\partial \sigma}$, $\lambda \geq 0$, and—together with $f(\sigma) \leq 0$ —to the Kuhn–Tucker conditions.

Elastoplasticity, in both small and finite strains, can be cast in the RIS form $(\mathcal{E}, \mathcal{R})$. This unifying variational structure clarifies the role of energy and dissipation, facilitates robust time-incremental schemes, and prepares the ground for the analytical tools developed in the next chapter.

Chapter 3

Abstract Theory of Rate-Independent Systems

In finite-strain elastoplasticity, the internal energy $\mathcal{E}$ and the dissipation distance $\mathcal{D}$ constitute the two fundamental pillars that govern the material response. These structural components are common across a broad class of elastoplastic models, including those based on small-strain theory, and the framework of Rate-Independent Systems (RIS) provides a robust mathematical foundation for describing their time-evolution. This chapter deliberately adopts a more abstract perspective and places these concepts within the framework of RIS, following the architecture of Mielke and Roubíček, 2015.

Guided by that text, we systematically review the fundamental ingredients of RIS—namely, the state space, the free-energy functional, and the dissipation distance—together with the two variational principles of energetic stability and energy balance.

RIS characterizes time evolution not through differential equations, but through a balance between the potential energetic gain and the dissipative cost required to reach a competing state. Its intrinsic incremental time discretization replaces the evolution problem by a sequence of static minimization problems,

$$(y_{n+1}, z_{n+1}) \in \arg \min_{(y,z)} \{ \mathcal{E}(t_{n+1}, y, z) + \mathcal{D}(z_n, z) \},$$

so that, on the analytical side, the compactness of the discrete solutions together with the lower semicontinuity of the functionals leads directly to the existence of energetic solutions in continuous time, while, on the computational side, each step can be handled using standard finite element or nonlinear optimization techniques.

The main result of this dissertation is a homogenization theorem for such a static minimization problem, formulated within the RIS setting and proved in a later chapter. This result represents the first step toward a rigorous treatment of the full evolutionary problem in the context of finite-strain elastoplasticity.

3.1 Introductory Examples

Before developing the abstract RIS theory, we briefly revisit two prototypical elasto-plastic models. No new modelling ingredients are added here; the aim is only to pin down the triplet $(\mathcal{E}, \mathcal{R}, q)$ that will recur throughout the chapter.

One-Dimensional Elasto-Plasticity in the RIS Framework We study a prismatic bar that undergoes isothermal, infinitesimal deformations. The total strain $\varepsilon(t) \in \mathbb{R}$ is additively decomposed into an elastic part $\varepsilon_e(t) := \varepsilon(t) - p(t)$ and a plastic strain

$p(t)$. With Young's modulus $E > 0$ and the yield stress $\sigma_y > 0$ we define the free energy and the rate-independent dissipation potential by

$$\mathcal{E}(t, \varepsilon, p) = \frac{1}{2} E (\varepsilon(t) - p(t))^2, \quad \mathcal{R}(\dot{p}) = \sigma_y |\dot{p}|.$$

Because $\mathcal{R}$ is positively one-homogeneous, $\mathcal{R}(\lambda \dot{p}) = \lambda \mathcal{R}(\dot{p})$ for every $\lambda \geq 0$, the accumulated dissipation $\mathcal{D}(p_1, p_2) = \sigma_y |p_2 - p_1|$ depends only on the increment $p_2 - p_1$ and not on the time required to reach it; the model is therefore rate-independent.

The subdifferential of $\mathcal{R}$ reads

$$\partial \mathcal{R}(v) = \begin{cases} \{\sigma_y \operatorname{sign} v\}, & v \neq 0, \\ [-\sigma_y, \sigma_y], & v = 0, \end{cases}$$

and the elastic stress is $\sigma(t) = E(\varepsilon(t) - p(t)) = -D_p \mathcal{E}$. Balancing these terms yields the rate-independent inclusion

$$0 \in \partial \mathcal{R}(\dot{p}) + D_p \mathcal{E}(t, \varepsilon, p) \iff \partial \mathcal{R}(\dot{p}) - \sigma(t) \ni 0, \quad \sigma(t) = E(\varepsilon(t) - p(t)). \quad (3.1)$$

Equation (3.1) simultaneously captures

$$|\sigma| < \sigma_y \implies \dot{p} = 0 \quad (\text{elastic regime}), \quad |\sigma| = \sigma_y \implies \dot{p} \neq 0 \quad (\text{plastic flow}),$$

thus providing a concise illustration of the Biot form $\partial \mathcal{R}(\dot{q}) + D_q \mathcal{E}(t, q) \ni 0$ that underpins rate-independent systems. See §2.4, for the derivation and energetic interpretation of the Biot inclusion (2.2). Here q denotes the collection of internal state variables; in the above example $q = p$, while in the finite-strain example below we take $q = (y, F_p)$.

Finite-Strain Plasticity in the RIS Framework In analogy with the small-strain elasto-plastic bar, finite-strain elasto-plastic models based on the Kröner–Lee multiplicative decomposition

$$F = F_e F_p$$

naturally fit into the framework of rate-independent systems (RIS). We define the deformation map $y : \Omega \rightarrow \mathbb{R}^3$ (with $F = \nabla y$) and the state space

$$q = (y, F_p) \in \mathcal{Q} := \{ (y, F_p) \mid y \in \mathcal{Y}, F_p \in \operatorname{GL}^+(3) \}.$$

The free energy is

$$\mathcal{E}(t, y, F_p) = \int_{\Omega} W(F F_p^{-1}) dx - \mathcal{W}_{\text{ext}}(t, y),$$

where W is a frame-indifferent elastic energy density depending on the elastic deformation $F_e = F F_p^{-1}$, assumed sufficiently convex for well-posedness.

Plastic flow is measured by the right logarithmic derivative

$$\xi = \dot{F}_p F_p^{-1} \in \mathfrak{gl}(3),$$

and plastic incompressibility $\text{tr } \xi = 0$ is enforced via the rate-independent dissipation potential

$$\mathcal{R}(\xi) = \begin{cases} \sigma_y \|\xi^{\text{dev}}\|, & \text{tr } \xi = 0, \\ +\infty, & \text{tr } \xi \neq 0, \end{cases}$$

where $\sigma_y > 0$ denotes the yield stress. Since $\mathcal{R}$ is positively one-homogeneous, the model is invariant under time rescaling and hence rate-independent.

The Biot inclusion

$$\partial \mathcal{R}(\dot{q}) + D_q \mathcal{E}(t, q) \ni 0$$

splits into the mechanical equilibrium and plastic flow laws:

$$\begin{cases} \partial_y \mathcal{E}(t, y, F_p) = 0 & \text{(quasi-static mechanical equilibrium),} \\ 0 \in \partial \mathcal{R}(\dot{F}_p F_p^{-1}) + D_{F_p} \mathcal{E}(t, y, F_p) & \text{(plastic flow).} \end{cases}$$

Remark 4. In both of these models—the small-strain bar model and the finite-strain formulation—the setting is quasi-static: neither inertial effects ($M\ddot{q}$) nor viscous dissipation ($F(\dot{q})$) are taken into account, where q denotes the collection of generalized state variables (e.g. displacements y and internal variables z). A more comprehensive model that incorporates these dynamic and viscous effects within the rate-independent framework will be developed in a later chapter.

3.2 Abstract Framework and Definitions

In this paper, we adopt the triplet $(\mathcal{Q}, \mathcal{E}, \mathcal{R})$, comprising the state space $\mathcal{Q}$, the time-dependent energy functional $\mathcal{E}(t, q)$, and the dissipation potential $\mathcal{R}$. Here $t \in [0, T]$ denotes the time parameter, and $q \in \mathcal{Q}$ represents the state variable of the system, which can include, depending on the model under consideration, both observable quantities (such as displacements) and internal variables (such as plastic strains). The governing equation is written in the unified Biot form

$$\partial \mathcal{R}(\dot{q}) + D_q \mathcal{E}(t, q) \ni 0.$$

Throughout this work, we build the various notions of solutions upon this triplet and analyze, by variational methods, their existence, possible uniqueness, and long-time behavior.

3.2.1 Axioms and the Mielke–Roubíček System

The preceding heuristic discussion motivates the introduction of the differential inclusion as a direct model for quasi-static, rate-independent material behavior. This formulation captures the essential property of rate-independence, in the sense that rescaling time does not affect the resulting evolution. At the same time, however, one must recognize that the model introduces the following mathematical difficulties:

- (a) Solutions are in general not unique; jump phenomena and hysteresis loops may occur even for smooth loadings ℓ .

- (b) Classical ODE theory no longer applies because the subdifferential $\partial\mathcal{R}$ is multi-valued and merely positively one-homogeneous.

To handle these issues one must decouple the structural axioms of a rate-independent system from any particular equation. Definition 1 below (Def. 1.2.1 in Mielke and Roubíček, 2015) does precisely that. Instead of prescribing a single trajectory, it assigns to every triplet $([t_1, t_2], q_1, \ell)$ a possibly non-singleton solution set $\mathcal{O}([t_1, t_2], q_1, \ell)$. The definition imposes five elementary requirements:

- (i) **Admissibility:** every trajectory in the set starts at the prescribed initial datum q_1 .
- (ii) **Rate-independence:** re-parametrising time leaves the graph of each trajectory invariant.
- (iii) **Concatenation Closure:** two successive solution pieces can be glued together and still form a solution.
- (iv) **Restriction Closure:** every time sub-interval of a solution is again a solution for the restricted data.
- (v) **Causality:** the current state depends solely on the past history of the loading ℓ .

Together these axioms provide an abstract system that is rich enough to accommodate numerous concrete models—plasticity, damage, fracture, dry friction—yet is general enough to admit variational existence results that do not rely on particular smoothness properties. They also separate the genuinely rate-independent content (ii) from additional structural assumptions (iii)–(v) that are indispensable once non-uniqueness enters the picture.

With this preparation we can now formalize the notion of an evolutionary system in the sense of Mielke–Roubíček. We introduce the following notation that will be used in the formulation of the evolutionary system. First, we fix the time interval $[0, T]$ and denote by

$$\mathcal{J}_{[0, T]} := \{ [t_1, t_2] \subset [0, T] \mid t_2 > t_1 \}$$

the collection of all non-degenerate closed subintervals.

The external loading will be a function $\ell \in F_0([0, T]; \mathcal{X})$, with $\mathcal{X}$ the loading space (F_0 being the function space for external loadings), and state trajectories will lie in $F_1([0, T]; \mathcal{Q})$, with $\mathcal{Q}$ the state space (F_1 being the function space for state trajectories). When $\mathcal{X}$ and $\mathcal{Q}$ are endowed with suitable metric structures, typical choices are $F_0 = \text{AC}([0, T]; \mathcal{X})$ and $F_1 = \text{BV}_d([0, T]; \mathcal{Q})$.

Given an initial condition $q_1 \in \mathcal{Q}$ at time t_1 and a loading $\ell \in F_0([0, T]; \mathcal{X})$, we denote by

$$\mathcal{O}([t_1, t_2], q_1, \ell) \subset F_1([t_1, t_2]; \mathcal{Q})$$

the set of all admissible trajectories over the interval $[t_1, t_2]$ starting from q_1 under the loading ℓ .

With this minimal notation in place, we proceed to define the notion of an evolutionary system.

Definition 1 (Rate-independent system). (Mielke and Roubíček, 2015, Def. 1.2.1) Consider an input-output system $\mathcal{O}$ from $\mathcal{Q} \times F_0([0, T]; \mathcal{X})$ to $F_1([0, T]; \mathcal{Q})$.

- (i) $\mathcal{O}$ is called admissible if $q(t_1) = q_1$ for all $q \in \mathcal{O}([t_1, t_2]; q_1, \ell)$.

- (ii) $\mathcal{O}$ is called rate-independent if for all $([t_1, t_2], q_1, \ell)$ and all strictly increasing continuous time reparameterizations $\alpha : [t_1, t_2] \rightarrow [t_1^*, t_2^*] \in \mathcal{J}_{[0, T]}$ with $\alpha(t_1) = t_1^*$ and $\alpha(t_2) = t_2^*$, the following relation holds:

$$q \in \mathcal{O}([t_1, t_2], q_1, \ell) \iff q \circ \alpha \in \mathcal{O}([t_1^*, t_2^*], q_1, \ell \circ \alpha).$$

- (iii) $\mathcal{O}$ is said to have the concatenation property, if

$$\begin{aligned} \hat{q} \in \mathcal{O}([t_1, t_2], q_1, \ell), \tilde{q} \in \mathcal{O}([t_2, t_3], \hat{q}(t_2), \ell), \\ \implies \hat{q} \bowtie \tilde{q} \in \mathcal{O}([t_1, t_3], q_1, \ell), \text{ where } (\hat{q} \bowtie \tilde{q})(t) = \begin{cases} \hat{q}(t) & \text{for } t \in [t_1, t_2], \\ \tilde{q}(t) & \text{for } t \in [t_2, t_3]. \end{cases} \end{aligned}$$

- (iv) $\mathcal{O}$ is said to have the restriction property, if

$$t_1 \leq t_2 < t_3 \leq t_4 \text{ and } q \in \mathcal{O}([t_1, t_4], q_1, \ell) \implies q|_{[t_2, t_3]} \in \mathcal{O}([t_2, t_3], q(t_2), \ell).$$

- (v) $\mathcal{O}$ is called causal, if

$$t_1 < t_2 \leq T \text{ and } \ell_1|_{[t_1, t_2]} = \ell_2|_{[t_1, t_2]} \implies \mathcal{O}([t_1, t_2], q_1, \ell_1) = \mathcal{O}([t_1, t_2], q_1, \ell_2).$$

3.2.2 Energetic Solutions and Stability

We start the analysis of RIS by introducing the convex dissipation potential

$$\mathcal{R} : \mathcal{Q} \longrightarrow [0, \infty),$$

whose convex subdifferential is defined by

$$\partial \mathcal{R}(v) := \{ \xi \in \mathcal{Q}^* \mid \mathcal{R}(w) \geq \mathcal{R}(v) + \langle \xi, w - v \rangle \quad \forall w \in \mathcal{Q} \}.$$

Combining this with the gradient of the stored energy $\mathcal{E}(t, q)$ leads to the multivalued inclusion which describes the dynamics of the rate-independent model in its most direct form mentioned above:

$$0 \in \partial \mathcal{R}(\dot{q}(t)) + D_q \mathcal{E}(t, q(t)). \quad (3.2)$$

The inclusion is, however, technically difficult to handle. Invoking the standard first-order optimality rule for convex functions,

$$\xi \in \partial \varphi(x) \iff \varphi(y) \geq \varphi(x) + \langle \xi, y - x \rangle \quad \forall y,$$

we find that (3.2) is equivalent, for almost every $t \in [0, T]$, to the evolutionary variational inequality

$$\langle D_q \mathcal{E}(t, q(t)), w - \dot{q}(t) \rangle + \mathcal{R}(w) - \mathcal{R}(\dot{q}(t)) \geq 0 \quad \forall w \in \mathcal{Q}. \quad (3.3)$$

This conversion relies only on convexity; the specific feature of rate-independence has not yet entered.

At this point we exploit the defining structure of a rate-independent model, namely the positive one-homogeneity

$$\mathcal{R}(\lambda v) = \lambda \mathcal{R}(v) \quad (\lambda > 0).$$

One-homogeneity endows the subdifferential $\partial\mathcal{R}$ with a cone structure and allows for a sharper evaluation of the dissipation term $\mathcal{R}(w) - \mathcal{R}(\dot{q})$ in (3.3).

Lemma 1. (Mielke and Roubíček, 2015, Lem. 1.3.1)

Let $\mathcal{R} : \mathcal{Q} \rightarrow [0, \infty]$ be lower semicontinuous, convex, and 1-homogeneous. Then

$$\partial\mathcal{R}(v) = \{\eta \in \Sigma \mid \mathcal{R}(v) = \langle \eta, v \rangle\}, \quad \text{where } \Sigma = \partial\mathcal{R}(0)$$

Moreover, we have the characterization

$$\xi \in \partial\mathcal{R}(v) \iff (\forall w \in \mathcal{Q} : \mathcal{R}(w) - \langle \xi, w \rangle \geq \mathcal{R}(v) - \langle \xi, v \rangle = 0).$$

Now we assume that the dissipation distance $\mathcal{D} : \mathcal{Q} \times \mathcal{Q} \rightarrow [0, \infty)$ is generated by dissipation potential $\mathcal{R} : \mathcal{Q} \rightarrow [0, \infty)$ according to

$$\mathcal{D}(z_1, z_2) = \mathcal{R}(z_2 - z_1), \quad z_1, z_2 \in \mathcal{Q}.$$

For simplicity, let us assume that $\mathcal{E}(t, \cdot)$ is convex, though not necessarily differentiable. Here, we interpret the second term on the right-hand side of (3.2) in the sense of the subdifferential, and denote it by ∂ . Later, in the case where (Gâteaux) differentiability is available, we will use D to distinguish it; however, unless otherwise stated, we will henceforth restrict ourselves to the differentiable case and use the symbol ∂ throughout. From (3.2), there exists $v \in \mathcal{Q}$ such that

$$v \in \partial\mathcal{R}(\dot{q}(t)) \quad \text{and} \quad -v \in \partial_q \mathcal{E}(t, q(t)),$$

which yields

$$\langle v, \tilde{q} - q(t) \rangle + \mathcal{E}(t, \tilde{q}) - \mathcal{E}(t, q(t)) \geq 0 \quad \forall \tilde{q} \in \mathcal{Q}.$$

By Lemma 1, it follows that $v \in \partial\mathcal{R}(\dot{q}(t)) \subset \Sigma$. On the other hand, we can estimate by Lemma 1

$$\langle v, \tilde{q} - q(t) \rangle \leq \sup_{\eta \in \Sigma} \langle \eta, \tilde{q} - q(t) \rangle \leq \mathcal{R}(\tilde{q} - q(t)) = \mathcal{D}(q(t), \tilde{q}).$$

Combining these observations, we arrive at the global stability condition

$$\mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \tilde{q}) + \mathcal{D}(q(t), \tilde{q}) \quad \forall \tilde{q} \in \mathcal{Q}. \quad (\text{S})$$

Condition (S) characterises an instantaneous energetic equilibrium without appealing to any differential formulation and hence reflects the time-scale separation that is typical for rate-independent models. Thus, Lemma 1 unifies the two descriptions—differential inclusion and energetic stability—and will serve as an indispensable bridge in the existence theory.

However, the stability condition alone is not sufficient to determine a unique evolution. This is because it contains no information on the work done by the external loading or on the dissipated energy accumulated over time.

Assume that $\mathcal{E}$ is differentiable and the (Gâteaux) chain rule applies. Then, from (3.2)

$$-D_q \mathcal{E}(t, q(t)) = v(t) \in \partial\mathcal{R}(\dot{q}(t)).$$

For $\dot{q}(t)$ and this $v(t) \in \partial\mathcal{R}(\dot{q}(t))$, Lemma 1 implies

$$\mathcal{R}(\dot{q}(t)) = \langle v(t), \dot{q}(t) \rangle. \quad (3.4)$$

By the chain rule,

$$\frac{d}{dt}\mathcal{E}(t, q(t)) = D_t\mathcal{E}(t, q(t)) + \langle D_q\mathcal{E}(t, q(t)), \dot{q}(t) \rangle = D_t\mathcal{E}(t, q(t)) - \langle v(t), \dot{q}(t) \rangle.$$

Substituting (3.4) yields

$$\frac{d}{dt}\mathcal{E}(t, q(t)) = D_t\mathcal{E}(t, q(t)) - \mathcal{R}(\dot{q}(t)).$$

Integrating this identity from 0 to t , we obtain

$$\mathcal{E}(t, q(t)) + \int_0^t \mathcal{R}(\dot{q}(s)) \, ds = \mathcal{E}(0, q(0)) - \int_0^t D_s\mathcal{E}(s, q(s)) \, ds, \quad (E)$$

which is the energy identity. The last term, $-\int_0^t D_s\mathcal{E}(s, q(s)) \, ds$, represents the work input due to time-dependent external loading (e.g., varying forces or boundary conditions). Due to the one-homogeneity of $\mathcal{R}$, this expression is invariant under time reparameterisation and thus preserves the rate-independent character of the model.

Taken together,

$$(S) \text{ “energetic stability”} \quad \text{and} \quad (E) \text{ “energy balance”}$$

form the two pillars of the energetic solution concept. Throughout we work in a quasi-static, rate-independent setting in which inertial and viscous effects are neglected; the absence of velocities and accelerations thus reflects a modelling choice rather than a physical claim. In this regime, solutions may exhibit jumps and the framework accommodates history-dependent phenomena such as plasticity, damage, or brittle fracture. To give a precise definition of an energetic solution, we impose the following conditions.

First, we decompose the basic state space $\mathcal{Q}$ into fast and slow components,

$$\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}, \quad (3.5)$$

where the state $q = (y, z)$ consists of a non-dissipative part y and a dissipative part z . The state space $\mathcal{Q}$ is equipped with a Hausdorff topology $\mathcal{T}_{\mathcal{Q}} = \mathcal{T}_{\mathcal{Y}} \times \mathcal{T}_{\mathcal{Z}}$. We denote convergence of sequences by $q_k \xrightarrow{\mathcal{Q}} q$, $y_k \xrightarrow{\mathcal{Y}} y$, and $z_k \xrightarrow{\mathcal{Z}} z$.

Stored Energy The stored-energy functional $\mathcal{E} : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R}_{\infty}$ depends on time $t \in [0, T]$ and state $q \in \mathcal{Q}$, and takes values in the extended reals $\mathbb{R} \cup \{\infty\}$. Here, the time variable t represents a very slow process time that modifies the system via changing loading conditions. The (effective) domain of $\mathcal{E}$ is

$$\text{Dom } \mathcal{E} := \{(t, q) \in [0, T] \times \mathcal{Q} \mid \mathcal{E}(t, q) < \infty\}, \quad \text{Dom } \mathcal{E}(t, \cdot) := \{q \in \mathcal{Q} \mid \mathcal{E}(t, q) < \infty\}.$$

Our standing assumptions on $\mathcal{E}$ are:

(E1) *Compactness of sublevels:* For every $t \in [0, T]$ and $C \in \mathbb{R}$, the sublevel set

$$\{q \in \mathcal{Q} \mid \mathcal{E}(t, q) \leq C\}$$

is compact in $\mathcal{Q}$.

(E2) *Energetic control of power*: There exist $c_{\mathcal{E}} \in \mathbb{R}$, $\lambda_{\mathcal{E}} \in L^1(0, T)$, and a null set $N_{\mathcal{E}} \subset [0, T]$ (i.e. $\mathcal{L}^1(N_{\mathcal{E}}) = 0$) such that:

- $\text{Dom } \mathcal{E} = [0, T] \times \text{Dom } \mathcal{E}(0, \cdot)$;
- for all $q \in \text{Dom } \mathcal{E}(0, \cdot)$ we have $\mathcal{E}(\cdot, q) \in W^{1,1}(0, T)$, the partial derivative $\partial_t \mathcal{E}(t, q)$ exists for $t \in [0, T] \setminus N_{\mathcal{E}}$, and

$$|\partial_t \mathcal{E}(t, q)| \leq \lambda_{\mathcal{E}}(t) (\mathcal{E}(t, q) + c_{\mathcal{E}}) \quad \text{for a.e. } t \in (0, T).$$

Dissipation Distance We consider a map $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, \infty]$ satisfying:

(D1) $\mathcal{D}$ is an extended pseudo-metric on $\mathcal{Z}$, i.e.:

- $\mathcal{D}(z_1, z_2) = 0$ if and only if $z_1 = z_2$ for all $z_1, z_2 \in \mathcal{Z}$;
- $\mathcal{D}(z_1, z_3) \leq \mathcal{D}(z_1, z_2) + \mathcal{D}(z_2, z_3)$ for all $z_1, z_2, z_3 \in \mathcal{Z}$ (triangle inequality).

(D2) $\mathcal{D}$ is lower semicontinuous on $\mathcal{Z} \times \mathcal{Z}$.

Total Dissipation For a given curve $z : [0, T] \rightarrow \mathcal{Z}$, we define the total dissipation on $[s, t]$ via

$$\text{Diss}_{\mathcal{D}}(z; [s, t]) := \sup \left\{ \sum_{j=1}^N \mathcal{D}(z(t_{j-1}), z(t_j)) \mid N \in \mathbb{N}, s = t_0 < t_1 < \dots < t_N = t \right\}$$

The functions are defined everywhere, and changing them at one point may increase the dissipation. Moreover, the dissipation is additive:

$$\text{Diss}_{\mathcal{D}}(z; [r, t]) = \text{Diss}_{\mathcal{D}}(z; [r, s]) + \text{Diss}_{\mathcal{D}}(z; [s, t]) \quad \text{for all } r < s < t.$$

Under the assumptions (E1)–(E2) and (D1)–(D2), we are prepared to define the notion of energetic solutions.

Definition 2 (Energetic solution). (Mielke and Roubíček, 2015, Def. 2.1.2) A function $q = (y, z) : [0, T] \rightarrow Q = Y \times Z$ is called an energetic solution of the rate-independent system $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ if $t \mapsto \partial_t \mathcal{E}(t, q(t))$ is integrable and if the global stability (S) and the energy balance (E) hold for all $t \in [0, T]$:

$$\forall \hat{q} = (\hat{y}, \hat{z}) \in Q : \quad \mathcal{E}(t, q(t)) \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(z(t), \hat{z}) \quad (\text{S})$$

$$\mathcal{E}(t, q(t)) + \text{Diss}_{\mathcal{D}}(z; [0, t]) = \mathcal{E}(0, q(0)) + \int_0^t \partial_t \mathcal{E}(\tau, q(\tau)) d\tau \quad (\text{E})$$

For subintervals $[t_0, t_1] \subset [0, T]$, we call $q : [t_0, t_1] \rightarrow Q$ an energetic solution of $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ if (S) and (E) hold for all $t \in [t_0, t_1]$, where the initial time 0 in (E) is replaced by t_0 .

Because this framework does not require differentiability of trajectories, it is well-suited to describing rate-independent phenomena with jumps, such as plasticity, damage, or brittle fracture. It will serve as the analytic foundation for the analysis developed in the subsequent chapters of the thesis.

The stability condition (S) is global in all of $\mathcal{Q}$, and it can be rephrased by defining the set $\mathcal{S}(t)$ of stable states at time t via

$$\mathcal{S}(t) := \{q \in \mathcal{Q} \mid \mathcal{E}(t, q) < \infty, \mathcal{E}(t, q) \leq \mathcal{E}(t, \hat{q}) + \mathcal{D}(q, \hat{q}) \text{ for all } \hat{q} \in \mathcal{Q}\}.$$

We call $\mathcal{S}(t)$ the stability set at time t for short. Then (S) simply means that $q(t) \in \mathcal{S}(t)$ for all $t \in [0, T]$. Then, we are now ready to formulate the central existence theorem for energetic solutions.

Theorem 1 (Existence of energetic solutions). (*Mielke and Roubíček, 2015, Thm. 2.1.6*)
Let $(\mathcal{Q}, \mathcal{E}, \mathcal{D})$ be a rate-independent system on the time interval $[0, T]$ and suppose that

- (I) the dissipation distance $\mathcal{D}$ fulfils (D1)–(D2);
- (II) the stored energy $\mathcal{E}$ fulfils (E1)–(E2);
- (III) the following compatibility conditions hold for every stable sequence $(t_m, q_m)_{m \in \mathbb{N}} \subset [0, T] \times \mathcal{Q}$ with $(t_m, q_m) \rightarrow (t, q)$ and $q_m \in \mathcal{S}(t_m)$:

$$t \in [0, T] \setminus N_{\mathcal{E}} \implies \partial_t \mathcal{E}(t, q) = \lim_{m \rightarrow \infty} \partial_t \mathcal{E}(t_m, q_m), \quad (\text{C1})$$

$$q \in \mathcal{S}(t), \quad (\text{C2})$$

where $N_{\mathcal{E}} \subset [0, T]$ is the measure-zero exceptional set from (E2).

- (IV) the topology of $\mathcal{Q}$ restricted to compact sets is separable and metrizable.

Then, for every initial datum $q_0 \in \mathcal{S}(0)$, there exists an energetic solution $q = (y, z) : [0, T] \rightarrow \mathcal{Q}$ to the initial-value problem $(\mathcal{Q}, \mathcal{E}, \mathcal{D}, q_0)$. The mapping q is measurable and satisfies, for almost every $t \in [0, T]$,

$$\partial_t \mathcal{E}(t, q(t)) = \partial_t \mathcal{E}(t, y(t), z(t)) = \mathcal{P}_{\text{red}}(t, z(t)),$$

where, for fixed $(t, z) \in [0, T] \times Z$, we introduce the reduced power

$$\mathcal{P}_{\text{red}}(t, z) := \sup_{y \in \arg \min_{y' \in Y} \mathcal{E}(t, y', z)} \partial_t \mathcal{E}(t, y, z). \quad (3.6)$$

3.2.3 Energetic Solution Existence via Incremental Scheme

The remainder of this subsection sketches the proof strategy. Instead of presenting every technical detail, we summarise the key ideas of Mielke–Roubíček’s discrete-to-continuous argument in six steps.

First, the time interval $[0, T]$ is subdivided by a partition $\tau = \{0 = t_0 < t_1 < \dots < t_N = T\}$. A discrete sequence $\{q_k\}_{k=0}^N$ is then generated recursively by solving at each time node the Incremental Minimisation Problem (IMP)

$$q_k \in \arg \min_{q \in \mathcal{Q}} \{\mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q)\}, \quad (k = 1, \dots, N). \quad (3.7)$$

Thus each discrete state is obtained by minimising the sum of the current energy and the dissipative cost of transition from the previous state. This construction forms the foundation of the existence theory.

Helly’s selection principle combined with the Arzelà–Ascoli theorem provides compactness: after introducing a suitable time interpolation of the discrete solutions

(e.g., piecewise constant or piecewise affine on each subinterval), a subsequence converges to a limit curve q (Step 2). Preservation of the stability condition (S) in the limit is ensured by the compatibility hypotheses (C1) and (C2) (Step 3). Passing to the limit in the discrete energy balance gives the upper energy inequality, whereas stability yields the complementary lower energy inequality (Steps 4–5). Since the two bounds coincide, the continuous energy equality (E) is recovered. Consequently, the limit curve q satisfies both (S) and (E) and is therefore an energetic solution (Step 6).

If, in addition, the minimiser of $\mathcal{E}(t, \cdot, z)$ is unique for every stable state, the convergence of the discrete solutions improves from weak convergence to strong convergence, and uniqueness of the energetic solution can be concluded.

The interplay of these six steps is summarised concisely in Table 3.1. We emphasise that the six-step discrete-to-continuous scheme outlined above is prototypical for existence proofs in rate-independent systems: almost all contemporary variants—including finite-strain plasticity models—follow the same pattern of incremental minimisation, BV-type compactness, and two-sided energy inequalities. Hence the blueprint presented here can be reused virtually unchanged when we examine more specialized system.

This variational structure not only serves as the foundation for the existence theory, but also plays a central role in the subsequent chapters of this dissertation, where we study the homogenization and multiscale behavior of such incremental problems.

Thus, we stress that the IMP is more than a numerical approximation scheme—it is a unifying principle that underpins both the analytical and computational aspects of rate-independent systems, and it provides the conceptual bridge to the static problems addressed later in this work.

TABLE 3.1: Outline of Steps 0–6 in the proof of the main existence theorem

Step	Keyword	Summary	Remarks
0	Time discretization & IMP	Solve an Incremental Minimization Problem at each time-step.	Fix a partition $\tau = \{0 = t_0 < \dots < t_N = T\}$ and compute $q_k \in \arg \min_{q \in \mathcal{Q}} \{\mathcal{E}(t_k, q) + \mathcal{D}(q_{k-1}, q)\}.$
1	A priori estimates	Uniform bounds for energy and dissipation.	Assumptions (E1) and (D1)–(D2) $\Rightarrow$ lower semicontinuity, compact sublevels, finite infimum $\Rightarrow$ existence of a minimiser. Discrete stability gives $\mathcal{E}(t_k, q_k) \leq \mathcal{E}(t_{k-1}, q_{k-1})$. Telescoping yields $\sup_k \mathcal{E}(t_k, q_k) + \text{Diss}_{\mathcal{D}}(q^\tau; [0, T]) \leq C,$
2	Helly selection	BV bound $\Rightarrow$ compactness.	independent of τ . The bound on $\text{Diss}_{\mathcal{D}}$ implies bounded BV-norm for z^τ . Helly/Arzelà–Ascoli provides $z^\tau \rightarrow z$ pointwise (subsequence). The fast variable is recovered by $y(t) \in \arg \min_y \mathcal{E}(t, (y, z(t)))$.
3	Limit stability	(C1) (C2) transfer stability to the limit.	(C2) $\Rightarrow$ the stable set $S(t)$ is closed; (C1) justifies exchanging limit and time derivative, whence $q(t) \in S(t)$ for all t .
4	Upper energy inequality	Discrete $(E)^{\text{discr}} \Rightarrow$ “ $\leq$ ” in the limit.	From $\mathcal{E}(t_k, q_k) + \sum_{j=1}^k \mathcal{D}(q_{j-1}, q_j)$ $= \mathcal{E}(0, q_0) + \sum_{j=1}^k \int_{t_{j-1}}^{t_j} \partial_t \mathcal{E}(s, q_{j-1}) \, ds$
5	Lower energy inequality	Stability $\Rightarrow$ “ $\geq$ ”.	and Fatou, obtain the upper inequality for the limit curve. Testing with admissible competitors and using the triangle inequality yields the complementary lower bound, completing the energy inequality.
6	Equality & conclusion	Combine bounds $\Rightarrow$ energetic solution.	Upper $\leq$ lower $\Rightarrow$ exact energy balance (E). Together with stability (S) this yields an energetic solution. Extra convexity/uniqueness of minimisers $\Rightarrow$ strong convergence and, possibly, uniqueness of the solution.

Chapter 4

Abstract Theory of Γ -convergence

In this Chapter, Γ -convergence is introduced as a fundamental tool for homogenization. In Chapter 5, we focus on composite materials with ε -periodic microstructures depending on a parameter $\varepsilon > 0$. The ideal goal would be to directly analyze the RIS describing deformation of such a material for small ε . However, due to the nonconvexity of both the stored energy density and the admissible set, combined with the rapid microscopic oscillations as $\varepsilon \downarrow 0$, a stable analysis within the classical RIS framework becomes intractable. The general theoretical framework for such problems is evolutionary Γ -convergence. Evolutionary Γ -convergence is, roughly speaking, a concept that provides a unified description of the limiting behavior of variational problems with time evolution. Not only does it guarantee that the energetic solutions of the original model converge to those of the limit system, but typically it also allows one to treat homogenization ($\varepsilon \rightarrow 0$) and the convergence of time-discrete solutions ($\tau \rightarrow 0$) simultaneously. Hence, the standard RIS framework and evolutionary Γ -convergence are connected through time discretization. We therefore begin with the time-discrete problems in the RIS framework.

The construction of energy solutions to RIS is based on incremental minimization via time discretization, followed by the passage to the continuous-time limit. Given a time partition $\{t_k\}$, one solves at each step the essentially static minimization problem

$$q_\varepsilon^k \in \arg \min_q \left\{ \mathcal{E}_\varepsilon(t_k, q) + \mathcal{D}_\varepsilon(q_\varepsilon^{k-1}, q) \right\}, \quad (4.1)$$

and, in the limit of the discrete solutions, obtains an energy solution satisfying stability (S) and energy-dissipation balance (E) (Mielke, 2011/2012; Mielke and Roubířek, 2015). Even at the level of the incremental minimization problems (4.1), the analytical obstacles caused by nonconvexity and rapid microscopic oscillations persist. In fact, for variational models of composites with ε -periodic microstructure the original ε -problem may fail to attain its minimum: minimizing sequences tend to develop finer and finer microstructures so that the infimum is finite but not achieved (Ball and James, 1992). In the setting of Chapter 5, too, the integrand $W(\nabla u)$ obtained is neither convex nor quasiconvex, hence the associated integral functional need not be weakly lower semicontinuous; moreover, the admissible set of gradients is, in general, not weakly sequentially closed. In such situations one is naturally led to a relaxation of the static problem. A classical device for this purpose is (static) Γ -convergence.

As a characterizing example of Γ -convergence, for a fixed functional F the Γ -limit of the constant family $F_\varepsilon \equiv F$ coincides with the largest lower-semicontinuous functional below F (the lower-semicontinuous envelope), hence it preserves the infimum; combined with equi-coercivity/compactness, this relaxed functional admits

minimizers (Braides, 2006; Dal Maso, 1993). In this sense, static Γ -convergence provides a principled relaxation framework that replaces a possibly non-attaining problem by a well-posed one without altering the variational lower bound.

On the other hand, the full ε -problem simultaneously resolves the fine $O(\varepsilon)$ and the coarse $O(1)$ spatial scales, which renders direct analysis and computation prohibitively expensive. Homogenization replaces it with an effective macroscopic model whose energy is the Γ -limit of the ε -energies; in periodic settings, while the exact solution u_ε carries both fine- and coarse-scale oscillations, the homogenized solution u_0 exhibits only coarse-scale behavior (Allaire, 1992; Berlyand and Owhadi, 2010). Crucially, this reduction is not merely qualitative but quantitative: for second-order elliptic problems with ε -periodic coefficients one has convergence $u_\varepsilon \rightarrow u_0$ and sharp operator-error bounds, with precise rates available in Suslina, 2012; Pakhnin and Suslina, 2012; Jikov, Kozlov, and Oleinik, 1994.

Hence, static Γ -convergence can be viewed as an effective tool to approximate and analyze the RIS procedure (4.1) for small ε .

In contrast to the classical notion of static Γ -convergence, the corresponding concept in the time-evolving RIS theory is that of evolutionary Γ -convergence. One instance of the general theory of evolutionary Γ -convergence is the case where the limit system can be characterized via the static Γ -limit of the energies together with Mosco convergence – which is a concept similar to ordinary Γ -convergence – of the dissipations (Mielke, Roubíček, and Stefanelli, 2008). That is, under the standard assumptions of uniform-in-time Γ -convergence of the family $\{\mathcal{E}_\varepsilon(t, \cdot)\}_\varepsilon$, Mosco convergence of the convex dissipation potentials $\{\mathcal{R}_\varepsilon\}_\varepsilon$ (this means the Mosco convergence of the dissipation $\mathcal{D}_\varepsilon$), as well as power control with respect to the time derivative of the energy, the existence of mutual recovery sequences, equi-coercivity, and the closedness of the stability set, ensures—through compactness and liminf/limsup estimates—the convergence of discrete solutions towards energy solutions of the limit system $(\mathcal{E}_0, \mathcal{D}_0)$ (Mielke, Roubíček, and Stefanelli, 2008; Mielke, 2014; Mielke, Rossi, and Savaré, 2012; Mielke, 2016).

In particular, one has

$$\mathcal{E}_0(t, \cdot) = \Gamma\text{-}\lim_{\varepsilon \rightarrow 0} \mathcal{E}_\varepsilon(t, \cdot), \quad \mathcal{D}_0 = \text{Mosco-}\lim_{\varepsilon \rightarrow 0} \mathcal{D}_\varepsilon \ (\Leftrightarrow \ \mathcal{R}_\varepsilon \xrightarrow{\text{M}} \mathcal{R}_0). \quad (4.2)$$

Hence, in this prototypical setting it is justified to employ the static Γ -limit directly in the incremental time-discrete minimization scheme, while simultaneously ensuring convergence towards energetic solutions of the limiting system; as a consequence, the original problem attains a substantially simpler form. In other words, evolutionary Γ -convergence can be regarded as a theory that, when dealing with time-evolution problems in settings such as the composite materials considered in this thesis—where direct analysis is practically intractable—transfers the problem to a limit problem that is expected to possess better properties.

Since it is difficult to directly determine the standard RIS energy solutions for small ε , the ultimate objective in the present setting of composite materials can be regarded as the establishment of evolutionary Γ -convergence, although the very establishment of evolutionary Γ -convergence itself entails considerable technical difficulties. In view of (4.2), determining the static Γ -limit constitutes a first step towards establishing the evolutionary Γ -limit.

In summary, in situations such as those considered in Chapter 5, where a direct analysis of the evolution problem via RIS is practically intractable, the significance of first establishing static Γ -convergence can be summarized in the following two points:

- I Numerical Advantage:** Through the static Γ -limit, the incremental problem (4.1) can be approximated by the homogenized incremental problem based on the effective energy $\mathcal{E}_0$ and (although not derived here) the effective dissipation $\mathcal{D}_0$, thereby stabilizing the computations and reducing the numerical cost for small ε (Dal Maso, 1993; Braides, 2002).
- II Prediction of the Evolutionary Limit:** The effective energy $\mathcal{E}_0$ obtained via static Γ -convergence is expected to coincide with the energy component of the evolutionary Γ -limit, and thus provides a structural preview of the limiting system.

4.1 Γ -Convergence

Motivation and Historical Background

In this section, we deal with the concept that a family of functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$ on a Banach space X converges, in the sense of approximating variational problems, to a functional $\mathcal{E}$, namely Γ -convergence. The concept of Γ -convergence was introduced by De Giorgi in the early 1970s (De Giorgi, 1975; De Giorgi and Spagnolo, 1973) to describe the asymptotic behavior of variational problems depending on small parameters. Since then, it has become a fundamental tool in a variety of fields including:

- homogenisation of composite materials,
- dimension reduction (e.g., plate and shell theories),
- phase transition models (Modica and Mortola, 1977; Braides, 2002),
- and rate-independent systems in elasto-plasticity and damage mechanics (Mielke and Roubíček, 2015; Francfort and Mielke, 2006a).

For a family of energy functionals $(\mathcal{E}_\varepsilon)_{\varepsilon>0}$, Γ -convergence provides two essential ingredients: it guarantees that the limit of (almost) minimizing sequences is a minimizer of the limit functional, and that the corresponding minimal values converge. These properties make Γ -convergence a central tool in the approximation of variational problems.

Throughout this chapter, we work in the reflexive Banach space X . Standard references include Dal Maso, 1993; Braides, 2002.

4.1.1 Definition of Γ -Convergence

We define the Γ -limit along a continuous parameter $\varepsilon \rightarrow 0+$ by requiring Γ -convergence along arbitrary sequence $(\varepsilon_j)_j$ such that $\varepsilon_j \rightarrow 0+$.

Definition 3 (Γ -limit along $\varepsilon \downarrow 0$). *Let $(X, \|\cdot\|_X)$ be a Banach space with its strong topology, and let $\{\mathcal{E}_\varepsilon\}_{\varepsilon>0}$ be functionals $\mathcal{E}_\varepsilon : X \rightarrow (-\infty, +\infty]$. We say that $\mathcal{E}_\varepsilon$ Γ -converges to $\mathcal{E} : X \rightarrow (-\infty, +\infty]$ in X as $\varepsilon \downarrow 0$ (written $\mathcal{E}_\varepsilon \xrightarrow{\Gamma} \mathcal{E}$) if the following holds:*

for every sequence $(\varepsilon_j)_j$ with $\varepsilon_j \downarrow 0$, the sequence $(\mathcal{E}_{\varepsilon_j})_j$ Γ -converges to $\mathcal{E}$ in the sense that:

Compactness If $(u_j)_j \subset X$ satisfies $\sup_j \mathcal{E}_{\varepsilon_j}(u_j) < \infty$, then there exists a subsequence (not relabeled) and $u \in X$ with $u_j \rightarrow u$ in X and $\mathcal{E}(u) < \infty$.

Liminf inequality For every $(u_j)_j \subset X$ with $u_j \rightarrow u$ in X ,

$$\mathcal{E}(u) \leq \liminf_{j \rightarrow \infty} \mathcal{E}_{\varepsilon_j}(u_j). \quad (4.3)$$

Recovery sequence For every $u \in X$ with $\mathcal{E}(u) < \infty$, there exists $(u_j)_j \subset X$ with $u_j \rightarrow u$ in X such that

$$\limsup_{j \rightarrow \infty} \mathcal{E}_{\varepsilon_j}(u_j) \leq \mathcal{E}(u).$$

This definition will serve as the basis for the analysis of Γ -limits in the subsequent chapters. We have defined Γ -convergence by requiring convergence along arbitrary discrete sequences. Alternatively, one may define it directly for the continuous parameter $\varepsilon \rightarrow 0^+$, and the two definitions are equivalent (see Dal Maso, 1993, Prop. 8, 3).

4.1.2 General Properties of Γ -Limits

We now collect several structural properties of Γ -convergence that are crucial for variational analysis. Proofs may be found in standard references such as Dal Maso, 1993; Braides, 2006.

Fundamental Theorem of Γ -Convergence

Definition 4 (Equi-coercivity). A sequence of functionals $(\mathcal{F}_\varepsilon)$ on X is said to be equi-coercive if, for every $M \in \mathbb{R}$, there exists a bounded set $K_M \subset X$ such that

$$\{u \in X : \mathcal{F}_\varepsilon(u) \leq M\} \subset K_M$$

for all $\varepsilon > 0$.

Equi-coercivity guarantees that energy-bounded sequences are relatively compact in X . This compactness property is essential to ensure that minimizing (or almost minimizing) sequences admit convergent subsequences, so that the limit problem is well-posed in terms of minimizers.

Theorem 2 (Fundamental Theorem). (Dal Maso, 1993, Thm. 7.8); see also Braides, 2006, §1, “fundamental theorem”. Let $(\mathcal{F}_\varepsilon)_{\varepsilon > 0}$ be an equi-coercive sequence of functionals on X that Γ -converges to $\mathcal{F}$. Assume that u_ε is a minimizer of $\mathcal{F}_\varepsilon$. Then:

1. (Convergence of minima)

$$\lim_{\varepsilon \rightarrow 0} \min_X \mathcal{F}_\varepsilon = \min_X \mathcal{F}.$$

2. (Compactness of minimizers) The sequence (u_ε) admits a converging subsequence.

3. (Stability of minimizers) Every cluster point is a minimizer of $\mathcal{F}$.

The result remains valid for almost minimizers, i.e., $\mathcal{F}_\varepsilon(u_\varepsilon) \leq \min \mathcal{F}_\varepsilon + o(1)$.

Locality and Lower Semicontinuity

Proposition 1 (Locality). (Dal Maso, 1993, Prop. 16.15) *If each $\mathcal{F}_\epsilon$ is a local functional (e.g., integral functional depending on u and ∇u), then the Γ -limit $\mathcal{F}$ is also local.*

Proposition 2 (Lower semicontinuity). (Dal Maso, 1993, Prop. 6.8) *Every Γ -limit is lower semicontinuous with respect to the strong topology of X .*

Stability under Continuous Perturbations

Proposition 3 (Continuous perturbations). (Braides, 2006, Rem. 2.2(1)) *Let $\mathcal{F}_\epsilon \xrightarrow{\Gamma} \mathcal{F}$ and suppose $\mathcal{G}_\epsilon \rightarrow \mathcal{G}$ uniformly on bounded subsets of X . Then*

$$\mathcal{F}_\epsilon + \mathcal{G}_\epsilon \xrightarrow{\Gamma} \mathcal{F} + \mathcal{G}.$$

4.2 Evolutionary Γ -Convergence

The concept of evolutionary Γ -convergence—sometimes called Γ -convergence for Energetic Rate-Independent Systems (ERIS)—was put forward by Mielke and co-workers in the early 2000s as a natural extension of De Giorgi’s static theory to time-dependent problems (Mielke and Roubíček, 2006a; Mielke and Roubíček, 2015). Just as static Γ -convergence controls the asymptotics of minimisers, the evolutionary variant governs the asymptotic behaviour of entire trajectories $(q_\epsilon)_{\epsilon>0}$ solving a system of ERIS $(Q, \mathcal{E}_\epsilon, \mathcal{D}_\epsilon)$ when a small parameter $\epsilon \rightarrow 0$ is present. In the last two decades, evolutionary Γ -convergence has become a key tool in the analysis of:

- fracture and damage mechanics (Francfort and Mielke, 2006a; Mielke and Roubíček, 2006b),
- two-scale (periodic) homogenisation (Mielke and Timofte, 2007; Davoli and Kreisbeck, 2022),
- dimension-reduction of thin structures (Davoli and Mora, 2013),
- numerical approximation and space–time discretisation schemes (Knees, 2019; Mielke, 2009),
- delamination and other interface problems (Kočvara, Mielke, and Roubíček, 2006).

In particular, in the small-deformation (linearization) regime introduced in Chapter 1, Mielke and Stefanelli rigorously show—within the ERIS framework—that the quasi-static finite-strain elastoplasticity system converges, in the sense of evolutionary Γ -convergence, to linearized plasticity: energetic solutions of the finite-strain model converge to the unique strong solution of the linearized model (Mielke and Stefanelli, 2013).

4.2.1 Definition of Evolutionary Γ -Convergence and the Basic Convergence Theorem

In the following, we define evolutionary Γ -convergence in the form of a closedness property for sequences of energetic solutions. We then state the ten structural assumptions that guarantee it, together with the basic limit theorem.

Let $\mathcal{Q} = Y \times Z$ be a metrizable space and $T > 0$. For $k \in \mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$ let

$$\mathcal{E}_k : [0, T] \times \mathcal{Q} \rightarrow \mathbb{R} \cup \{+\infty\}, \quad \mathcal{D}_k : Z \times Z \rightarrow [0, \infty]$$

with $\mathcal{D}_k$ a (possibly non-symmetric) quasidistance on Z . For $q = (y, z) \in \mathcal{Q}$ and $\tilde{q} = (\tilde{y}, \tilde{z}) \in \mathcal{Q}$ we write, with a standard abuse, $\mathcal{D}_k(q, \tilde{q}) := \mathcal{D}_k(z, \tilde{z})$. Here $k \in \mathbb{N}$ is the discrete parameter, as in Mielke–Roubíček (Mielke and Roubíček, 2015). For the continuous parameter $\varepsilon (\rightarrow 0)$, the definition and assumptions/theorems in this and the next section apply to any sequence $(\varepsilon_k)_k$ with $\varepsilon_k \rightarrow 0+$. Moreover, when referring to (static) Γ -convergence here, we mean that Definition 3 holds for the given $(\varepsilon_k)_k$. All properties of Γ -convergence established in the previous section continue to hold in this case.

The stability set at time t is

$$\mathcal{S}_k(t) := \left\{ q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \leq \mathcal{E}_k(t, \tilde{q}) + \mathcal{D}_k(q, \tilde{q}) \quad \forall \tilde{q} \in \mathcal{Q} \right\}.$$

A stable sequence $(t_l, q_{k_l}) \rightarrow (t, q)$ means $t_l \rightarrow t$ and $q_{k_l} \in \mathcal{S}_{k_l}(t_l)$ with $q_{k_l} \rightarrow q$ in $\mathcal{Q}$. Analogously, a sequence is stable for the limit system if $q_l \in \mathcal{S}_\infty(t_l)$.

For a curve $q = (y, z) : [0, T] \rightarrow \mathcal{Q}$ we define the $\mathcal{D}_k$ -dissipation

$$\text{Diss}_k(q; [s, t]) := \sup_{\Pi} \sum_{i=0}^{N-1} \mathcal{D}_k(z(t_i), z(t_{i+1})), \quad 0 \leq s \leq t \leq T,$$

where the supremum runs over all partitions $\Pi : s = t_0 < \dots < t_N = t$.

Energetic Solutions (recall). A curve $q_k = (y_k, z_k) : [0, T] \rightarrow \mathcal{Q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)$ if for all $t \in [0, T]$:

$$(S) \quad q_k(t) \in \mathcal{S}_k(t),$$

$$(E) \quad \mathcal{E}_k(t, q_k(t)) + \text{Diss}_k(q_k; [0, t]) = \mathcal{E}_k(0, q_k(0)) + \int_0^t \partial_t \mathcal{E}_k(s, q_k(s)) \, ds.$$

Definition 5 (Evolutionary Γ -convergence). We say that the family $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)_{k \in \mathbb{N}}$ evolutionarily Γ -converges to $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ and write

$$(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)_{k \in \mathbb{N}} \xrightarrow{\text{ev-}\Gamma} (\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty) \quad \text{as } k \rightarrow \infty,$$

if the following closedness property holds:

For every sequence of energetic solutions q_k with well-prepared initial data

$$q_k(0) \rightarrow q_\infty(0), \quad \mathcal{E}_k(0, q_k(0)) \rightarrow \mathcal{E}_\infty(0, q_\infty(0)),$$

whenever (possibly along a subsequence, not relabelled)

$$q_k(t) \rightarrow q_\infty(t) \quad \text{in } \mathcal{Q} \text{ for every } t \in [0, T],$$

then q_∞ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$.

Remark 5 (On the pointwise convergence assumption). The condition $q_k(t) \rightarrow q_\infty(t)$ for every $t \in [0, T]$ is not restrictive. Indeed, as shown in the proof of Theorem 3 in (Mielke and Roubíček, 2015), every sequence of energetic solutions possesses a subsequence whose z -component converges pointwise. This is the only convergence needed in the argument. For

the corresponding statement on the y -component we refer to Mielke and Roubíček, 2015, Rem. 2.1.8 and 2.1.9.

We now introduce the basic convergence theorem in Mielke and Roubíček, 2015. It is proved under the ten structural assumptions (4.4a)–(4.4s).

Quasidistance (4.4a)

$$\forall k \in \mathbb{N}_\infty, \forall z, \tilde{z}, \hat{z} \in Z : \quad (4.4b)$$

$$\mathcal{D}_k(z, \tilde{z}) = 0 \Leftrightarrow z = \tilde{z},$$

$$\mathcal{D}_k(z, \hat{z}) \leq \mathcal{D}_k(z, \tilde{z}) + \mathcal{D}_k(\tilde{z}, \hat{z}).$$

Lower semicontinuity of $\mathcal{D}_k$ (4.4c)

$$\mathcal{D}_k \text{ is lower semicontinuous on } Z \times Z. \quad (4.4d)$$

Γ -liminf for $\mathcal{D}_k$ (4.4e)

$$\forall \text{ stable } (t_l, q_{k_l}) \rightarrow (t, q), (\tilde{t}_l, \tilde{q}_{k_l}) \rightarrow (\tilde{t}, \tilde{q}) : \quad (4.4f)$$

$$\mathcal{D}_\infty(q, \tilde{q}) \leq \liminf_{l \rightarrow \infty} \mathcal{D}_{k_l}(q_{k_l}, \tilde{q}_{k_l}).$$

Compact energy sublevels (4.4g)

$$\forall t \in [0, T], \forall E \in \mathbb{R} : \quad (4.4h)$$

$$(i) \forall k : \{q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \leq E\} \text{ is compact,}$$

$$(ii) \bigcup_{k \geq 1} \{q \in \mathcal{Q} \mid \mathcal{E}_k(t, q) \leq E\} \text{ is relatively compact.}$$

Separability & metrizable (4.4i)

$$\text{Each sublevel } \{\mathcal{E}(t, \cdot) \leq E\} \text{ is compact, separable, and metrizable.} \quad (4.4j)$$

Uniform power control (4.4k)

$$\text{Dom } \mathcal{E}_k = [0, T] \times \text{Dom } \mathcal{E}_k(0, \cdot), \quad (4.4l)$$

$$\exists c_{\mathcal{E}} \in \mathbb{R}, \exists \lambda_{\mathcal{E}} \in L^1(0, T), \exists N_{\mathcal{E}} \subset [0, T], \mathcal{L}^1(N_{\mathcal{E}}) = 0 :$$

$$\forall k, \forall q \in \text{Dom } \mathcal{E}_k(0, \cdot) : \mathcal{E}_k(\cdot, q) \in W^{1,1}(0, T),$$

$$t \in [0, T] \setminus N_{\mathcal{E}} \Rightarrow |\partial_t \mathcal{E}_k(t, q)| \leq \lambda_{\mathcal{E}}(t)(\mathcal{E}_k(t, q) + c_{\mathcal{E}}).$$

Γ -liminf for $\mathcal{E}_k$ (4.4m)

$$\mathcal{E}_\infty(t, q) \leq \liminf_{l \rightarrow \infty} \mathcal{E}_{k_l}(t_l, q_{k_l}) \quad \text{for all stable } (t_l, q_{k_l}) \rightarrow (t, q). \quad (4.4n)$$

Power semicontinuity I (4.4o)

$$\forall t \in [0, T] \setminus N_{\mathcal{E}}, \forall \text{ stable } (t_l, q_{k_l}) \rightarrow (t, q) : \quad (4.4p)$$

$$\partial_t \mathcal{E}_\infty(t, q) \geq \limsup_{l \rightarrow \infty} \partial_t \mathcal{E}_{k_l}(t_l, q_{k_l}).$$

Power semicontinuity II (4.4q)

$$\forall t \in [0, T] \setminus N_{\mathcal{E}}, \forall \text{ stable } (t_l, \tilde{q}_l) \rightarrow (t, q) \text{ with } \tilde{q}_l \in \mathcal{S}_\infty(t_l) : \quad (4.4r)$$

$$\partial_t \mathcal{E}_\infty(t, q) \geq \limsup_{l \rightarrow \infty} \partial_t \mathcal{E}_\infty(t, \tilde{q}_l).$$

Stability upper semicontinuity (4.4s)

$$\forall \text{ stable } (t_l, q_{k_l}) \rightarrow (t, q), q_{k_l} \in \mathcal{S}_{k_l}(t_l) : q \in \mathcal{S}_\infty(t). \quad (4.4t)$$

Theorem 3 (Evolutionary Γ -limit). (Mielke and Roubíček, 2015, Th. 2.4.10) Assume (4.4). Then the family $(\mathcal{Q}, \mathcal{E}_k, \mathcal{D}_k)_{k \in \mathbb{N}}$ evolutionarily Γ -converges to $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$ in the sense of Definition 5.

Moreover, for any energetic solutions q_k with well-prepared initial data, if (after extraction, not relabelled) $q_k(t) \rightarrow q_\infty(t)$ in $\mathcal{Q}$ for every $t \in [0, T]$, then q_∞ is an energetic solution of the limit system and the following convergences hold (with $N_\mathcal{E}$ from (4.4k)):

$$\begin{aligned} \mathcal{E}_k(t, q_k(t)) &\rightarrow \mathcal{E}_\infty(t, q_\infty(t)) && \text{for all } t \in [0, T], \\ \text{Diss}_k(q_k; [0, t]) &\rightarrow \text{Diss}_\infty(q_\infty; [0, t]) && \text{for all } t \in [0, T], \\ \partial_t \mathcal{E}_k(t, q_k(t)) &\rightarrow \partial_t \mathcal{E}_\infty(t, q_\infty(t)) && \text{for all } t \in [0, T] \setminus N_\mathcal{E}, \\ \partial_t \mathcal{E}_k(\cdot, q_k(\cdot)) &\rightarrow \partial_t \mathcal{E}_\infty(\cdot, q_\infty(\cdot)) && \text{in } L^1(0, T). \end{aligned} \quad (4.5)$$

Remark 6 (Static Γ -limits versus mutual recovery). *Theorem 3 does not require full static Γ -convergence of $(\mathcal{E}_k, \mathcal{D}_k)$: the proof supplies the needed Γ -limsup via a mutual-recovery construction inside the stability sets. In applications, however, the technically decisive and verifiable hypothesis is the static Γ -convergence of the energy functionals,*

$$\mathcal{E}_k(t, \cdot) \xrightarrow{\Gamma} \mathcal{E}_\infty(t, \cdot) \quad \forall t \in [0, T],$$

(with respect to the intrinsic topology generated by stable sequences). This delivers the Γ -liminf parts in (4.4e)–(4.4m) and the closedness of stability, so that, together with mutual-recovery (and any standard assumption on the dissipation, e.g. Mosco convergence of convex, l.s.c. 1-homogeneous potentials $\mathcal{R}_k$ implying the corresponding convergence of $\mathcal{D}_k$), the convergence of energetic solutions follows. This is one of the main motivations for the static Γ -convergence results for the energies proved later in this thesis.

4.2.2 Solutions of (IMP^k) Converge to Solutions of Γ -Limit

In this section we consider a framework where the time step vanishes ($|\Pi^k| \rightarrow 0$) while, simultaneously, the model $(\mathcal{E}_k, \mathcal{D}_k)$ converges to a limit pair $(\mathcal{E}_\infty, \mathcal{D}_\infty)$. Under the standing assumptions (see (IV) of Theorem 1 and (4.4)), the time-continuization and the model limit behave consistently: any subsequential macroscopic limit of discrete minimizers yields an energetic solution of the limit Rate-Independent System (RIS) $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty)$.

Let $\mathcal{Q}$ be the state space endowed with a topology/metric ρ . When convenient we write $q = (y, z) \in Y \times Z$. For a partition $\Pi^k = \{0 = t_0^k < \dots < t_{N_k}^k = T\}$ with mesh size $|\Pi^k| := \max_j (t_j^k - t_{j-1}^k)$, the fully implicit incremental minimization problem (IMP^k) reads

$$q_j^k \in \text{Argmin}_{q \in \mathcal{Q}} \left\{ \mathcal{E}_k(t_j^k, q) + \mathcal{D}_k(q_{j-1}^k, q) \right\} \quad (j = 1, \dots, N_k), \quad (4.6)$$

given $q_0^k \in \mathcal{Q}$. We use the right-continuous, piecewise constant interpolant $\underline{q}^k : [0, T] \rightarrow \mathcal{Q}$ defined by

$$\underline{q}^k(t) = q_{j-1}^k \quad \text{for } t \in [t_{j-1}^k, t_j^k), \quad \underline{q}^k(T) = q_{N_k}^k. \quad (4.7)$$

Theorem 4 (Convergence of discrete minimizers to an energetic solution). *(Mielke and Roubíček, 2015, Th. 2.4.13) Assume (IV) and (4.4), let $|\Pi^k| \rightarrow 0$, and suppose the initial data satisfy*

$$q_0^k \xrightarrow{\rho} q_0 \quad \text{and} \quad \mathcal{E}_k(0, q_0^k) \rightarrow \mathcal{E}_\infty(0, q_0) \in \mathbb{R}. \quad (4.8)$$

Then:

- (i) Each IMP^k admits at least one solution and the interpolants q^k possess a subsequence (not relabeled) and a curve $q : [0, T] \rightarrow \mathcal{Q}$ such that, for every $t \in [0, T]$ (and for a.e. t in the statement about time derivatives),

$$\underline{z}^k(t) \rightarrow z(t), \quad (4.9)$$

$$\text{Diss}_k(q^k; [0, t]) \rightarrow \text{Diss}_\infty(q; [0, t]), \quad (4.10)$$

$$\mathcal{E}_k(t, q^k(t)) \rightarrow \mathcal{E}_\infty(t, q(t)), \quad (4.11)$$

$$\partial_t \mathcal{E}_k(t, q^k(t)) \rightarrow \partial_t \mathcal{E}_\infty(t, q(t)) \quad \text{for a.e. } t \in [0, T]. \quad (4.12)$$

In particular, $\partial_t \mathcal{E}_k(\cdot, q^k(\cdot)) \rightarrow \partial_t \mathcal{E}_\infty(\cdot, q(\cdot))$ in $L^1(0, T)$. The limit curve q is an energetic solution of the initial value problem $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty, q_0)$, i.e., it satisfies stability (S) and the energy–dissipation balance (E).

- (ii) If for every t and z the map $y \mapsto \mathcal{E}_\infty(t, y, z)$ admits a unique minimizer $y = y(t, z)$ (and the same structure is consistent along the approximations), then (4.9) extends to the full state: for all $t \in [0, T]$,

$$q^k(t) = (\underline{y}^k(t), \underline{z}^k(t)) \rightarrow \tilde{q}(t) = (Y(t, z(t)), z(t)),$$

with convergence in the topology of $\mathcal{Q}$ (pointwise in time).

- (iii) If, in addition, $y(t) \in \text{Argmin}_y \mathcal{E}_\infty(t, y, z(t))$ holds for all t and the reduced power is compatible, namely

$$\partial_t \mathcal{E}_\infty(t, y(t), z(t)) = \mathcal{P}_{\text{red}}^\infty(t, z(t)) \quad \text{for a.e. } t \in [0, T],$$

then any such limit $\tilde{q}$ is an energetic solution of $(\mathcal{Q}, \mathcal{E}_\infty, \mathcal{D}_\infty, q_0)$.

Remark 7 (Quasi-minimization at the discrete level). Instead of exact minimizers in (3.7), one may admit quasi-minimizers with errors $\eta_j^k \geq 0$ satisfying

$$\mathcal{E}_k(t_j^k, q_j^k) + \mathcal{D}_k(q_{j-1}^k, q_j^k) \leq \inf_{q \in \mathcal{Q}} \{\mathcal{E}_k(t_j^k, q) + \mathcal{D}_k(q_{j-1}^k, q)\} + \eta_j^k, \quad \sum_{j=1}^{N_k} \eta_j^k \rightarrow 0.$$

Then Theorem 4 remains valid.

Interpretation and Two Convergence Levels. The construction of energetic solutions of RIS—both for a fixed model and in the presence of an evolutionary Γ -limit—reduces to chaining static incremental minimizations in time. Hence solving the incremental problems is central analytically and computationally. In the setting, such as in Chapter 5 where we deal with composite materials consisting of periodically repeated thin layers of different properties, each static step is itself delicate due to the combination of nonconvexity and ε -periodic microstructure. Identifying the effective energy and effective dissipation by a prior static Γ /Mosco limit is therefore highly meaningful.

(A) *On-trajectory convergence (under the assumptions of Theorem 4 only).*

With $|\Pi^k| \rightarrow 0$ and the standing assumptions (power control, closedness of stability, initial consistency, etc.), one obtains along the discrete solutions

$$\mathcal{E}_k(t_j^k, q_j^k) + \mathcal{D}_k(q_{j-1}^k, q_j^k) \rightarrow \mathcal{E}_\infty(t, q(t)) + \mathcal{D}_\infty(q(t^-), q(t)),$$

together with $\underline{q}^k(\cdot) \rightarrow q(\cdot)$. Here $\mathcal{E}_\infty(t, \cdot)$ denotes the limiting energy functional obtained by the evolutionary Γ -limit, while $\mathcal{D}_\infty(\cdot, \cdot)$ is the limiting dissipation distance. In particular, the limit q is an energetic solution. This convergence is asserted along the actually computed discrete trajectory.

(B) *Incremental functional convergence (with time-uniform static Γ /Mosco).*

If, in addition, $\mathcal{E}_k(t, \cdot)$ Γ -converges to $\mathcal{E}_\infty(t, \cdot)$ uniformly in t with equi-coercivity, and $\mathcal{D}_k$ Mosco-converges to $\mathcal{D}_\infty$ (together with mutual recovery and power compatibility), then for every reference state $\bar{q}_k \rightarrow \bar{q}$ and every t ,

$$\inf_{q \in \mathcal{Q}} \{ \mathcal{E}_k(t, q) + \mathcal{D}_k(\bar{q}_k, q) \} \longrightarrow \min_{q \in \mathcal{Q}} \{ \mathcal{E}_\infty(t, q) + \mathcal{D}_\infty(\bar{q}, q) \},$$

and (quasi-)minimizers are stable. In particular, the effective incremental problem

$$\min_{q \in \mathcal{Q}} \{ \mathcal{E}_\infty(t, q) + \mathcal{D}_\infty(\bar{q}, q) \}$$

describes the limit independently of the approximating trajectory. In this typical homogenization setting one can identify

$$\mathcal{E}_\infty \equiv E_0, \quad \mathcal{D}_\infty \equiv D_0,$$

where E_0 and D_0 are the effective energy and effective dissipation respectively. That is, $\mathcal{E}_\infty$ and $\mathcal{D}_\infty$ are obtained by the static Γ /Mosco limit (see, e.g., (Dal Maso, 1993; Braides, 2002; Mielke and Roubíček, 2015)). Computationally, once E_0 and D_0 have been determined, one may solve the macroscopic incremental problem without resolving the microstructure at each step.

Chapter 5

Homogenization of Layered Elastoplastic Materials

In this chapter, we consider a static homogenization problem for a composite crystalline material composed of two types of parallel layers: a rigid layer and a soft layer. The results presented here are based on our previous work (Ishikawa and Švadlenka, 2025). In the soft layer, two slip systems are available, whereas in the rigid layer only local rotations are permitted. As the deformation modes allowed by the given energy density indicate, this problem cannot be treated as a standard variational minimization problem in the static setting. We therefore carry out a homogenization procedure (see Section 4.1). Our main motivation, however, is not merely to solve this static problem. The ultimate goal is to analyze a time-evolving model (RIS). Therefore, as emphasized at the beginning of Chapter 4, the homogenization result here is not intended merely to approximate the static problem via Γ -convergence, but rather to approximate energy solutions within the RIS framework and to anticipate the form of evolutionary Γ -convergence. Concerning the latter aspect, we expect a situation as in case (B) in 4.2.2, where the evolutionary Γ -limit is obtained from the static Γ -limit and provides an ideal approximation of the original (pre-homogenized) RIS model for small layer-period thickness. Even if such a situation does not occur, however, the static Γ -limit still has a clear significance, as it approximates the time-discrete problems of the RIS model.

As a main result, for orthogonal slip systems we derive the complete homogenized energy, and for the case of a general angle between slip directions we present partial results. The motivation of this work is to establish a more realistic homogenization model capable of describing materials in which multiple slip directions interact. Our approach builds upon the work of Christowiak and Kreisbeck, 2017, which treats the single-slip case, and relies on a modification of the classical laminate construction. Several additional challenges arise, however, due to the nonconvex constraints inherent in the composite energy.

To introduce the setting of the problem, let $\Omega \subset \mathbb{R}^2$ be a simply connected Lipschitz domain representing (the reference configuration of) an elastoplastic material in plane undergoing deformation, and $u : \Omega \rightarrow \mathbb{R}^2$ be the function representing the deformation. Let $\lambda \in (0, 1)$ represent the fraction of the softer layer in the periodic structure, and for the unit cell $Y = [0, 1]^2$ define the subsets

$$Y_{soft} := [0, 1) \times [0, \lambda) \subset Y \quad \text{and} \quad Y_{rig} := Y \setminus Y_{soft}, \quad (5.1)$$

which represent the soft and rigid parts, respectively. We extend this unit structure to the whole $\mathbb{R}^2$ and scale the oscillation width with $\epsilon > 0$, which represents the layer

thickness. Then the hard and soft layers $\epsilon Y_{rig} \cap \Omega$ and $\epsilon Y_{soft} \cap \Omega$ are parallel to e_1 , as shown in Figure 5.1.

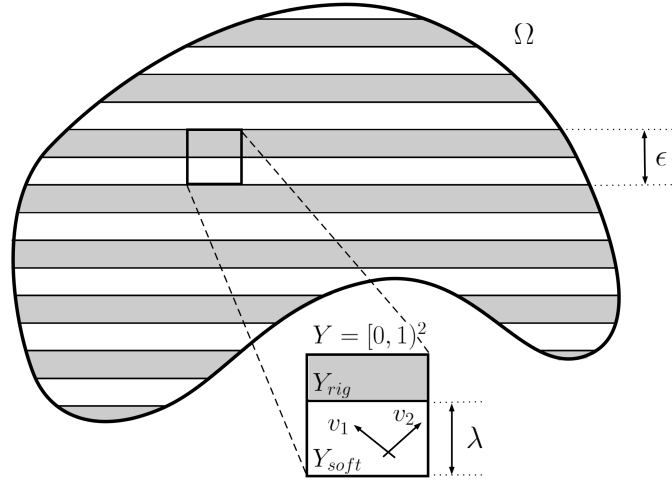


FIGURE 5.1: A periodic structure with two layers of thin rigid and soft components is shown. There are two slip systems in the soft layer with slip directions v_1 and v_2 .

Following the finite-strain crystal plasticity model in Section 2.3, we adopt the Kröner–Lee multiplicative decomposition

$$\nabla u = F_e F_p$$

as a fundamental assumption. Here, the elastic part F_e describes local rotation and stretching of the crystal structure, and the inelastic part F_p accounts for local plastic deformation due to dislocations. For simplicity, we consider only local rotations for F_e , so values of F_e are restricted to $SO(2)$, which is a reasonable approximation for metallic materials (Conti, Dolzmann, and Kreisbeck, 2013a). We also assume that there is no plastic deformation in the hard layer, i.e., $F_p = I$ on $\epsilon Y_{rig} \cap \Omega$. These assumptions, though rather restrictive, are reasonable in light of the results in Conti, Dolzmann, and Kreisbeck, 2013b, and allow to obtain explicit formulae for the homogenized energy functional. In the soft layers $\epsilon Y_{soft} \cap \Omega$ plastic glide can occur along one of two active slip systems $(v_1, v_1^\perp), (v_2, v_2^\perp)$ with slip directions given by unit vectors $v_1, v_2 \in \mathbb{R}^2$. Here $v^\perp$ denotes the counterclockwise rotation of $v \in \mathbb{R}^2$ by $\pi/2$. Hence, $F_p = I + \gamma_1 v_1 \otimes v_1^\perp + \gamma_2 v_2 \otimes v_2^\perp$ in $\epsilon Y_{soft} \cap \Omega$, where $\gamma_1, \gamma_2 \in \mathbb{R}$ corresponds to the amount of slip in each slip direction and satisfies $\gamma_1 \gamma_2 = 0$ a.e. in Ω (Conti and Theil, 2005; Gurtin, 2000).

If the slip direction at a point is v_i ($i = 1, 2$), the deformation gradient at that point is restricted to the set

$$\begin{aligned} \mathcal{M}_i &:= \left\{ F \in \mathbb{R}^{2 \times 2} : F = R(I + \gamma_i v_i \otimes v_i^\perp), R \in SO(2), \gamma_i \in \mathbb{R} \right\} \\ &= \left\{ F \in \mathbb{R}^{2 \times 2} : \det F = 1, |F v_i| = 1 \right\}. \end{aligned}$$

In the rigid layer, both γ_1 and γ_2 vanish, so that the deformation gradient is restricted to $SO(2) = \mathcal{M}_1 \cap \mathcal{M}_2$.

In our model, only rotation is allowed in the rigid layer, so we have for the energy density in the rigid layer

$$W_{rig}(F) = \begin{cases} 0 & \text{if } F \in SO(2) \\ \infty & \text{otherwise.} \end{cases}$$

This corresponds to the gradient being restricted to $SO(2)$ in ϵY_{rig} . On the other hand, using the two-slip model in Conti, Dolzmann, and Kreisbeck, 2013b; Conti, Dolzmann, and Kreisbeck, 2015, the energy density in the soft layer is

$$W_{soft}(F) = \begin{cases} \gamma_1^2 + \gamma_2^2 & \text{if } F \in \mathcal{M} \\ \infty & \text{otherwise,} \end{cases}$$

where $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$. Note that since slip in only one direction is allowed at every point, $\gamma_1^2 + \gamma_2^2$ is equal either to γ_1^2 or to γ_2^2 . The energy W_{soft} does not include a linear term in the slips, whose presence would be desirable in order to address real world problems, in particular, evolution problems involving dissipation. However, the problem with both a linear and a quadratic term is still open even on the level of relaxation problems with one slip (Conti, Dolzmann, and Kreisbeck, 2015).

Summarizing the above, we can define the energy functional $E_\epsilon : W^{1,2}(\Omega; \mathbb{R}^2) \cap L_0^2(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$E_\epsilon(u) := \begin{cases} \int_{\Omega} (\gamma_1^2 + \gamma_2^2) dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \text{ where} \\ & \nabla u = R(I + \gamma_1 v_1 \otimes v_1^\perp + \gamma_2 v_2 \otimes v_2^\perp), \\ & R \in L^\infty(\Omega; SO(2)), \gamma_i \in L^2(\Omega), \\ & \gamma_i = 0 \text{ a.e. in } \epsilon Y_{rig} \cap \Omega, \gamma_1 \gamma_2 = 0 \text{ a.e. in } \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

The condition $\gamma_1 \gamma_2 = 0$ means that only one of the slip systems is active at each material point.

We remark that for orthogonal slip directions one can write

$$\gamma_1^2 + \gamma_2^2 = |(\nabla u)v_1|^2 + |(\nabla u)v_2|^2 - 2.$$

Hence, the functional E_ϵ can be rephrased as

$$E_\epsilon(u) = \begin{cases} \int_{\Omega} W(\nabla u) dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \nabla u \in \mathcal{M} \\ & \text{and } \nabla u \in SO(2) \text{ in } \epsilon Y_{rig} \cap \Omega, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$W(F) = \begin{cases} |Fv_1|^2 + |Fv_2|^2 - 2 & \text{if } F \in \mathcal{M}, \\ \infty & \text{otherwise.} \end{cases} \quad (5.2)$$

The homogenization in this chapter is carried out within the framework of Γ -convergence introduced in Section 4.1. This approach allows us to characterize the effective energy E without explicitly tracking rapid oscillations caused by the fine-scale structure, and is widely used in the homogenization of composite materials.

For orthogonal slip systems ($v_2 = v_1^\perp$), we show that $(E_\epsilon)_\epsilon$ Γ -converges with respect to the strong L^2 topology, and we provide a complete characterization of the

homogenized energy. For general angles, we obtain partial results via the evaluation of the rank-one (laminate) envelope based on an iterated (Kohn and Strang, 1986a; Kohn and Strang, 1986b) lamination construction.

Before explaining our main results, we introduce the static homogenization result of Christowiak and Kreisbeck, 2017, which served as the starting point of our study. This result is based on exactly the same physical setting as described above, but it treats the case of a single slip system which slip system is specified by a unit vector $s \in \mathbb{R}^2$ and $m = s^\perp$. Deformations u are constrained by the multiplicative ansatz

$$\nabla u = R(I + \gamma s \otimes m), \quad R \in SO(2),$$

with $\gamma \in L^2(\Omega)$ allowed on the soft layer and $\gamma = 0$ on the rigid layer. The ε -scale energy functional is $E_\varepsilon : W^{1,2}(\Omega; \mathbb{R}^2) \cap L_0^2(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R} \cup \{\infty\}$ as

$$E_\varepsilon(u) := \begin{cases} \int_\Omega \gamma^2 dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \text{ where} \\ & \nabla u = R(I + \gamma s \otimes m), \\ & R \in L^\infty(\Omega; SO(2)), \gamma \in L^2(\Omega), \\ & \gamma = 0 \text{ a.e. in } \varepsilon Y_{rig} \cap \Omega, \\ \infty & \text{otherwise.} \end{cases} \quad (5.3)$$

Under this setting, the following basic homogenization result was obtained, together with a homogenization result including an additional linear dissipation term in the special case where the slip system is parallel to the layers.

Theorem 5 (Homogenization via Γ -convergence — single-slip case). (*Christowiak and Kreisbeck, 2017, Th. 1.1*) As $\varepsilon \rightarrow 0$, the functionals $E_\varepsilon^{(s)}$ Γ -converge with respect to the strong L^2 -topology to

$$E(u) = \begin{cases} \frac{s_1^2}{\lambda} \int_\Omega \gamma^2 dx - 2s_1 s_2 \int_\Omega \gamma dx, & u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \\ & \nabla u = R(I + \gamma e_1 \otimes e_2), R \in SO(2), \\ & \gamma \in L^2(\Omega), \gamma(x) \in K_{s,\lambda} \text{ a.e. in } \Omega \\ +\infty, & \text{otherwise,} \end{cases}$$

where $s = (s_1, s_2)$ and the constraint set is

$$K_{s,\lambda} = \begin{cases} \{0\}, & s = e_2, \\ \mathbb{R}, & s = e_1, \\ [-2s_1/(s_2\lambda), 0], & s_1 s_2 > 0, \\ [0, -2s_1/(s_2\lambda)], & s_1 s_2 < 0. \end{cases}$$

Theorem 6 (Adding a linear term for $s = e_1$). (*Christowiak and Kreisbeck, 2017, Th. 4.1*) For $s = e_1$, i.e., when the slip direction is parallel to the layers, we consider the family with a

linear contribution in the energy on the soft layer,

$$E_\epsilon^\tau(u) = \begin{cases} \int_\Omega \gamma^2 + \tau |\gamma| dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \text{ where} \\ & \nabla u = R(I + \gamma e_1 \otimes e_2), \\ & R \in L^\infty(\Omega; SO(2)), \gamma \in L^2(\Omega), \\ & \gamma = 0 \text{ a.e. in } \epsilon Y_{rig} \cap \Omega, \\ \infty & \text{otherwise.} \end{cases}$$

Its Γ -limit is

$$E^\tau(u) = \begin{cases} \frac{1}{\lambda} \int_\Omega \gamma^2 dx + \tau \int_\Omega |\gamma| dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \nabla u = R(\mathbb{I} + \gamma e_1 \otimes e_2) \\ & R \in SO(2), \gamma \in L^2(\Omega) \\ \infty & \text{otherwise.} \end{cases}$$

The added linear term plays the role of a dissipation term in the static problem and this result can be interpreted as an approximation of the first step in a time-discrete scheme. Note that in the Γ -limit the admissible slip direction is restricted to be parallel to the layers, regardless of the slip direction in the original model, and that the integrand of the Γ -limit coincides with the rank-one convex envelope of the integrand in the original model.

We now proceed to present our main results. As in the result mentioned above, the slip direction is restricted to be parallel to the layers.

Theorem 7 (Homogenization via Γ -convergence – orthogonal case).

Let $v_2 = v_1^\perp$. Then the family $(E_\epsilon)_\epsilon$ Γ -converges as $\epsilon \rightarrow 0$ with respect to the strong $L^2(\Omega; \mathbb{R}^2)$ -topology to the functional

$$E(u) = \begin{cases} \lambda \int_\Omega W_{hom}(N) dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \nabla u = R(I + \gamma e_1 \otimes e_2), \\ & R \in SO(2), \gamma \in L^2(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Here, N is given by the formula $\lambda N + (1 - \lambda)R = \nabla u$, and $W_{hom} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ is defined by

$$W_{hom}(F) := \begin{cases} |Fv_1^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_1| \leq 1, \\ |Fv_2^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_2| \leq 1, \\ \chi(|Fv_3|) & \text{if } \det F = 1, |Fv_1|, |Fv_2| > 1 \text{ and } Fv_1 \cdot Fv_2 > 0, \\ \chi(|Fv_3^\perp|) & \text{if } \det F = 1, |Fv_1|, |Fv_2| > 1 \text{ and } Fv_1 \cdot Fv_2 < 0, \\ \infty & \text{otherwise,} \end{cases} \quad (5.4)$$

where $v_3 = -(v_1 + v_2)/|v_1 + v_2| = -(v_1 + v_2)/\sqrt{2}$, and the function χ is defined by

$$\chi(z) = \left((2z^2 - 1)_+^{1/2} - 1 \right)_+^2, \quad z \in \mathbb{R}.$$

The symbol $(a)_+^s$ for $a \in \mathbb{R}$ and $s > 0$ means $(\max\{a, 0\})^s$.

The function W_{hom} appearing in Theorem 7 coincides with the rank-one convex envelope of W in (5.2) obtained in Conti, Dolzmann, and Kreisbeck, 2013b. Note that

if $\det F = 1$ then $|Fv_1|, |Fv_2| > 1$ and $Fv_1 \cdot Fv_2 = 0$ cannot happen at the same time (see proof of Lemma 5). Moreover, since $|Fv_3| \geq |Fv_3^\perp|$ is equivalent to $Fv_1 \cdot Fv_2 \geq 0$ (see Lemma 7), we can simply write

$$W_{hom}(F) = \chi(\max\{|Fv_3|, |Fv_3^\perp|\}) \quad \text{if } \det F = 1, |Fv_1|, |Fv_2| > 1.$$

Theorem 8 (Homogenization via Γ -convergence – general case).

Let (v_1, v_2) form a right-handed system with angle $2\theta \in (\pi/2, \pi)$. Let the family $(E_\epsilon)_\epsilon$ Γ -converge with respect to the strong $L^2(\Omega; \mathbb{R}^2)$ -topology to an integral functional $E(u) : L_0^2(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ whose integrand is a function of ∇u . Then E can be represented as

$$E(u) = \begin{cases} \lambda \int_{\Omega} W_{hom}(N) dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \nabla u = R(I + \gamma e_1 \otimes e_2), \\ & R \in SO(2), \gamma \in L^2(\Omega), \\ \infty & \text{otherwise.} \end{cases}$$

Here, N is given by the formula $\lambda N + (1 - \lambda)R = \nabla u$, and $W_{hom} : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ is given in part by

$$W_{hom}(F) := \begin{cases} |Fv_1^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_1| = 1, \\ |Fv_2^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_2| = 1, \\ h(|Fv_3|) & \text{if } \det F = 1, |Fv_1|, |Fv_2| < 1, \\ h(|Fv_3|) & \text{if } \det F = 1, |Fv_1|, |Fv_2| > 1 \text{ and } Fv_1 \cdot Fv_2 > 0, \\ h^\perp(|Fv_3^\perp|) & \text{if } \det F = 1, |Fv_1|, |Fv_2| > 1 \text{ and } Fv_1 \cdot Fv_2 < 0, \\ \infty & \text{if } \det F \neq 1, \end{cases}$$

where $v_3 = -(v_1 + v_2)/|v_1 + v_2|$, and the functions $h, h^\perp$ are defined by

$$h(z) := \left(\left(\frac{z^2}{\sin^2 \theta} - 1 \right)_+^{1/2} - \frac{\cos \theta}{\sin \theta} \right)_+^2, \quad h^\perp(z) := \left(\left(\frac{z^2}{\cos^2 \theta} - 1 \right)_+^{1/2} - \frac{\sin \theta}{\cos \theta} \right)_+^2.$$

Note that this is a conditional and partial statement in the sense that it is assumed that the Γ -limit is an integral functional, while $W_{hom}(F)$ remains unknown for matrices F with either $|Fv_1| > 1, |Fv_2| < 1$ or $|Fv_1| < 1, |Fv_2| > 1$. On the other hand, in the orthogonal case $\theta = \pi/4$, we always have $\min\{|Fv_1|, |Fv_2|\} \geq 1$ for $\det F = 1$, and both functions h and $h^\perp$ reduce to the function χ from Theorem 7.

The basic flow of the proof when the slip systems are orthogonal builds upon the one proposed in Christowiak and Kreisbeck, 2017. In particular, in the proof of existence of recovery sequences, we approximate a limit deformation gradient (precisely, the matrix N in Theorem 7) as a rank-one convex combination of matrices with finite energy. Since the set of admissible deformations is widened by increasing the number of slip systems to two, the construction of rank-one convex combinations needs to be revised. In addition, due to the higher complexity of the homogenized functional, the simple approach adopted in Christowiak and Kreisbeck, 2017 of reducing the construction of recovery sequences for general admissible functions for the Γ -limit to the case of piecewise affine functions does not work. Similarly, the fact that, unlike the problem with only one slip system, the homogenized functional E has a different form from the original functional E_ϵ makes the proof of $\liminf$ inequality difficult. Namely, the different form of the homogenized functional does not allow to use the same method as in Corollary 2.5 of Christowiak and Kreisbeck, 2017 to pass to limit along a sequence of non-admissible deformations when proving

the liminf inequality. Another essential complication in the proof of liminf inequality is that a sequence converging to an admissible deformation can choose from two slip directions at each point.

In this paper, we addressed the above issues in the following way. When proving the existence of recovery sequences, we first show that, in general, if piecewise affine functions are dense in the set of admissible functions for the Γ -limit, it is sufficient to deal with the case of piecewise affine limit functions. Subsequently, the piecewise affine case is realized through the construction of rank-one convex combinations, referring to Conti, Dolzmann, and Kreisbeck, 2013b. In order to prove the liminf inequality, the energies of convergent sequences sliding in different directions at each point are collectively evaluated by a judiciously chosen function f , defined in Lemma 2. The important point is that f can be chosen as a convex function. This function f plays an intermediary role to connect the different forms of the original energy density W and its homogenization W_{hom} . We show that the extra terms arising due to this discrepancy between the densities can be estimated employing the function f . The same approach generalizes to the problem with arbitrary angles between slip directions.

Section 5.2 is devoted to the search for generalized convex envelopes (Dacorogna, 2008) of the energy density function in the general case of arbitrary slip systems. Due to the interference of the slip systems, the envelopes are identified using certain symmetries only in a certain subset of $\mathbb{R}^{2 \times 2}$. Based on this information, we then reveal the form of the Γ -limit for arbitrary slip systems summarized in Theorem 8.

5.1 Homogenization for Orthogonal Slips

In this section, we present the details of proof of the main result, Theorem 7, which deals with the special case of orthogonal slip systems. We start with compactness, continue with the proof of liminf inequality and finish with a construction of recovery sequences, as outlined after the statement of main theorems.

5.1.1 Compactness

To begin with, we look at the asymptotic behavior of material with stiff layers in the limit of vanishing layer width ϵ . It has been described fully in the following fundamental proposition quoted from Christowiak and Kreisbeck, 2017.

Proposition 4 (Weak limit of deformation of materials with stiff layers).

Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Suppose that the sequence $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ satisfies $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ as $\epsilon \rightarrow 0$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\det \nabla u = 1$ a.e. in Ω , and

$$\nabla u_\epsilon \in SO(2) \quad \text{a.e. in } \Omega \cap \epsilon Y_{rig} \quad \forall \epsilon > 0,$$

with Y_{rig} defined in (5.1). Then there exists a matrix $R \in SO(2)$ and a function $\gamma \in L^2(\Omega)$ such that

$$\nabla u = R(I + \gamma e_1 \otimes e_2). \quad (5.5)$$

Furthermore,

$$\nabla u_\epsilon \mathbb{1}_{\epsilon Y_{rig} \cap \Omega} \rightharpoonup |Y_{rig}| R \quad \text{in } L^2(\Omega; \mathbb{R}^2). \quad (5.6)$$

Remark 8. The function γ in (5.5) is independent of x_1 in the sense that its distributional derivative $\partial_1 \gamma$ vanishes (Christowiak and Kreisbeck, 2017).

Since any weakly convergent sequence $(u_\epsilon)_\epsilon$ of bounded energy $(E_\epsilon)_\epsilon$ fulfills the assumptions of this proposition, one obtains a necessary condition on the form of deformations for the homogenized material. Note that the conclusion of the proposition is independent of the structure of slip systems in soft layers.

To prove compactness, we recall that the original energy has the form

$$E_\epsilon(u) = \begin{cases} \int_\Omega W(\nabla u) dx & \text{if } u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2) \text{ and } \nabla u \in K_\epsilon, \\ \infty & \text{otherwise,} \end{cases}$$

for a set $K_\epsilon \subset L^2(\Omega; \mathbb{R}^{2 \times 2})$, where $W : \mathbb{R}^{2 \times 2} \rightarrow [0, \infty]$ has quadratic growth, i.e., $W(F) \geq \mu|F|^2 - c$ for some $\mu > 0$. When $(u_j)_j$ is a sequence satisfying $E_{\epsilon_j}(u_j) < c$ for some $c > 0$ and all j , this implies that $(\|\nabla u_j\|_{L^2})_j$ is uniformly bounded. Since $\int_\Omega u_j dx = 0$, $(u_j)_j$ is uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^2)$ due to Poincaré-Wirtinger inequality. Therefore, we can choose a subsequence $(u_j)_j$ (not relabeled) to be weakly convergent in $W^{1,2}(\Omega; \mathbb{R}^2)$. Finally, using Rellich-Kondrachov theorem, we find that $(u_j)_j$ converges also strongly in $L^2(\Omega; \mathbb{R}^2)$ to a limit $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L_0^2(\Omega; \mathbb{R}^2)$. To see that $E(u)$ is finite, it is enough to note that the convergence $u_j \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ implies $\det \nabla u_j \xrightarrow{*} \det \nabla u$ in the sense of measures and to apply Proposition 4 (cf. Morrey, 1966; Reshetnyak, 1968).

5.1.2 Lower Bound

The following fact, which is a generalization of Corollary 2.5 from Christowiak and Kreisbeck, 2017, plays an important role in the proof of the liminf inequality (4.3).

Proposition 5. *Let Ω be a cube, to say $\Omega = Q = (0, l)^2$ for $l > 0$, and let $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ be such that $E_\epsilon(u_\epsilon) \leq c$ for all $\epsilon > 0$ and $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for some $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with gradient of the form (5.5). If, in addition, u is finitely piecewise affine, then*

$$\liminf_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) \geq \lambda \int_\Omega W_{\text{hom}} \left(\frac{1}{\lambda} (\nabla u - (1 - \lambda)R) \right) dx,$$

where W_{hom} is defined in (5.4).

In order to prove the assertion, we prepare a lemma, which will be used also later on.

Lemma 2. *Define the function $f : \mathbb{R}^{2 \times 2} \rightarrow [0, \infty)$ by*

$$f(F) := \max \left\{ (|Fv_1|^2 - 1)_+, (|Fv_2|^2 - 1)_+, \chi \left(\max\{|Fv_3|, |Fv_3^\perp|\} \right) \right\}.$$

Then f is convex and coincides with W on $\mathcal{M}$ and with W_{hom} on the set $\mathcal{N} := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1\}$. Thus, W_{hom} is continuous on $\mathcal{N}$.

Proof. Since the maximum of convex functions is convex and $(|Fv_1|^2 - 1)_+, (|Fv_2|^2 - 1)_+$ are convex functions, it is only necessary to confirm that the third term is a convex function. This can be seen from the fact that $\chi(z)$ is nondecreasing for $z \geq 0$ and $F \mapsto |Fv|$ is convex for any $v \in \mathbb{R}^2$.

If $F \in \mathcal{M}$ by symmetry we can assume that $F = R(I + \gamma v_1 \otimes v_2)$. Then $|Fv_1|^2 - 1 = 0$, $|Fv_2|^2 - 1 = \gamma^2$ and a short calculation shows that $\chi(\max\{|Fv_3|, |Fv_3^\perp|\}) = \chi(\sqrt{1 + |\gamma| + \gamma^2/2}) = \gamma^2$, which implies that $f(F) = \gamma^2 = W(F)$. The coincidence of f and W_{hom} on $\mathcal{N}$ follows from the results in Conti, Dolzmann, and Kreisbeck, 2013b (Step 2 in the proof of Theorem 1.1). $\square$

Proof of Proposition 5. In the following we can assume that u is affine in Ω . If otherwise, we can slightly generalize Ω to be a cuboid instead of a cube, and apply the same argument to each affine part of u . Indeed, since by Remark 8 the limit is a function of x_2 only, each piecewise affine region is a cuboid.

Since $\nabla u_\epsilon \rightharpoonup \nabla u$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$, and, by (5.6), $\nabla u_\epsilon \mathbb{1}_{\epsilon Y_{rig}} \rightharpoonup (1 - \lambda)R$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$, it follows that

$$\nabla u_\epsilon \mathbb{1}_{\epsilon Y_{soft}} \rightharpoonup \nabla u - (1 - \lambda)R \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \quad (5.7)$$

Define the sets Ω_ϵ^i and Ω_ϵ^0 via

$$\Omega_\epsilon^i := (0, l) \times ((i-1)\epsilon, i\epsilon) \quad (i = 1, \dots, \lfloor l/\epsilon \rfloor), \quad \Omega_\epsilon^0 := \Omega \setminus \bigcup_{i=1}^{\lfloor l/\epsilon \rfloor} \Omega_\epsilon^i.$$

Then $|\Omega_\epsilon^0| < l\epsilon$ and since $|\int_{\Omega'} \nabla u_\epsilon dx| \leq |\Omega'|^{1/2} \|\nabla u_\epsilon\|_{L^2(\Omega)}$ for $\Omega' \subset \Omega_\epsilon^0$, we deduce

$$\int_{\Omega_\epsilon^0 \cap \epsilon Y_{soft}} \nabla u_\epsilon dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \quad (5.8)$$

In the same fashion, $\max\{(|\nabla u_\epsilon v_1|^2 - 1)_+, (|\nabla u_\epsilon v_2|^2 - 1)_+\} \leq |\nabla u_\epsilon|^2$ and $\chi(\max\{|\nabla u_\epsilon v_3|, |\nabla u_\epsilon v_3^\perp|\}) \leq 2|\nabla u_\epsilon|^2$ imply

$$\int_{\Omega_\epsilon^0} f(\nabla u_\epsilon) dx \leq \int_{\Omega_\epsilon^0} 2|\nabla u_\epsilon|^2 dx \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

Considering that $f(\nabla u_\epsilon) = 0$ a.e. in $\epsilon Y_{rig} \cap \Omega$ and that $W(\nabla u_\epsilon) = f(\nabla u_\epsilon)$ a.e. in $\epsilon Y_{soft} \cap \Omega$ by Lemma 2, we have

$$\begin{aligned} \liminf_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) &= \liminf_{\epsilon \rightarrow 0} \int_{\epsilon Y_{soft} \cap \Omega} W(\nabla u_\epsilon) dx = \liminf_{\epsilon \rightarrow 0} \int_{\Omega} f(\nabla u_\epsilon) dx \\ &= \liminf_{\epsilon \rightarrow 0} \left(\sum_{i=1}^{\lfloor l/\epsilon \rfloor} \int_{\Omega_\epsilon^i \cap \epsilon Y_{soft}} f(\nabla u_\epsilon) dx + \int_{\Omega_\epsilon^0} f(\nabla u_\epsilon) dx \right) \\ &\geq \liminf_{\epsilon \rightarrow 0} \lambda \epsilon l \sum_{i=1}^{\lfloor l/\epsilon \rfloor} f \left(\frac{1}{\lambda \epsilon l} \int_{\Omega_\epsilon^i \cap \epsilon Y_{soft}} \nabla u_\epsilon dx \right) \\ &\geq \liminf_{\epsilon \rightarrow 0} \lambda \epsilon l \cdot \lfloor l/\epsilon \rfloor \cdot f \left(\frac{1}{\lfloor l/\epsilon \rfloor} \sum_{i=1}^{\lfloor l/\epsilon \rfloor} \frac{1}{\lambda \epsilon l} \int_{\Omega_\epsilon^i \cap \epsilon Y_{soft}} \nabla u_\epsilon dx \right), \end{aligned}$$

where we have used the continuous and discrete form of Jensen's inequality in the last two steps, respectively. From (5.8), we may continue as

$$\begin{aligned} &= \liminf_{\epsilon \rightarrow 0} \lambda |\Omega| \cdot f \left(\frac{1}{\lfloor l/\epsilon \rfloor} \frac{1}{\lambda \epsilon l} \int_{\Omega \cap \epsilon Y_{soft}} \nabla u_\epsilon dx \right) \\ &= \lambda |\Omega| \cdot f \left(\frac{1}{\lambda} (\nabla u - (1 - \lambda)R) \right) \\ &= \lambda |\Omega| \cdot W_{hom} \left(\frac{1}{\lambda} (\nabla u - (1 - \lambda)R) \right). \end{aligned}$$

Here, we employed (5.7) and the continuity of f in the second equality. $\square$

We proceed to the proof of the liminf inequality. Let Ω be a cube, that is, $\Omega = Q = (0, l)^2$ for $l > 0$. The proof for the general case is done by approximating Ω by a finite number of cubes and taking the supremum in the resulting liminf inequality over all such approximations.

Let $(\epsilon_j)_j$ fulfill $\epsilon_j \rightarrow 0$ as $j \rightarrow \infty$, and let $(u_j)_j$ be a sequence with $u_j \rightarrow u$ as $j \rightarrow \infty$ in $L^2(\Omega; \mathbb{R}^2)$ such that $(E_{\epsilon_j}(u_j))_j$ is bounded. Then, similarly as in the compactness proof (Section 5.1.1), the limit $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ satisfies $\nabla u = R(I + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and $\gamma \in L^2(\Omega)$.

If γ is piecewise constant, Corollary 5 implies the liminf inequality, i.e.,

$$\liminf_{j \rightarrow \infty} E_{\epsilon_j}(u_j) \geq \int_{\Omega} W_{hom} \left(\frac{1}{\lambda} (\nabla u - (1 - \lambda)R) \right) dx = E(u).$$

The general case $\gamma \in L^2(\Omega)$ can be reduced to the previous one through approximating γ by piecewise constant functions. Since γ is essentially a function of x_2 only (cf. Remark 8), one can approximate γ in one-dimension and extend it constantly in the x_1 -direction. Let $\gamma^{(\epsilon)} \in C_0^\infty(0, l)$ be such a one-dimensional proxy for γ fulfilling $\|\gamma^{(\epsilon)} - \gamma\|_{L^2(\Omega)} \leq \epsilon$, and $(\zeta_k^{(\epsilon)})_k \subset L^2(\Omega)$ be a approximation by simple functions for $\gamma^{(\epsilon)}$ such that $\|\zeta_k^{(\epsilon)} - \gamma^{(\epsilon)}\| < 1/k$. By a diagonal argument we obtain a subsequence $\zeta_k := \zeta_k^{(\epsilon_k)}$ that satisfies

$$\zeta_k \rightarrow \gamma \quad \text{in } L^2(\Omega) \quad \text{as } k \rightarrow \infty, \quad \zeta_k = \sum_{i=1}^{n_k} \zeta_{k,i} \mathbb{1}_{(t_{k,i-1}, t_{k,i})},$$

where $(t_{k,i-1}, t_{k,i})_i$ are divisions $0 = t_{k,0} < t_{k,1} < \dots < t_{k,n_k} = l$ of the interval $(0, l)$. This approximation can be constructed so that the subdivisions are nested. For ζ_k obtained in this way, let $w_k \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L_0^2(\Omega; \mathbb{R}^2)$ be given by $\nabla w_k = R(I + \zeta_k e_1 \otimes e_2)$ for $k \in \mathbb{N}$.

Adapting the method from Müller, 1987, we construct $(\omega_j)_j, (\omega_{k,j})_j \subset W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2)$ weakly converging to u and w_k , respectively:

$$\omega_j \rightharpoonup u, \quad \omega_{k,j} \rightharpoonup w_k \quad \forall k, \quad \text{both in } W^{1,2}(\Omega; \mathbb{R}^2) \quad \text{as } j \rightarrow \infty. \quad (5.9)$$

Furthermore, we require that $(\omega_j)_j, (\omega_{k,j})_j$ are defined so that

$$\nabla \omega_{k,j} = \nabla \omega_j \quad \text{in } \epsilon_j Y_{rig} \cap \Omega \quad \text{for all } j, k \in \mathbb{N}, \quad (5.10)$$

and so that there is a nondecreasing sequence of natural numbers $(k(j))_j$ diverging to ∞ as $j \rightarrow \infty$, for which

$$\|\nabla \omega_{k,j} - \nabla \omega_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \leq \frac{1}{\lambda} \|\zeta_k - \zeta_{k(j)}\|_{L^2(\Omega)} + \frac{c}{\sqrt{k(j)}} \quad (5.11)$$

holds for all $k \in \mathbb{N}$ and all $j \geq j_0$, where j_0 may depend on k . Here, c is a constant independent of j, k .

To satisfy these conditions, we define $(\omega_{k,j})_j$ with zero mean by

$$\nabla \omega_{k,j} = R + \sum_{i=1}^{n_k} (N_{k,i} - R) \mathbb{1}_{\epsilon_j Y_{soft} \cap \Omega} \mathbb{1}_{(0,l) \times ([\epsilon_j^{-1} t_{k,i-1}] \epsilon_j, [\epsilon_j^{-1} t_{k,i}] \epsilon_j)}, \quad (5.12)$$

with $N_{k,i} \in \mathcal{N}$ defined from $\zeta_{k,i}$ by

$$\lambda N_{k,i} + (1 - \lambda)R = R(I + \zeta_{k,i}e_1 \otimes e_2). \quad (5.13)$$

Note that $\omega_{k,j}$ is well defined by its gradient since $N_{k,i}$ and R rank-one connect along all segments $[0, l] \times m\epsilon_j$, $m = 1, \dots, \lfloor l/\epsilon_j \rfloor$ (see Figure 5.2).

Thanks to the averaging lemma on weak convergence of periodic functions, combined with the fact that the measure of the region around the lines $[0, l] \times t_{k,i}$, where $\nabla\omega_{k,j}$ deviates from a periodic function, vanishes for $j \rightarrow \infty$ and fixed k , we have

$$\nabla\omega_{k,j} \rightharpoonup \sum_{i=1}^{n_k} (\lambda N_{k,i} + (1 - \lambda)R) \mathbb{1}_{(0,l) \times (t_{k,i-1}, t_{k,i})} = \nabla w_k \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2})$$

for every $k \in \mathbb{N}$ as $j \rightarrow \infty$. Thus the second condition in (5.9) is fulfilled.

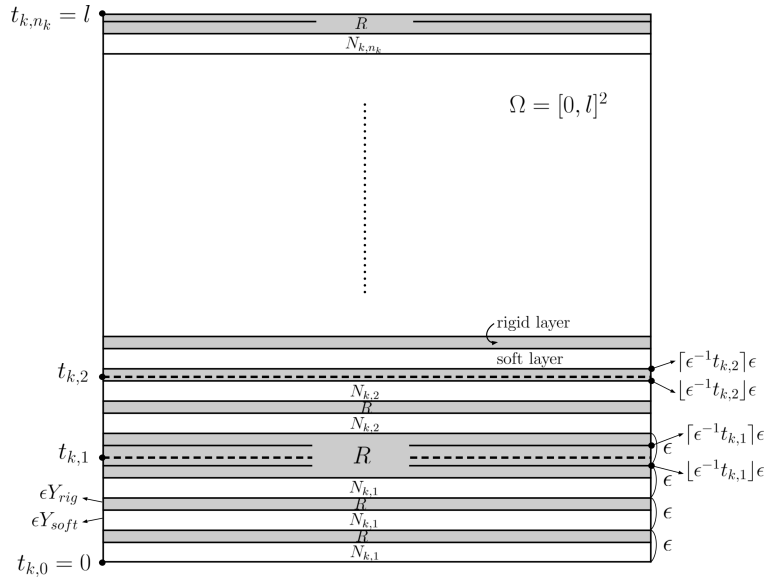


FIGURE 5.2: A sketch of the function $\nabla\omega_{k,j}$ defined in (5.12). For simplicity we omit the index j in ϵ_j . As shown by gray shading, the value of $\nabla\omega_{k,j}$ becomes R not only in rigid layers but also in certain soft layers, namely those which contain a nodal point $t_{k,i}$.

The idea to realize also the first condition in (5.9), together with (5.10) and (5.11) is to use a diagonal argument to define $\omega_j := \omega_{k(j),j}$. Then (5.10) trivially holds. Furthermore, to achieve (5.11), we set $\tau_k := \min_h (t_{k,h} - t_{k,h-1})$ and estimate

$$\begin{aligned} \|\nabla\omega_{k,j} - \nabla\omega_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} &= \|\nabla\omega_{k,j} - \nabla\omega_{k(j),j}\|_{L^2(\Omega)} \\ &\leq \left\| \sum_{i=1}^{n_k} \sum_{h=1}^{n_{k(j)}} (N_{k,i} - N_{k(j),h}) \mathbb{1}_{(0,l) \times (t_{k(i),h-1}, t_{k(i),h})} \mathbb{1}_{(0,l) \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega)} \\ &\quad + \sqrt{\frac{2\epsilon_j}{\tau_{k(j)}}} \left\| \sum_{i=1}^{n_k} \sum_{h=1}^{n_{k(j)}} (N_{k,i} - R) \mathbb{1}_{(0,l) \times (t_{k(i),h-1}, t_{k(i),h})} \mathbb{1}_{(0,l) \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega)}. \end{aligned}$$

The second term accounts for the discrepancy between $\nabla\omega_{k,j}$ and $\nabla\omega_{k(j),j}$ possibly occurring in an ϵ_j -neighborhood of the jump lines $t_{k(j),h}$, where we assume that j is large enough so that $k(j) > k$. Note that since the divisions $(t_{k,i})_i$ are nested when

k increases, the subintervals corresponding to $\zeta_{k(j)}$ are then a subdivision of those corresponding to ζ_k . Accordingly,

$$\begin{aligned}
& \|\nabla \omega_{k,j} - \nabla \omega_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \\
& \leq \frac{1}{\lambda} \left\| \sum_{i=1}^{n_k} \sum_{h=1}^{n_{k(j)}} (\zeta_{k,i} - \zeta_{k(j),h}) \mathbb{1}_{(0,I) \times (t_{k(j),h-1}, t_{k(j),h})} \mathbb{1}_{(0,I) \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega)} \\
& \quad + \frac{1}{\lambda} \sqrt{\frac{2\epsilon_j}{\tau_{k(j)}}} \left\| \sum_{i=1}^{n_k} \zeta_{k,i} \mathbb{1}_{(0,I) \times (t_{k,i-1}, t_{k,i})} \right\|_{L^2(\Omega)} \\
& = \frac{1}{\lambda} \|\zeta_k - \zeta_{k(j)}\|_{L^2(\Omega)} + \frac{1}{\lambda} \sqrt{\frac{2\epsilon_j}{\tau_{k(j)}}} \|\zeta_k\|_{L^2(\Omega)}. \tag{5.14}
\end{aligned}$$

Therefore, to have (5.9) and (5.11), it is enough to find $(k(j))_j$ so that

$$\frac{\epsilon_j}{\tau_{k(j)}} \leq \frac{1}{k(j)} \quad \forall j \quad \text{and} \quad \omega_{k(j),j} \rightharpoonup u \quad \text{in } W^{1,2}(\Omega; \mathbb{R}^2) \quad \text{as } j \rightarrow \infty.$$

We next construct such a sequence $(k(j))_j$. Noting that $(\omega_{k,j})_{k,j}$ and $(w_k)_k$ are uniformly bounded in $W^{1,2}(\Omega; \mathbb{R}^2)$, we let d denote a distance metrizing the weak topology of $W^{1,2}(\Omega; \mathbb{R}^2)$ in some closed sphere containing $(\omega_{k,j})_{k,j}$, $(w_k)_k$ and u . Taking into account the weak convergence of $\omega_{p,j}$ to w_p and the fact that $(\epsilon_j/\tau_p)_j$ is a decreasing sequence for fixed p , we can find an increasing sequence $(j(p))_p \subset \mathbb{N}$ such that $j(1) = 1$ and for $p \geq 2$,

$$d(\omega_{p,j}, w_p) \leq \frac{1}{p} \quad \text{and} \quad \frac{\epsilon_j}{\tau_p} \leq \frac{1}{p} \quad \text{when } j \geq j(p).$$

Then we define $k(j)$ for $j \in \mathbb{N}$ as follows:

$$k(j) = p \quad \text{if } j(p) \leq j \leq j(p+1) - 1.$$

For any $\delta > 0$, take $p \in \mathbb{N}$ sufficiently large so that $1/p < \min\{1, \delta\}$ and so that for any $k \geq p$ one has $d(w_k, u) < \delta$. Then $j \geq j(p)$ implies

$$d(\omega_{k(j),j}, u) \leq d(\omega_{k(j),j}, w_{k(j)}) + d(w_{k(j)}, u) < 2\delta,$$

and thus $\omega_j = \omega_{k(j),j} \rightharpoonup u$ as $j \rightarrow \infty$. In addition, by construction of $(k(j))_j$, $\epsilon_j/\tau_{k(j)} < 1/k(j)$ for $j \geq j(2)$, which in combination with (5.14) yields (5.11).

Now we set

$$z_{k,j} = u_j - \omega_j + \omega_{k,j}$$

for $j, k \in \mathbb{N}$. By (5.10), $f(\nabla z_{k,j}) = 0$ a.e. in $\epsilon_j Y_{rig} \cap \Omega$, where f is the function defined in Lemma 2. By Proposition 4, for each $k \in \mathbb{N}$,

$$\nabla z_{k,j} \mathbb{1}_{\epsilon_j Y_{rig}} \rightharpoonup (1 - \lambda)R \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}) \quad \text{as } j \rightarrow \infty.$$

Also, thanks to (5.9), $z_{k,j}$ weakly converges to the piecewise affine function w_k in $W^{1,2}(\Omega; \mathbb{R}^2)$ as $j \rightarrow \infty$.

We now show the liminf inequality using the proof of Corollary 5 and the following lemma.

Lemma 3. Let $F = A + D$, where $D = R\gamma_1 e_1 \otimes e_2$ and either $A = R(I + \gamma_2 v_2 \otimes v_1)$ or $A = R(I + \gamma_2 v_1 \otimes v_2)$ for some γ_1, γ_2 and $R \in SO(2)$. Then there exists a constant c , such that

$$f(F) \leq |F|^2 - 2 + cg(A, D)(|A| + g(A, D)),$$

where $g(A, D)$ stands for $(|D|^{\frac{1}{4}} + |D|^{\frac{1}{2}})(|A|^{\frac{1}{4}} + |A|^{\frac{1}{2}} + |D|^{\frac{1}{4}} + |D|^{\frac{1}{2}})$.

Proof. We focus on the case $|Av_2| = 1$, i.e., $A = R(I + \gamma_2 v_2 \otimes v_1)$ since the remaining case $|Av_1| = 1$ is analogous. We can also assume $R = I$ since rotations do not change any of the terms appearing in the inequality to be proved. First, we denote the components of v_i by $(v_i^{(1)}, v_i^{(2)})$ and note that $\det F = 1 - \gamma_1 \gamma_2 v_2^{(2)} v_1^{(2)}$ and thus, in view of $|A| = \sqrt{2 + \gamma_2^2}$, $|D| = |\gamma_1|$, we have

$$|1 - (\det F)^2| \leq |D|^2 |A|^2 + 2|D||A|. \quad (5.15)$$

Recalling that

$$f(F) = \max \left\{ (|Fv_1|^2 - 1)_+, (|Fv_2|^2 - 1)_+, \chi \left(\max\{|Fv_3|, |Fv_3^\perp|\} \right) \right\},$$

it is enough to prove the following 3 inequalities:

$$(|Fv_1|^2 - 1)_+ \leq |F|^2 - 2 + c(|D| + |D||A|), \quad (5.16)$$

$$(|Fv_2|^2 - 1)_+ \leq |F|^2 - 2 + c|D||A|, \quad (5.17)$$

$$\chi \left(\max\{|Fv_3|, |Fv_3^\perp|\} \right) \leq |F|^2 - 2 + cg(A, D)(|A| + g(A, D)). \quad (5.18)$$

When $|Fv_1| \geq 1$, since $|F|^2 = |Fv_2|^2 + |Fv_1|^2$, inequality (5.16) follows from

$$-|Fv_2|^2 + 1 = -|Av_2 + Dv_2|^2 + 1 \leq -(1 - |Dv_2|)^2 + 1 \leq 2|D|.$$

On the other hand, when $|Fv_1| < 1$, (5.16) is again true due to $|Av_1| \geq 1$ and

$$-|F|^2 + 2 \leq -(|Av_1| - |Dv_1|)^2 - (1 - |Dv_2|)^2 + 2 \leq 2(|D| + |D||A|).$$

The reasoning for the second inequality (5.17) is similar and we omit it.

To address the third inequality (5.18), we first notice that $|Fv_3|^2 + |Fv_3^\perp|^2 = |F|^2 = |I + (\gamma_1 e_1 \otimes e_2 + \gamma_2 v_2 \otimes v_1)|^2 \geq |I|^2 = 2$, which implies $\max\{|Fv_3|^2, |Fv_3^\perp|^2\} \geq 1$, and hence we can omit the plus subscripts in $\chi(z) = ((2z^2 - 1)_+^{1/2} - 1)_+^2$ when $z^2 = \max\{|Fv_3|^2, |Fv_3^\perp|^2\}$. For brevity, we set $x := |Fv_1|^2$, $\delta_1 := |Fv_2|^2 - 1$, $\delta_2 := |Fv_1|^2(|Fv_2|^2 - 1) + 1 - (\det F)^2$, and write

$$\begin{aligned} & \chi \left(\max\{|Fv_3|, |Fv_3^\perp|\} \right) \\ &= \left(\sqrt{|Fv_1|^2 + |Fv_2|^2 + 2\sqrt{|Fv_1|^2 |Fv_2|^2 - (\det F)^2} - 1} - 1 \right)^2 \\ &= \left(\sqrt{x + \delta_1 + 2\sqrt{x - 1 + \delta_2} - 1} \right)^2. \end{aligned}$$

For now we assume that $|Fv_1|^2 \geq 1$ to estimate

$$\begin{aligned}
& \left(\sqrt{x + \delta_1 + 2\sqrt{x-1+\delta_2}} - 1 \right)^2 - (|Fv_1|^2 - 1) \\
& \leq \left(\sqrt{x + |\delta_1| + 2\sqrt{x-1} + 2\sqrt{|\delta_2|}} - 1 \right)^2 - (x-1) \\
& \leq \left(\sqrt{x-1} + \sqrt{|\delta_1| + 2\sqrt{|\delta_2|}} \right)^2 - (\sqrt{x-1})^2 \\
& \leq \sqrt{|\delta_1| + 2\sqrt{|\delta_2|}} \left(2\sqrt{x-1} + \sqrt{|\delta_1| + 2\sqrt{|\delta_2|}} \right),
\end{aligned}$$

where we used the fact that $\sqrt{x + 2\sqrt{x-1}} = \sqrt{x-1} + 1$. It remains to show that $x-1 = |Fv_1|^2 - 1$, δ_1 and δ_2 have the right order. In view of (5.15) this follows from

$$\begin{aligned}
|Fv_1|^2 - 1 &= |(D+A)v_1|^2 - 1 \leq |D+A|^2 - 1 \leq (|D| + |A|)^2, \\
|\delta_1| &= ||Dv_2 + Av_2|^2 - 1| \leq 2|Dv_2| + |Dv_2|^2 \leq 2|D| + |D|^2, \\
|\delta_2| &\leq (|D| + |A|)^2 (2|D| + |D|^2) + |D|^2|A|^2 + 2|D||A| \leq c(g(A, D))^4.
\end{aligned}$$

Next, when $|Fv_1|^2 < 1$, we set $x := |Fv_2|^2 > 1$, $\delta_1 := |Fv_1|^2 - 1 < 0$ and $\delta_2 := |Fv_2|^2(|Fv_1|^2 - 1) + 1 - (\det F)^2$, where

$$\delta_{2+} := \max\{0, \delta_2\} \leq \max\{0, 1 - (\det F)^2\} \leq 2|D||A|.$$

Then an analogous estimation process leads to

$$\chi \left(\max\{|Fv_3|, |Fv_3^\perp|\} \right) \leq |Fv_2|^2 - 1 + 2\delta_{2+}^{1/4} \left(\sqrt{2(x-1)} + \delta_{2+}^{1/4} \right),$$

which is the desired estimate in view of having already estimated $|Fv_2|^2 - 1$. $\square$

In view of (5.12), we know that $\nabla\omega_{k,j}$ is equal to R in rigid layers and either $N_{k,i} = R(I + \frac{\zeta_{k,i}}{\lambda} e_1 \otimes e_2)$ or R in soft layers. Since $\omega_j = \omega_{k(j),j}$ is a subsequence, it satisfies the same property and thus

$$\nabla\omega_j - \nabla\omega_{k,j} = \begin{cases} 0 & \text{in } \Omega \cap \epsilon_j Y_{rig} \\ R \frac{\gamma}{\lambda} e_1 \otimes e_2 & \text{in } \Omega \cap \epsilon_j Y_{soft} \end{cases}$$

for some $\gamma \in \mathbb{R}$.

Hence, we may apply Lemma 3 to $F := \nabla z_{k,j}$ decomposed as $F = A + D$, where $A := \nabla u_j \in \mathcal{M}$ and $D := \nabla\omega_{k,j} - \nabla\omega_j$. Combined with Hölder's inequality, with the boundedness of $\|\nabla u_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})}$, $\|\zeta_k\|_{L^2(\Omega)}$ and with the estimate (5.11), this yields

$$\begin{aligned}
\int_{\Omega} f(\nabla z_{k,j}) dx &\leq \int_{\Omega} (|\nabla z_{k,j}|^2 - 2) dx + c \|\nabla\omega_{k,j} - \nabla\omega_j\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})}^{1/4} \\
&\leq \int_{\Omega} (|\nabla z_{k,j}|^2 - 2) dx + c \left(\|\zeta_k - \zeta_{k(j)}\|_{L^2(\Omega)} + \frac{1}{\sqrt{k(j)}} \right)^{1/4}.
\end{aligned}$$

Using triangle inequality and the uniform boundedness of $\|\nabla\omega_j - \nabla\omega_{k,j}\|_{L^2}$, $\|\nabla z_{k,j}\|_{L^2}$ and $\|\zeta_k - \zeta_{k(j)}\|_{L^2}$, we get

$$\begin{aligned}
\liminf_{j \rightarrow \infty} E_{\epsilon_j}(u_j) &= \liminf_{j \rightarrow \infty} \int_{\Omega} (|\nabla z_{k,j} + (\nabla\omega_j - \nabla\omega_{k,j})|^2 - 2) \, dx \\
&\geq \liminf_{j \rightarrow \infty} \left(\int_{\Omega} (|\nabla z_{k,j}|^2 - 2) \, dx - c \|\nabla\omega_j - \nabla\omega_{k,j}\|_{L^2(\Omega; \mathbb{R}^{2 \times 2})} \right) \\
&\geq \liminf_{j \rightarrow \infty} \left(\int_{\Omega} f(\nabla z_{k,j}) \, dx - c \left(\|\zeta_k - \zeta_{k(j)}\|_{L^2(\Omega)} + \frac{1}{\sqrt{k(j)}} \right)^{1/4} \right) \\
&\geq \liminf_{j \rightarrow \infty} \int_{\Omega} f(\nabla z_{k,j}) \, dx - c \|\zeta_k - \gamma\|_{L^2(\Omega)}^{1/4} \\
&\geq E(w_k) - c \|\zeta_k - \gamma\|_{L^2(\Omega)}^{1/4},
\end{aligned}$$

where the last estimate can be shown as in the proof of Corollary 5 due to the weak convergence of $(z_{k,j})_j$ to the piecewise affine function w_k together with the convergence $\nabla z_{k,j} \mathbb{1}_{\epsilon_j Y_{rig}} \rightharpoonup (1 - \lambda)R$. Therefore, taking limes inferior as $k \rightarrow \infty$, we arrive at

$$\liminf_{j \rightarrow \infty} E_{\epsilon_j}(u_j) \geq E(u)$$

because $E(w_k) = \int_{\Omega} f(\nabla w_k) \, dx$, which is a lower semicontinuous functional thanks to the convexity of f (Benešová and Kružík, 2017; Kristensen, 2015). $\square$

5.1.3 Recovery Sequence

Let $u \in W^{1,2}(\Omega; \mathbb{R}^2) \cap L_0^2(\Omega; \mathbb{R}^2)$ be such that $\nabla u = R(I + \gamma e_1 \otimes e_2) = \lambda N + (1 - \lambda)R$ with $R \in SO(2)$, $N \in \mathcal{N}$ and $\gamma \in L^2(\Omega)$. The goal of this section is to show the existence of a sequence $(u_j)_j \subset L_0^2(\Omega; \mathbb{R}^2)$ such that $u_j \rightarrow u$ in $L^2(\Omega; \mathbb{R}^2)$ and $E_{\epsilon_j}(u_j) \rightarrow E(u)$. The idea of constructing such a sequence would be to take a function $w \in W^{1,\infty} \cap L_0^\infty(Y; \mathbb{R}^2)$ with $\nabla w = R \mathbb{1}_{Y_{rig}} + N \mathbb{1}_{Y_{soft}}$ in Y , extend it periodically and set $\nabla u_\epsilon(x) = \nabla w(\frac{x}{\epsilon})$ for $x \in \Omega$. Then the laminate ∇u_ϵ converges weakly to $\lambda N + (1 - \lambda)R$, as desired. However, since N may not belong to the admissible set $\mathcal{M}$, this construction needs to be refined by grafting into the soft layer another laminate between two matrices from $\mathcal{M}$ to replace the matrix N (see Figure 5.4).

To realize this goal, we first prove in Section 5.1.3 (Lemma 4) that it is in fact sufficient to construct recovery sequences for piecewise affine limit functions u , i.e., piecewise constant functions γ . This result can be generalized to the L^p -setting for $p > 2$. Then we address the construction of suitable laminates approximating N in Section 5.1.3 and, finally, construct recovery sequences for constant and piecewise constant functions γ .

Recovery Sequence for General $\gamma \in L^2(\Omega)$

Lemma 4. *Let $(E_\epsilon)_\epsilon : W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ be a functional family and let $E : L_0^2(\Omega; \mathbb{R}^2) \rightarrow [0, \infty]$ have the form*

$$E(u) = \begin{cases} \int_{\Omega} W_{hom}(\nabla u) \, dx & \text{if } u \in K, \\ \infty & \text{otherwise.} \end{cases}$$

Here, K is a subset of $W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2)$ such that the set of finitely piecewise affine functions belonging to K is dense in K . Further, the integrand $W_{\text{hom}} : \mathbb{R}^{2 \times 2} \rightarrow [0, \infty]$ is a continuous function on $\mathcal{L} := \{\nabla u(x) : u \in K, x \in \Omega\}$ with the growth $W_{\text{hom}}(A) \leq \mu|A|^2 + c$ for some $\mu, c > 0$ and all $A \in \mathcal{L}$.

If for any finitely piecewise affine $u \in K$ there exists $(u_j)_j \subset W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2)$ such that $u_j \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ and $E_{\varepsilon_j}(u_j) \rightarrow E(u)$, then such a sequence exists for any $u \in K$.

Proof. For a given $u \in K$, let $(w_k)_k \subset K \cap W^{1,\infty}(\Omega; \mathbb{R}^2)$ be a sequence of finitely piecewise affine functions such that $w_k \rightarrow u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ as $k \rightarrow \infty$. We show that

$$\lim_{k \rightarrow \infty} E(w_k) = E(u). \quad (5.19)$$

For α with $0 < \alpha \leq 1$ we define

$$\Omega_k^\alpha := \{x \in \Omega : |(\nabla w_k - \nabla u)(x)| > \alpha\} \quad \text{and} \quad \Omega_k^{\alpha,0} := \Omega \setminus \Omega_k^\alpha. \quad (5.20)$$

Then

$$\lim_{k \rightarrow \infty} |\Omega_k^\alpha| = 0. \quad (5.21)$$

Now, let $k \in \mathbb{N}$ be fixed and consider u as a function over $\Omega_k^{1,0}$. By assumption, W_{hom} is uniformly continuous on the set $\mathcal{L} \cap \{F \in \mathbb{R}^{2 \times 2} : |F| \leq \|\nabla u\|_{L^\infty(\Omega_k^{1,0}; \mathbb{R}^{2 \times 2})} + 1\}$. Note that by (5.20), $\nabla u(x)$ and $\nabla w_{k'}(x)$ belong to this set for all $x \in \Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}$ and for all $k' \in \mathbb{N}$. Therefore, for any $\varepsilon > 0$ one can find a sufficiently small $\alpha(k, \varepsilon)$ such that for every $k' \in \mathbb{N}$,

$$|W_{\text{hom}}(\nabla w_{k'}) - W_{\text{hom}}(\nabla u)| < \varepsilon \quad \text{in } \Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}.$$

It follows that

$$\int_{\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}} W_{\text{hom}}(\nabla w_{k'}) \, dx < \varepsilon |\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}| + \int_{\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}} W_{\text{hom}}(\nabla u) \, dx. \quad (5.22)$$

Hence, by the growth assumption on W_{hom} ,

$$E(w_{k'}) \leq \int_{\Omega_k^{1,0} \cup (\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0})} (\mu |\nabla w_{k'}|^2 + c) \, dx + \int_{\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}} W_{\text{hom}}(\nabla w_{k'}) \, dx. \quad (5.23)$$

The convergence $\nabla w_{k'} \rightarrow \nabla u$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ together with (5.21) imply that the first term on the right side converges to $\int_{\Omega_k^1} (\mu |\nabla u|^2 + c) \, dx$ when $k' \rightarrow \infty$. Similarly, in view of (5.22),

$$\limsup_{k' \rightarrow \infty} \int_{\Omega_k^{1,0} \cap \Omega_{k'}^{\alpha,0}} W_{\text{hom}}(\nabla w_{k'}) \, dx \leq \varepsilon |\Omega_k^{1,0}| + \int_{\Omega_k^{1,0}} W_{\text{hom}}(\nabla u) \, dx,$$

so that (5.23) yields

$$\limsup_{k' \rightarrow \infty} E(w_{k'}) \leq \int_{\Omega_k^1} (\mu |\nabla u|^2 + c) \, dx + \varepsilon |\Omega| + \int_{\Omega} W_{\text{hom}}(\nabla u) \, dx.$$

Taking $\varepsilon \rightarrow 0+$ and $k \rightarrow \infty$, we get from (5.21),

$$\limsup_{k' \rightarrow \infty} E(w_{k'}) \leq \int_{\Omega} W_{\text{hom}}(\nabla u) \, dx.$$

An analogous argument, exploiting the nonnegativity of W_{hom} in place of the growth condition, leads to the opposite inequality for the lower limit, and we conclude

$$\lim_{k' \rightarrow \infty} E(w_{k'}) = E(u). \quad (5.24)$$

By assumption, we can obtain a recovery sequence $(w_{k,j})_j$ for w_k such that $\lim_{j \rightarrow 0} E_{\epsilon_j}(w_{k,j}) = E(w_k)$ and $w_{k,j} \rightharpoonup w_k$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ as $j \rightarrow \infty$ for all $k \in \mathbb{N}$. By Rellich-Kondrachov theorem, $(w_{k,j})_j$ can be assumed to converge to w_k strongly in $L^2(\Omega; \mathbb{R}^2)$. Combining this with $\lim_{k \rightarrow \infty} \|w_k - u\|_{L^2(\Omega; \mathbb{R}^2)} = 0$ and with (5.19), we have

$$\lim_{k \rightarrow \infty} \lim_{j \rightarrow 0} \left(\|w_{k,j} - u\|_{L^2(\Omega; \mathbb{R}^2)} + |E_{\epsilon_j}(w_{k,j}) - E(u)| \right) = 0.$$

A simplified version of the diagonalization argument introduced in the proof of the liminf inequality (see also Attouch, 1984, Cor. 1.18), yields the existence of a sequence $(u_j)_j$ such that $u_j = w_{k(j),j}$ and

$$u_j \rightarrow u \text{ in } L^2(\Omega; \mathbb{R}^2) \quad \text{and} \quad E_{\epsilon_j}(u_j) \rightarrow E(u)$$

as $j \rightarrow 0$. It remains to show that u_j converges to u weakly in $W^{1,2}(\Omega; \mathbb{R}^2)$. This follows by noting that any subsequence of $(u_j)_j$ contains a subsequence weakly converging to u . $\square$

We verify that W_{hom} given by (5.4) satisfies the assumptions of Lemma 4. Indeed, since $K = \{u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2) : \nabla u = R(I + \gamma e_1 \otimes e_2), R \in SO(2), \gamma \in L^2(\Omega)\}$, taking an arbitrary $u \in K$ we see that $\nabla u = R(I + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and this γ is independent of x_1 due to $0 = \text{curl } \nabla u = -(\partial \gamma / \partial x_1) R e_1$ (see Remark 8). Hence we may use a one-dimensional approximation of γ by a sequence of simple functions $(\zeta_k)_k$, i.e., $\zeta_k \rightarrow \gamma$ in $L^2(\Omega)$. Then the vanishing curl of $R(I + \zeta_k e_1 \otimes e_2)$ guarantees the existence of a piecewise affine $w_k \in W^{1,2}(\Omega; \mathbb{R}^2)$ with $\nabla w_k = R(I + \zeta_k e_1 \otimes e_2)$ and with mean value zero. In combination with Poincaré's inequality we have $w_k \rightarrow u$ in $W^{1,2}(\Omega; \mathbb{R}^{2 \times 2})$, as desired. Moreover, the continuity of W_{hom} on $\mathcal{L}$ follows from Lemma 2 and the relation $\mathcal{L} \subset \mathcal{N}$.

Remark 9. For the specific form of energy that we have, one can directly show that it is locally Lipschitz continuous in γ (see below for details). Namely, there is a constant c depending only on λ such that

$$|W_{hom}(\gamma_1) - W_{hom}(\gamma_2)| \leq c(1 + |\gamma_1| + |\gamma_2|)|\gamma_1 - \gamma_2| \quad \forall \gamma_1, \gamma_2 \in \mathbb{R},$$

where we abuse notation to write $W_{hom}(\gamma)$ instead of $W_{hom}(\nabla u)$ when $\nabla u = R(I + \gamma e_1 \otimes e_2)$. This fact allows for a simpler proof in the sense that the argument leading to (5.24) can be skipped.

(Proof of Remark 9.) Let (a, b) stand for the components of the vector v_1 . Then v_2 has components $(-b, a)$. For $\nabla u = R(I + \gamma e_1 \otimes e_2)$ we have $N = R(I + (\gamma/\lambda)e_1 \otimes e_2)$ and thus

$$\begin{aligned} |Nv_2|^2 &= 1 - 2ab\frac{\gamma}{\lambda} + a^2\frac{\gamma^2}{\lambda^2}, & |Nv_1|^2 &= 1 + 2ab\frac{\gamma}{\lambda} + b^2\frac{\gamma^2}{\lambda^2}, \\ |N(v_1 \pm v_2)|^2 &= 2 \pm 2\frac{\gamma}{\lambda}(a^2 - b^2) + \frac{\gamma^2}{\lambda^2}(1 \pm 2ab). \end{aligned}$$

Then for $ab > 0$,

$$\begin{aligned}
|Nv_2| \leq 1 &\Leftrightarrow 0 \leq \gamma \leq 2\lambda \frac{b}{a} \\
|Nv_1| \leq 1 &\Leftrightarrow -2\lambda \frac{a}{b} \leq \gamma \leq 0 \\
|Nv_2|, |Nv_1| \geq 1 &\Leftrightarrow \gamma \geq 2\lambda \frac{b}{a} \text{ or } \gamma \leq -2\lambda \frac{a}{b} \\
|N(v_1 + v_2)| \geq |N(v_1 - v_2)| &\Leftrightarrow \gamma \geq \max \left\{ 0, \lambda \frac{b^2 - a^2}{ab} \right\} \\
&\quad \text{or } \gamma \leq \min \left\{ 0, \lambda \frac{b^2 - a^2}{ab} \right\},
\end{aligned}$$

where the last inequality $|N(v_1 + v_2)| \geq |N(v_1 - v_2)|$ always holds when $|Nv_2| \geq 1$ and $|Nv_1| \geq 1$. An analogous calculation is done for the case $ab < 0$ leading to the following expression for W_{hom} when $ab > 0$:

$$W_{hom}(N) = \begin{cases} 2ab \frac{\gamma}{\lambda} + b^2 \frac{\gamma^2}{\lambda^2} & \text{when } 0 \leq \gamma \leq 2\lambda \frac{b}{a} \\ -2ab \frac{\gamma}{\lambda} + a^2 \frac{\gamma^2}{\lambda^2} & \text{when } -2\lambda \frac{a}{b} \leq \gamma \leq 0 \\ \left(\sqrt{(1 + 2(a^2 - b^2) \frac{\gamma}{\lambda} + (1 + 2ab) \frac{\gamma^2}{\lambda^2})_+} - 1 \right)^2 & \text{otherwise,} \end{cases}$$

and when $ab < 0$:

$$W_{hom}(N) = \begin{cases} 2ab \frac{\gamma}{\lambda} + b^2 \frac{\gamma^2}{\lambda^2} & \text{when } 2\lambda \frac{b}{a} \leq \gamma \leq 0 \\ -2ab \frac{\gamma}{\lambda} + a^2 \frac{\gamma^2}{\lambda^2} & \text{when } 0 \leq \gamma \leq -2\lambda \frac{a}{b} \\ \left(\sqrt{(1 + 2(b^2 - a^2) \frac{\gamma}{\lambda} + (1 - 2ab) \frac{\gamma^2}{\lambda^2})_+} - 1 \right)^2 & \text{otherwise.} \end{cases}$$

The $+$ subscript can be removed in the given ranges because, for example in the setting $ab > 0$, for $\gamma \geq 2\lambda(b/a)$ one can estimate

$$1 + 2(a^2 - b^2) \frac{\gamma}{\lambda} + (1 + 2ab) \frac{\gamma^2}{\lambda^2} = \left(1 + \frac{\gamma}{\lambda}\right)^2 + 2ab \frac{\gamma}{\lambda} \left(\frac{\gamma}{\lambda} - 2\frac{b}{a}\right) \geq 1,$$

and similarly for the remaining cases.

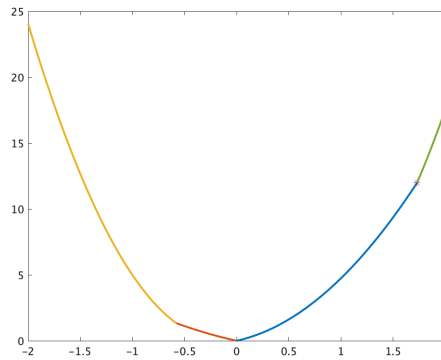


FIGURE 5.3: Graph of W_{hom} as function of γ for $v_1 = (1/2, \sqrt{3}/2)$ and $\lambda = 0.5$.

Estimating the derivative with respect to γ when $ab > 0$, we see that

- for $0 \leq \gamma \leq 2\lambda \frac{b}{a}$: $\left| \frac{\partial W_{hom}}{\partial \gamma} \right| = \left| 2\frac{ab}{\lambda} + 2\frac{b^2}{\lambda^2}\gamma \right| \leq \frac{2}{\lambda^2}(1 + |\gamma|)$
- for $-2\lambda \frac{a}{b} \leq \gamma \leq 0$: $\left| \frac{\partial W_{hom}}{\partial \gamma} \right| = \left| -2\frac{ab}{\lambda} + 2\frac{a^2}{\lambda^2}\gamma \right| \leq \frac{2}{\lambda^2}(1 + |\gamma|)$
- otherwise: setting $R = \sqrt{1 + 2(b^2 - a^2)\frac{\gamma}{\lambda} + (1 - 2ab)\frac{\gamma^2}{\lambda^2}} \geq 1$,

$$\left| \frac{\partial W_{hom}}{\partial \gamma} \right| = \left| 2(R - 1) \frac{\frac{b^2 - a^2}{\lambda} + \frac{1 - 2ab}{\lambda^2}\gamma}{R} \right| \leq \frac{2}{\lambda^2}(1 + |\gamma|).$$

The computation is analogous for $ab < 0$. This means that there is a constant $c(\lambda)$ such that for all $\gamma_1, \gamma_2 \in \mathbb{R}$,

$$|W_{hom}(\gamma_1) - W_{hom}(\gamma_2)| \leq c(1 + |\gamma_1| + |\gamma_2|)|\gamma_1 - \gamma_2|.$$

□

Laminate Constructions

Thanks to Lemma 4, it suffices to address the case of piecewise affine functions u , or equivalently, piecewise constant functions γ . As mentioned above, the underlying idea requires the construction of simple laminates between matrices with finite energy E_ϵ . For this purpose, in this section we first prove that the domain $\mathcal{N}$ of the homogenized functional E , i.e., the set of matrices with unit determinant, is contained in the rank-one convex hull of $\mathcal{M}$, that is, any $N \in \mathcal{N}$ can be represented by a convex combination of matrices in $\mathcal{M}$ that are rank-one connected. In this way, we can construct a simple laminate whose gradient lies in $\mathcal{M}$ and which, with period approaching to zero, weakly converges to a given $N \in \mathcal{N}$.

Lemma 5. *For a given $N \in \mathcal{N}$, there are $F_+, F_- \in \mathcal{M}$ satisfying $\text{rank}(F_+ - F_-) = 1$ and $\mu \in [0, 1]$ such that*

$$N = \mu F_+ + (1 - \mu) F_- \quad (5.25)$$

and

- (i) if $\min\{|Nv_1|, |Nv_2|\} > 1$ and $Nv_1 \cdot Nv_2 > 0$ then $N(v_1 + v_2) = F_+(v_1 + v_2) = F_-(v_1 + v_2)$;
- (ii) if $\min\{|Nv_1|, |Nv_2|\} > 1$ and $Nv_1 \cdot Nv_2 < 0$ then $N(v_1 - v_2) = F_+(v_1 - v_2) = F_-(v_1 - v_2)$;
- (iii) if $|Nv_1| \leq 1$ then $Nv_2 = F_+v_2 = F_-v_2$;
- (iv) if $|Nv_2| \leq 1$ then $Nv_1 = F_+v_1 = F_-v_1$.

Proof. First, let us confirm that the above cases (i)–(iv) cover all possibilities, namely, that $\min\{|Nv_1|, |Nv_2|\} > 1$ and $Nv_1 \cdot Nv_2 = 0$ cannot hold at the same time. Indeed, if $Nv_1 \cdot Nv_2 = 0$ then Nv_1 and Nv_2 are orthogonal, and since $\det N = |Nv_1||Nv_2| = 1$, we get $\min\{|Nv_1|, |Nv_2|\} \leq 1$, which contradicts our assumption.

To begin with, we focus on the case (i) and introduce the function $\varphi(t), t \in \mathbb{R}$ following the idea of Conti, Dolzmann, and Kreisbeck, 2013b:

$$\varphi(t) := |N_t(v_1 - v_2)|^2 - |N_t(v_1 + v_2)|^2, \quad \text{where } N_t := N(I + t(v_1 + v_2) \otimes (v_1 - v_2)).$$

φ is a quadratic function with $\varphi(0) = -4Nv_1 \cdot Nv_2 < 0$, and while $N_t(v_1 + v_2)$ is a constant with respect to t , the term $|N_t(v_1 - v_2)|$ diverges to $+\infty$ when $t \rightarrow \pm\infty$. Hence, we can determine two finite values t_-, t_+ with $t_- < 0 < t_+$ as

$$t_- = \inf A, \quad t_+ = \sup A,$$

where

$$A := \{t \in \mathbb{R} : \min\{|N_tv_1|, |N_tv_2|\} > 1 \text{ and } \varphi(t) < 0\}.$$

Since $|N_tv_1|^2 - 1, |N_tv_2|^2 - 1$ are quadratic functions of t , and so is φ , the set A is a union of a finite number of open intervals, and is not empty because $0 \in A$.

We see that $t_- = \inf A$ implies that $\min\{|N_{t_-}v_1|, |N_{t_-}v_2|\} = 1$ or $\varphi(t_-) = 0$ holds but, in fact, in either case $\min\{|N_{t_-}v_1|, |N_{t_-}v_2|\} = 1$ holds. Indeed, when $\varphi(t_-) = 0$, the vectors $N_{t_-}v_1$ and $N_{t_-}v_2$ are orthogonal, which in view of $\det N_{t_-} = 1$ implies $\min\{|N_{t_-}v_1|, |N_{t_-}v_2|\} \leq 1$ and so $\min\{|N_{t_-}v_1|, |N_{t_-}v_2|\} = 1$. From this and the definition of $\mathcal{M}$, we have that $N_{t_-} \in \mathcal{M}$ and an analogous argument leads to $N_{t_+} \in \mathcal{M}$.

Now we define $F_{\pm} := N_{t_{\pm}}$. Since $N = N_0$ is the point that internally divides the line segment $N_t, t \in [t_-, t_+]$ connecting F_- and F_+ into $-t_- : t_+$, setting $\mu = -t_- / (t_+ - t_-)$ yields (5.25). In addition, the property (i) is satisfied because $N_t(v_1 + v_2)$ is constant in t .

The case (ii) is proved in an analogous way, considering the function

$$\varphi(t) = |N_t(v_1 + v_2)|^2 - |N_t(v_1 - v_2)|^2, \quad \text{where } N_t = N(I + t(v_1 - v_2) \otimes (v_1 + v_2)).$$

The argument for cases (iii) and (iv) can be reduced to the setting of single slip, which is proved in Christowiak and Kreisbeck, 2017, Lem. 3.3. In particular, when $N \in \mathcal{M}$, we may take $F_+ = F_- = N$. $\square$

We notice that by the definition of t_-, t_+ , the function $t \mapsto W_{\text{hom}}(N_t)$ is constant in $[t_-, t_+]$. Moreover, for $F \in \mathcal{M}$ we have $W_{\text{hom}}(F) = |Fv_1|^2 + |Fv_2|^2 - 2 = W(F)$. Hence, when $\min\{|Nv_1|, |Nv_2|\} > 1$ we have

$$W_{\text{hom}}(N) = W(F_+) = W(F_-). \quad (5.26)$$

This is true also when $\min\{|Nv_1|, |Nv_2|\} \leq 1$, as shown in Christowiak and Kreisbeck, 2017, Lem. 3.3.

The following two theorems are taken from Conti and Theil, 2005; Müller and Šverák, 1999, respectively. In combination, they allow us to approximate a matrix $N \in \mathcal{N}$ by a simple laminate of finite energy, while preserving boundary values.

Theorem 9. (Conti and Theil, 2005, Thm. 4) Let $F_+, F_- \in \mathbb{R}^{2 \times 2}$ with $\det F_+ = \det F_- = 1$ and $\text{rank}(F_+ - F_-) = 1$ and $p \in \mathbb{R}^2$ be such that $|F_+p| = |F_-p|$ and $F_+p \neq F_-p$. Moreover, let $\Omega \subset \mathbb{R}^2$ be a bounded domain and fix $\mu \in (0, 1)$.

Then for any $\delta > 0$, there are $h_\delta > 0$ and $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ such that the restriction to Ω_δ of any simple laminate between the gradients F_+ and F_- with weights μ and $1 - \mu$ and period $h < h_\delta$, can be extended to a finitely piecewise affine function $v_\delta : \Omega \rightarrow \mathbb{R}^2$ so that $\nabla v_\delta = \mu F_+ + (1 - \mu)F_-$ on $\partial\Omega$, $\det \nabla v_\delta = 1$, $|(\nabla v_\delta)p| \leq |F_+p| = |F_-p|$, and $\text{dist}(\nabla v_\delta, [F_+, F_-]) \leq \delta$ on Ω , where $[F_+, F_-] = \{tF_+ + (1 - t)F_- : t \in [0, 1]\}$.

Note that the vector p appearing in the theorem exists and is unique up to scaling, cf. the proof in Conti and Theil, 2005.

Theorem 10. (Müller and Šverák, 1999, Thm. 1.3) Let $\mathcal{K} \subset \{F \in \mathbb{R}^{2 \times 2} : \det F = 1\}$. Suppose that $(U_i)_i$ is an in-approximation of $\mathcal{K}$, i.e., the sets U_i are open in $\{F \in \mathbb{R}^{2 \times 2} : \det F = 1\}$ and uniformly bounded, U_i is contained in the rank-one convex hull of U_{i+1} for every $i \in \mathbb{N}$, and $(U_i)_i$ converges to $\mathcal{K}$ in the following sense: if $F_i \in U_i$ and $|F_i - F| \rightarrow 0$ as $i \rightarrow \infty$ then $F \in \mathcal{K}$.

Then for any $F \in U_1$ and open domain $\Omega \subset \mathbb{R}^2$, there exists $u \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that $\nabla u \in \mathcal{K}$ a.e. in Ω and $u = Fx$ on $\partial\Omega$.

Using Lemma 5 and the above two theorems we prove the following approximation result that will be the basis for our construction of recovery sequence.

Corollary 1. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If $N \in \mathcal{N} \setminus \mathcal{M}$, let $F_+, F_- \in \mathcal{M}$ and $\mu \in (0, 1)$ be as in Lemma 5, otherwise let $F_+ = F_- = N \in \mathcal{M}$ and $\mu \in (0, 1)$.

Then for every $\delta > 0$ there exists $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ and $u_\delta \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that u_δ coincides with a simple laminate between F_+ and F_- with weights μ and $1 - \mu$ and period $h_\delta < \delta$ in Ω_δ , $\nabla u_\delta \in \mathcal{M}$ a.e. in Ω , and $u_\delta = Nx$ on $\partial\Omega$. Moreover, there is a constant c depending only on N , such that for any $\delta \in (0, 1)$,

$$|\nabla u_\delta| \leq c \quad \text{a.e. in } \Omega. \quad (5.27)$$

In particular, $\nabla u_\delta \rightharpoonup N$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ as $\delta \rightarrow 0$.

Proof. The proof requires addressing the four cases for N in Lemma 5 separately. Here we deal with the case (i) only, since a similar argument applies to case (ii), while cases (iii) and (iv) were treated in Christowiak and Kreisbeck, 2017, Cor. 3.7. For the given F_+, F_- and $\delta > 0$, Theorem 9 with this δ and p taken as $v_1 + v_2$ yields a finitely piecewise affine function v_δ and a set Ω_δ . Since $\text{dist}(\nabla v_\delta, [F_+, F_-]) \leq \delta$ and $0 < \delta < 1$, there is $c > 0$ independent of δ , such that

$$|\nabla v_\delta q| < c \quad \text{a.e. in } \Omega \quad \text{for } q = v_1 + v_2, v_1 - v_2, v_1, v_2.$$

The desired function u_δ is obtained as a result of applying Theorem 10 to modify v_δ in the finitely many subsets of $\Omega \setminus \Omega_\delta$ where v_δ is affine (and ∇v_δ does not belong to $\mathcal{M}$). Take any such subset S and first assume that ∇v_δ satisfies the conditions of case (i) in Lemma 5, i.e., $\min\{|\nabla v_\delta v_1|, |\nabla v_\delta v_2|\} > 1$ and $\nabla v_\delta v_1 \cdot \nabla v_\delta v_2 > 0$ on S . We apply Theorem 10 to the in-approximation $(U_i^\delta)_i$ of $\mathcal{K} := \mathcal{M} \cap \{F \in \mathbb{R}^{2 \times 2} : |F(v_1 + v_2)| \leq c, Fv_1 \cdot Fv_2 \geq 0\}$ defined as

$$U_i^\delta := \{F : \det F = 1, |F(v_1 + v_2)| < c, |Fv_1| > 1, |Fv_2| > 1, Fv_1 \cdot Fv_2 > 0\} \\ \cap \left\{ F \in \mathbb{R}^{2 \times 2} : |Fv_1| < 1 + 2^{-(i-1)} \text{ or } |Fv_2| < 1 + 2^{-(i-1)} \right\}, \quad i \in \mathbb{N}.$$

By shifting the index i if necessary, we may assume that $\nabla v_\delta \in U_1^\delta$ in S .

In order to apply Theorem 10, it remains to show that $(U_i^\delta)_i$ is indeed an in-approximation of $\mathcal{K}$. When $|F_i - F| \rightarrow 0$ for $F_i \in U_i^\delta$, it is easy to see that $\det F = 1$ and $|F(v_1 + v_2)| \leq c$ and $Fv_1 \cdot Fv_2 \geq 0$. In addition, we see that either $|Fv_1| = 1$ or $|Fv_2| = 1$ holds. Indeed, assuming that there is a subsequence $(F_{i_k})_k$ that satisfies $1 < |F_{i_k} v_1| < 1 + 2^{-(i_k-1)}$ for all k , we obtain $|Fv_1| = 1$. If this assumption is false and there is only a finite set of values k for which $1 < |F_{i_k} v_1| < 1 + 2^{-(i_k-1)}$ holds, from the definition of U_i^δ there must be an infinite subsequence $(F_{i_k})_k$ satisfying $1 < |F_{i_k} v_2| < 1 + 2^{-(i_k-1)}$ for all k . But then $|Fv_2| = 1$.

Next, we prove that U_i^δ is contained in the rank-one convex hull of U_{i+1}^δ . Let $G \in U_i^\delta$ and set $G_t = G(I + t(v_1 + v_2) \otimes (v_1 - v_2))$. Then we have $|G_t(v_1 + v_2)| = |G(v_1 + v_2)| < c$ for any $t \in \mathbb{R}$. Defining t_-, t_+ as in Lemma 5, in view of the continuity of $|G_t v_1|, |G_t v_2|$ as functions of t , one can find $t_-^{i+1} > t_-, t_+^{i+1} < t_+$ close enough to t_-, t_+ so that $\min\{|G_{t_-^{i+1}} v_1|, |G_{t_-^{i+1}} v_2|\} < 1 + 2^{-i}$ and $\min\{|G_{t_+^{i+1}} v_1|, |G_{t_+^{i+1}} v_2|\} < 1 + 2^{-i}$. Also, due to the definitions of t_- and t_+ , $G_{t_-^{i+1}} v_1 \cdot G_{t_+^{i+1}} v_2 > 0$. Then $G_{t_-^{i+1}}, G_{t_+^{i+1}} \in U_{i+1}^\delta$, and G can be written as a rank-one convex combination of $G_{t_-^{i+1}}$ and $G_{t_+^{i+1}}$.

Now Theorem 10 allows us to modify v_δ in S to obtain a function u_δ , which satisfies $\nabla u_\delta \in \mathcal{M}$ and $|\nabla u_\delta(v_1 + v_2)| \leq c$ in S .

The case when $\min\{|\nabla v_\delta v_1|, |\nabla v_\delta v_2|\} > 1$ and $\nabla v_\delta v_1 \cdot \nabla v_\delta v_2 < 0$ is handled analogously. One just needs to modify $\mathcal{K}, U_i^\delta$, and G_t appropriately by interchanging the roles of $v_1 + v_2$ and $v_1 - v_2$, and switching the inequality $Fv_1 \cdot Fv_2 > 0$ to $Fv_1 \cdot Fv_2 < 0$ in the definition of U_i^δ .

The case when $\min\{|\nabla v_\delta v_1|, |\nabla v_\delta v_2|\} < 1$ in a subset S can be reduced to the proof of Conti and Theil, 2005, Lem. 2. For example, when $|\nabla v_\delta v_1| < 1$, this yields $\nabla u_\delta \in \mathcal{M}_1$ satisfying $|\nabla u_\delta v_2| \leq c$ in S . Similarly, when $|\nabla v_\delta v_2| < 1$, we have $\nabla u_\delta \in \mathcal{M}_2$ satisfying $|\nabla u_\delta v_1| \leq c$ in S .

Finally, to prove (5.27), we recall that ∇u_δ coincides with ∇v_δ in Ω_δ and thus satisfies $|(\nabla u_\delta)(v_1 + v_2)| = |N(v_1 + v_2)|$ a.e. in Ω_δ . On the other hand, ∇u_δ on subsets S of $\Omega \setminus \Omega_\delta$ was constructed so that it fulfills one of the following four conditions: (i) $\nabla u_\delta \in \mathcal{M}$ and $|(\nabla u_\delta)(v_1 + v_2)| \leq c$, (ii) $\nabla u_\delta \in \mathcal{M}$ and $|(\nabla u_\delta)(v_1 - v_2)| \leq c$, (iii) $\nabla u_\delta \in \mathcal{M}_1$ and $|\nabla u_\delta v_2| \leq c$, (iv) $\nabla u_\delta \in \mathcal{M}_2$ and $|\nabla u_\delta v_1| \leq c$. Therefore, for $\delta < 1$, in either of the four cases both $|\nabla u_\delta v_2|$ and $|\nabla u_\delta v_1|$ are uniformly bounded with respect to δ in terms of N , from which (5.27) follows. $\square$

Recovery Sequence for Constant γ

Let $\gamma \in \mathbb{R}$ be constant and define $N \in \mathcal{N}$ through

$$\lambda N + (1 - \lambda)R = R(I + \gamma e_1 \otimes e_2). \quad (5.28)$$

With this definition, $Ne_1 = Re_1$ holds, and thus laminates parallel to the direction of e_1 can be constructed. For $\epsilon \in (0, 1)$ let $\varphi_\epsilon^1 \in W^{1,\infty}((0, 1) \times (0, \lambda); \mathbb{R}^2)$ be obtained by applying Corollary 1 to $\Omega = (0, 1) \times (0, \lambda) \subset \mathbb{R}^2$ and N with $\delta = \epsilon$. We then define

$$\varphi_\epsilon(x) := \sum_{i \in \mathbb{Z}^2} \left(\varphi_\epsilon^1(x - i) - N(x - i) \right) \mathbb{1}_{i+(0,1) \times (0,\lambda)}, \quad x \in \mathbb{R}^2,$$

so that φ_ϵ is Y -periodic. Next we set $z_\epsilon = w + \varphi_\epsilon$, where $w \in W_{loc}^{1,\infty} \cap L_0^\infty(\mathbb{R}^2; \mathbb{R}^2)$ is such that $\nabla w = R \mathbb{1}_{Y_{rig}} + N \mathbb{1}_{Y_{soft}}$ (see Figure 5.4). Because $\nabla \varphi_\epsilon \rightharpoonup 0$ in $L_{loc}^2(\mathbb{R}^2; \mathbb{R}^{2 \times 2})$ as $\epsilon \rightarrow 0$,

$$\nabla z_\epsilon \rightharpoonup \nabla w \quad \text{in } L_{loc}^2(\mathbb{R}^2; \mathbb{R}^{2 \times 2}). \quad (5.29)$$

If $u_\epsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with mean value zero is determined by

$$\nabla u_\epsilon(x) = \nabla z_\epsilon \left(\frac{x}{\epsilon} \right), \quad x \in \Omega,$$

the functions u_ϵ are admissible for E_ϵ in view of $\nabla u_\epsilon = R \in SO(2)$ in $\epsilon Y_{rig} \cap \Omega$ and $\nabla u_\epsilon \in \mathcal{M}$ a.e. in Ω .

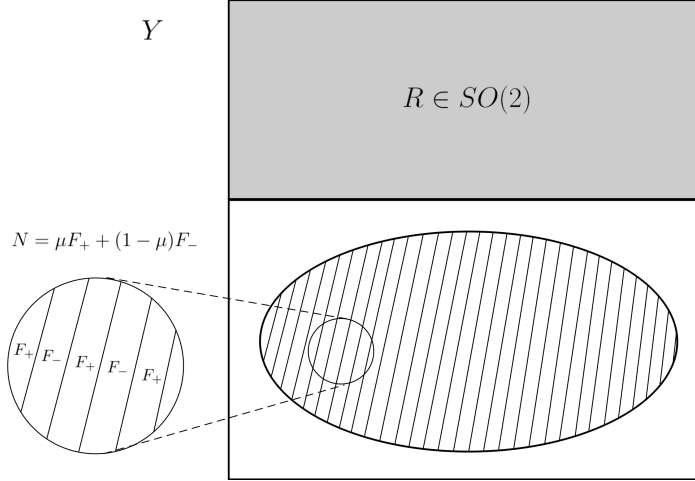


FIGURE 5.4: Structure of ∇z_ϵ in the unit cell Y . Corollary 1 is used to graft a simple laminate into the soft layer, so that the surrounding area is smaller than ϵ . Here, $\mu \in (0, 1)$ and F_+, F_- are as in Lemma 5.

By (5.29), when $\epsilon \rightarrow 0$,

$$\nabla u_\epsilon \rightharpoonup \int_Y \nabla w \, dx = \lambda N + (1 - \lambda)R = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}). \quad (5.30)$$

This weak convergence is proved using a nonstandard averaging lemma (see Lemma 6 below). Recalling (5.27) yields uniform boundedness of $\|\nabla z_\epsilon\|_{L^\infty}$ for $0 < \epsilon < 1$. It follows from the continuity of W on $\mathcal{M}$ that $|W(\nabla u_\epsilon)| = |W(\nabla z_\epsilon(\epsilon^{-1}x))| \leq c$ a.e. in Ω for some $c > 0$ and for all $0 < \epsilon < 1$. We wish to investigate the limit

$$\lim_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) = \lim_{\epsilon \rightarrow 0} \int_\Omega W(\nabla z_\epsilon(\epsilon^{-1}x)) \mathbb{1}_{\epsilon Y_{\text{soft}} \cap \Omega} \, dx. \quad (5.31)$$

Let $\Omega_\epsilon \subset \Omega$ be the open set where ∇u_ϵ is a simple laminate. Then ∇u_ϵ takes value in $\{F_+, F_-\}$ on Ω_ϵ and in $\mathcal{M}$ on $\Omega_\epsilon^0 := (\epsilon Y_{\text{soft}} \cap \Omega) \setminus \Omega_\epsilon$. Using (5.26),

$$(5.31) = \lim_{\epsilon \rightarrow 0} \left(\int_{\epsilon Y_{\text{soft}} \cap \Omega} W_{\text{hom}}(N) \, dx + \int_{\Omega_\epsilon^0} \left(W(\nabla z_\epsilon(\epsilon^{-1}x)) - W_{\text{hom}}(N) \right) \, dx \right).$$

The first term converges to $\lambda |\Omega| W_{\text{hom}}(N)$ when $\epsilon \rightarrow 0$ due to the averaging lemma. On the other hand, the second term converges to 0 when $\epsilon \rightarrow 0$ due to $|\Omega_\epsilon^0| \rightarrow 0$ and the uniform boundedness of the L^∞ -norm of the integrand. We conclude that

$$\lim_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) = \lambda |\Omega| W_{\text{hom}}(N) = E(u).$$

Here we provide a detailed account of the nonstandard averaging lemma used above. The standard averaging lemma is a special case of the following claim.

Lemma 6 (Averaging lemma for “doubly oscillating” sequences).

Let Ω be a bounded open set in $\mathbb{R}^n$, and let $g_\epsilon \in L^2_{\text{loc}}(\mathbb{R}^n)$ be Y -periodic functions, where Y is an n -cube. If $g_\epsilon \rightharpoonup g$ in $L^2(Y)$, then

$$g_\epsilon \left(\frac{x}{\epsilon} \right) \rightharpoonup \langle g \rangle := \frac{1}{|Y|} \int_Y g(x) \, dx \quad \text{in } L^2(\Omega).$$

Proof. We prove this theorem by generalizing the proof for the standard averaging lemma in Berlyand and Rybalko, 2018, see also Lukkassen and Wall, 2002. Since $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, we only need to prove the following for every $\theta \in C_0^\infty(\Omega)$:

$$\int_{\Omega} g_\epsilon \left(\frac{x}{\epsilon} \right) \theta(x) dx \rightarrow \langle g \rangle \int_{\Omega} \theta(x) dx. \quad (5.32)$$

First, we show the uniform boundedness of $g_\epsilon(x/\epsilon)$ in $L^2(\Omega)$. To this end, we introduce the notation $\epsilon Y_i := x_i^\epsilon + \epsilon Y$, where $x_i^\epsilon \in \epsilon \mathbb{Z}^n \cap \Omega$, i.e., $\{x_i^\epsilon, i \in \mathbb{Z}\}$ is the set of grid points separated by the distance ϵ . Then

$$\int_{\Omega} |g_\epsilon \left(\frac{x}{\epsilon} \right)|^2 dx = \sum_i \int_{\epsilon Y_i \cap \Omega} |g_\epsilon \left(\frac{x}{\epsilon} \right)|^2 dx \leq \frac{|\Omega|}{\epsilon^n} \int_Y |g_\epsilon(y)|^2 \epsilon^n dy = |\Omega| \|g_\epsilon\|_{L^2(Y)}^2.$$

Since $g_\epsilon \rightharpoonup g$, there is $C > 0$ such that

$$\|g_\epsilon \left(\frac{x}{\epsilon} \right)\|_{L^2(\Omega)} \leq C. \quad (5.33)$$

We define a piecewise constant interpolation of θ by

$$\theta_\epsilon(x) = \theta(x_i^\epsilon), \quad x \in \epsilon Y_i \cap \Omega.$$

By Cauchy-Schwarz inequality and (5.33),

$$\int_{\Omega} |g_\epsilon \left(\frac{x}{\epsilon} \right) (\theta(x) - \theta_\epsilon(x))| dx \leq \|g_\epsilon \left(\frac{x}{\epsilon} \right)\|_{L^2(\Omega)} \|\theta - \theta_\epsilon\|_{L^2(\Omega)} \leq C \|\theta - \theta_\epsilon\|_{L^2(\Omega)}. \quad (5.34)$$

Since θ has a compact support in Ω , making ϵ small enough we can ignore cubes ϵY_i intersecting the boundary $\partial\Omega$. Thus,

$$\int_{\Omega} g_\epsilon \left(\frac{x}{\epsilon} \right) \theta_\epsilon(x) dx = \sum_i \int_{\epsilon Y_i} g_\epsilon \left(\frac{x}{\epsilon} \right) \theta_\epsilon(x) dx = \langle g_\epsilon \rangle \int_{\Omega} \theta_\epsilon(x) dx. \quad (5.35)$$

In view of $\|\theta - \theta_\epsilon\|_{L^2(\Omega)} \rightarrow 0$ and $\langle g_\epsilon \rangle \int_{\Omega} \theta_\epsilon(x) dx \rightarrow \langle g \rangle \int_{\Omega} \theta(x) dx$ as $\epsilon \rightarrow 0$, the convergence (5.32) follows from (5.34) and (5.35). $\square$

Recovery Sequence for Piecewise Constant γ

In this step, we prove the existence of recovery sequence for (finitely) piecewise affine u . To begin with, assume that $\Omega = (0, l)^2 \subset \mathbb{R}^2$ with $l > 0$. Since γ is essentially a function of x_2 only, γ can be expressed as a one-dimensional simple function

$$\gamma(x_1, t) = \sum_{i=1}^n \gamma_i \mathbb{1}_{(t_{i-1}, t_i)}(t), \quad t \in (0, l),$$

with $\gamma_i \in \mathbb{R}$, $i = 1, \dots, n$ and $0 = t_0 < t_1 < \dots < t_n = l$.

Let $N_i \in \mathcal{N}$ be the matrix corresponding to γ_i via (5.28). We define $u_\epsilon \in W^{1,2}(\Omega; \mathbb{R}^2)$ with mean value zero by

$$\nabla u_\epsilon := R + \sum_{i=1}^n (\nabla u_{i,\epsilon} - R) \mathbb{1}_{\epsilon Y_{\text{sof}} \cap \Omega} \mathbb{1}_{\mathbb{R} \times ([\epsilon^{-1} t_{i-1}] \epsilon, [\epsilon^{-1} t_i] \epsilon)},$$

where $(u_{i,\epsilon})_\epsilon \subset W^{1,2}((0, l) \times (t_{i-1}, t_i); \mathbb{R}^2)$ for $i \in \{1, \dots, n\}$ are the recovery sequences corresponding to γ_i constructed in Section 5.1.3. Note that u_ϵ is well defined

since $\nabla u_{i,\epsilon}$ and R rank-one connect along all segments $[0, l] \times j\epsilon$, $j = 1, \dots, \lfloor l/\epsilon \rfloor$ (also see Figure 5.2).

For $\varphi \in L^2(\Omega)$,

$$\begin{aligned} \int_{(0,l) \times (t_{i-1}, t_i)} (\nabla u_\epsilon - \nabla u) \varphi \, dx &= \int_{(0,l) \times ([\epsilon^{-1}t_{i-1}] \epsilon, [\epsilon^{-1}t_i] \epsilon)} (\nabla u_{i,\epsilon} - \nabla u) \varphi \, dx \\ &\quad + \int_{(0,l) \times (t_{i-1}, t_i) \setminus (0,l) \times ([\epsilon^{-1}t_{i-1}] \epsilon, [\epsilon^{-1}t_i] \epsilon)} (R - \nabla u) \varphi \, dx \end{aligned}$$

By (5.30) the first term on the right-hand side converges to 0 as $\epsilon \rightarrow 0$, while the second term converges to 0 due to vanishing measure of integration domain. It follows that

$$\nabla u_\epsilon \rightharpoonup \sum_{i=1}^n (\lambda N_i + (1 - \lambda)R) \mathbb{1}_{(0,l) \times (t_{i-1}, t_i)} = \nabla u \quad \text{in } L^2(\Omega; \mathbb{R}^{2 \times 2}).$$

Similarly, for the energy contributions, it follows that

$$\begin{aligned} \int_{(0,l) \times (t_{i-1}, t_i)} W(\nabla u_\epsilon) \, dx &= \int_{(0,l) \times ([\epsilon^{-1}t_{i-1}] \epsilon, [\epsilon^{-1}t_i] \epsilon)} W(\nabla u_{i,\epsilon}) \, dx \\ &\quad + \int_{(0,l) \times (t_{i-1}, t_i) \setminus (0,l) \times ([\epsilon^{-1}t_{i-1}] \epsilon, [\epsilon^{-1}t_i] \epsilon)} W(R) \, dx. \end{aligned}$$

As $\epsilon \rightarrow 0$ the first term converges to $\lambda(t_i - t_{i-1})W_{\text{hom}}(N_i)$ in the same way as in Section 5.1.3, and the second term is equal to 0. This implies

$$\lim_{\epsilon \rightarrow 0} \int_{\Omega} W(\nabla u_\epsilon) \, dx = \lambda \sum_i^n (t_i - t_{i-1}) W_{\text{hom}}(N_i) = E(u).$$

To generalize to a simply-connected Lipschitz domain Ω , just fill Ω with overlapping cubes and glue the gradients properly.

Remark 10. We remark that routine technical modifications can be done to deal with the problem with $W_{\text{soft}}(F) = \gamma_1^p + \gamma_2^p$ for $F \in \mathcal{M}$, where $p > 2$. On the other hand, the variant of our problem, which consists in adding a linear term $|\gamma_1| + |\gamma_2|$ to the energy density, remains fully open for the setting of more than one slip system (but see Davoli, Ferreira, and Kreisbeck, 2021 for an analysis in the BV space). This term represents dissipative contribution and is thus vital for addressing the evolution problem (Carstensen, Hackl, and Mielke, 2002; Davoli and Francfort, 2015; Ortiz and Repetto, 1999), for example, in the framework of rate-independent systems (Mielke and Roubířek, 2015; Davoli, Kružík, and Pelech, 2022; Stefanelli, 2009). For the problem with one slip system, the authors of Christowiak and Kreisbeck, 2017 were able to identify the Γ -limit in the particular case of slip direction parallel to the layers (see Theorem 6) by noticing a special structure leading to strong convergence of rotations and hence weak convergence of slips. However, for two slip systems, the trick to show strong convergence of rotations is not available. On the other hand, numerical simulations of Drozdenko et al., 2022 were successful in capturing the real phenomenon using the homogenization analysis of Christowiak and Kreisbeck, 2017 but dealing with the linear term a posteriori.

5.2 Envelopes for General Slip Systems

Our homogenization result in the previous Section relies on the construction of generalized convex envelopes. For the problem with a single slip direction or the problem

with two orthogonal slip directions, the rank-one convex envelope of energy density in the setting of rigid plasticity have in common the usage of first-order laminates, i.e., rank-one convex combinations of matrices with finite energy. When there are two orthogonal slip systems, first-order laminates yield the rank-one convex envelope (Conti, Dolzmann, and Kreisbeck, 2013b), which allows for the full identification of the corresponding Γ -limit. In this case, this rank-one convex envelope in fact matches the polyconvex envelope and the quasiconvex envelope as well. This is not true for the spatial dimension 3 (Conti, Dolzmann, and Kreisbeck, 2013b). In this section, we extend the results from Conti, Dolzmann, and Kreisbeck, 2013b to arbitrary two-slip systems. Although the basic ideas are essentially contained in Conti and Dolzmann, 2016; Conti, Dolzmann, and Kreisbeck, 2013b, we believe that for the convenience of the reader it is worthwhile to provide the particular details related to the setting considered in this paper.

To fix the notation, let unit vectors v_1, v_2 be such that (v_1, v_2) is a right-handed system and the angle between v_1 and v_2 is 2θ with $\pi/4 < \theta < \pi/2$. Further, let

$$v_3 := -\frac{v_1 + v_2}{|v_1 + v_2|} \quad (5.36)$$

denote the unit vector dividing in half the larger angle between v_1 and v_2 . For later use we note that

$$v_1 \cdot v_2 = \cos 2\theta, \quad v_1 \cdot v_3 = v_2 \cdot v_3 = -\cos \theta, \quad v_1 \cdot v_3^\perp = -v_2 \cdot v_3^\perp = \sin \theta. \quad (5.37)$$

Considering the energy density

$$W(F) = \begin{cases} |F|^2 - 2 & \text{if } F \in \mathcal{M}_1 \cup \mathcal{M}_2, \\ \infty & \text{otherwise,} \end{cases}$$

where

$$\mathcal{M}_i := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fv_i| = 1\}, \quad i = 1, 2,$$

the main result of this section reads as follows.

Proposition 6. *The rank-one convex envelope W^{rc} , the polyconvex envelope W^{pc} and the quasiconvex envelope W^{qc} of W fulfill*

$$\begin{aligned} W^{rc}(F) = W^{pc}(F) = W^{qc}(F) &= \begin{cases} h(|Fv_3|) & \text{if } F \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1 \cap \mathcal{N}_2}, \\ h^\perp(|Fv_3^\perp|) & \text{if } F \in \overline{\mathcal{A}_\perp} \end{cases} \\ &= \begin{cases} |Fv_1^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_1| = 1, \\ |Fv_2^\perp|^2 - 1 & \text{if } \det F = 1, |Fv_2| = 1, \\ h(|Fv_3|) & \text{if } F \in \mathcal{A} \cup (\mathcal{N}_1 \cap \mathcal{N}_2), \\ h^\perp(|Fv_3^\perp|) & \text{if } F \in \mathcal{A}_\perp. \end{cases} \end{aligned}$$

Furthermore,

$$W^{rc}(F) = W^{pc}(F) = +\infty \quad \text{if } \det F \neq 1.$$

Here the sets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{A}, \mathcal{A}_\perp$ are defined by

$$\begin{aligned}\mathcal{N}_i &:= \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fv_i| < 1\}, \quad i = 1, 2, \\ \mathcal{A} &:= \left\{F \in \mathbb{R}^{2 \times 2} : \det F = 1, \min_{i=1,2} |Fv_i| > 1, Fv_1 \cdot Fv_2 > 0\right\}, \\ \mathcal{A}_\perp &:= \left\{F \in \mathbb{R}^{2 \times 2} : \det F = 1, \min_{i=1,2} |Fv_i| > 1, Fv_1 \cdot Fv_2 < 0\right\}.\end{aligned}$$

We recall that h and $h^\perp$ were defined in Theorem 8. For the purposes of this Section, we extend them as (see Figure 5.5),

$$\begin{aligned}h^*(z) &:= \frac{1 + z^2 - 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta} - 2 \quad \text{for } z \in [\sin \theta, \infty), \\ h^{\perp*}(z) &:= \frac{1 + z^2 - 2 \sin \theta \sqrt{z^2 - \cos^2 \theta}}{\cos^2 \theta} - 2 \quad \text{for } z \in [\cos \theta, \infty).\end{aligned}$$

Then one easily checks that

$$h(z) = \begin{cases} 0 & \text{if } z \in [0, 1] \\ h^*(z) & \text{if } z \geq 1 \end{cases}, \quad h^\perp(z) = \begin{cases} 0 & \text{if } z \in [0, 1] \\ h^{\perp*}(z) & \text{if } z \geq 1 \end{cases}, \quad (5.38)$$

and that

$$h(z) \leq h^*(z) \text{ for } z \in [\sin \theta, \infty), \quad \text{and} \quad h^\perp(z) \leq h^{\perp*}(z) \text{ for } z \in [\cos \theta, \infty). \quad (5.39)$$

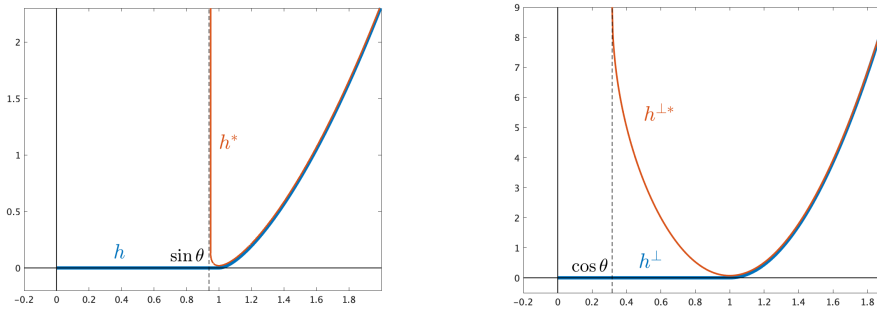


FIGURE 5.5: The graphs of functions h, h^* and $h^\perp, h^{\perp*}$ for the particular case of slip directions $v_1 = (1, 0), v_2 = (-0.8, 0.6)$.

Remark 11. The condition $Fv_1 \cdot Fv_2 \geq 0$ in the definition of $\mathcal{A}$ and $\mathcal{A}_\perp$ appears already in Theorem 7 and serves the purpose of dividing the set $\{F : \det F = 1, |Fv_1| > 1, |Fv_2| > 1\}$ into two parts $\mathcal{A}$ and $\mathcal{A}_\perp$, as shown in Figure 5.6. It is equivalent to $|Fv_3|/|Fv_3^\perp| \geq \sin \theta / \cos \theta$, as shown in Lemma 7 below.

We introduce two formulas that will be useful in the sequel. First, for $F \in \mathbb{R}^{2 \times 2}$ and $a, b \in \mathbb{R}^2$, one has

$$|Fa|^2 |Fb|^2 = |Fa \cdot Fb|^2 + (\det F)(a^\perp \cdot b)^2. \quad (5.40)$$

Second, for $F \in \mathbb{R}^{2 \times 2}$ and unit vectors a, b with $a^\perp \cdot b \neq 0$, one has

$$|F|^2 = \frac{|Fa|^2 + |Fb|^2 - 2(a \cdot b)(Fa \cdot Fb)}{(a^\perp \cdot b)^2}. \quad (5.41)$$

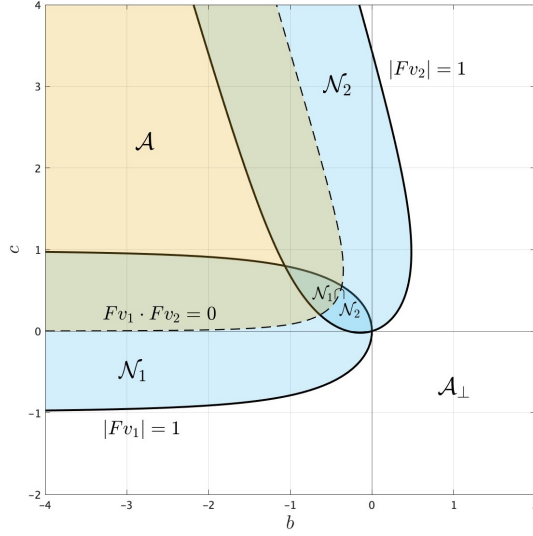


FIGURE 5.6: The sets $\mathcal{N}_1, \mathcal{N}_2, \mathcal{A}, \mathcal{A}_\perp$ shown for $v_1 = (1, 0), v_2 = (-0.8, 0.6)$ using the b, c -coordinate system defined by $(F^T F)^{1/2} = \begin{pmatrix} a+b & c \\ c & a-b \end{pmatrix}$ with $a = \sqrt{1+b^2+c^2}$. Each point in the (b, c) plane corresponds to the orbit of one matrix under $SO(2)$ on matrices with determinant 1.

Lemma 7. *The conditions $Fv_1 \cdot Fv_2 \geq 0$ and $|Fv_3|/|Fv_3^\perp| \geq \sin \theta / \cos \theta$ are equivalent. In addition, the following holds:*

$$\begin{aligned} |Fv_3| &\geq 1 & \text{for } F \in \overline{\mathcal{A}} \cup (\overline{\mathcal{N}_1 \cap \mathcal{N}_2}), \\ |Fv_3^\perp| &\geq 1 & \text{for } F \in \overline{\mathcal{A}_\perp}. \end{aligned}$$

Proof. First, we calculate

$$\begin{aligned} |Fv_3|^2 &= \frac{|Fv_1 + Fv_2|^2}{|v_1 + v_2|^2} = \frac{|Fv_1|^2 + |Fv_2|^2 + 2Fv_1 \cdot Fv_2}{2 + 2v_1 \cdot v_2}, \\ |Fv_3^\perp|^2 &= \frac{|Fv_1 - Fv_2|^2}{|v_1 - v_2|^2} = \frac{|Fv_1|^2 + |Fv_2|^2 - 2Fv_1 \cdot Fv_2}{2 - 2v_1 \cdot v_2}, \end{aligned} \quad (5.42)$$

which together with (5.37) implies that, if $Fv_3^\perp \neq 0$,

$$\frac{|Fv_3|^2}{|Fv_3^\perp|^2} \cdot \frac{\cos^2 \theta}{\sin^2 \theta} = \frac{|Fv_1|^2 + |Fv_2|^2 + 2Fv_1 \cdot Fv_2}{|Fv_1|^2 + |Fv_2|^2 - 2Fv_1 \cdot Fv_2}.$$

Hence, the conditions $Fv_1 \cdot Fv_2 \geq 0$ and $|Fv_3|/|Fv_3^\perp| \geq \sin \theta / \cos \theta$ are equivalent.

To prove the second part of the Lemma, we use (5.40) to get

$$|Fv_1 \cdot Fv_2|^2 = |Fv_1|^2 |Fv_2|^2 - (v_1^\perp \cdot v_2)^2 = |Fv_1|^2 |Fv_2|^2 - 1 + (v_1 \cdot v_2)^2. \quad (5.43)$$

Now if $F \in \overline{\mathcal{A}}$ then $|Fv_1|, |Fv_2| \geq 1$, $Fv_1 \cdot Fv_2 \geq 0$, and thus in view of $v_1 \cdot v_2 = \cos 2\theta < 0$ (5.42) implies

$$|Fv_3|^2 \geq \frac{|Fv_1|^2 + |Fv_2|^2}{2} \geq 1.$$

On the other hand, if $F \in \overline{\mathcal{N}_1 \cap \mathcal{N}_2}$, (5.42) yields

$$|Fv_3^\perp|^2 \leq \frac{1 - Fv_1 \cdot Fv_2}{1 - v_1 \cdot v_2} \leq 1,$$

because $v_1 \cdot v_2 < 0$ and $|Fv_1 \cdot Fv_2|^2 \leq (v_1 \cdot v_2)^2$ by (5.43), so irrespective of the sign of $Fv_1 \cdot Fv_2$, it is true that $v_1 \cdot v_2 \leq Fv_1 \cdot Fv_2$. Therefore, $|Fv_3| \geq 1$.

Finally, if $F \in \overline{\mathcal{A}^\perp}$ then $Fv_1 \cdot Fv_2 \leq 0$, and

$$|Fv_3^\perp|^2 \geq \frac{1 - Fv_1 \cdot Fv_2}{1 - v_1 \cdot v_2} \geq 1,$$

because $v_1 \cdot v_2 < 0$ and $|Fv_1 \cdot Fv_2|^2 \geq (v_1 \cdot v_2)^2$, again by (5.43). $\square$

We shall prove Proposition 6 in a series of lemmas. There are two ways to construct first order laminates: between two matrices on $\mathcal{M}_i$ and between a matrix on $\mathcal{M}_1$ and another matrix on $\mathcal{M}_2$. In the first case, the optimal laminate is in the direction v_i , which follows from the analysis of the single-slip problem (Conti and Theil, 2005). In the latter case, the following lemma provides the stepping stone for identifying the optimal directions for first order laminates in the set $\mathcal{A} \cup \mathcal{A}_\perp = \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fv_1| > 1, |Fv_2| > 1\}$.

Lemma 8. *If $F \in \mathcal{A}_\perp$ we define for $t \in \mathbb{R}$,*

$$F_t = F(I + t v_3^\perp \otimes v_3), \quad (5.44)$$

and if $F \in \mathcal{A}$ we define for $t \in \mathbb{R}$,

$$F_t = F(I + t v_3 \otimes v_3^\perp). \quad (5.45)$$

Then the equations

$$|F_t v_1| = 1 \quad (5.46)$$

and

$$|F_t v_2| = 1 \quad (5.47)$$

have each two distinct solutions. The two solutions of each of these two equations have the same sign, and the sign of the two solutions of (5.46) is opposite to the sign of the two solutions of (5.47).

Proof. We first prove that there are two distinct solutions to each of (5.46), (5.47) and that these solutions have the same sign. We focus on the first case (5.44) since the second case (5.45) follows by switching the roles of v_3 and $v_3^\perp$. Equation (5.46), when squared, is equivalent to

$$|Fv_1|^2 - 1 + 2t(v_3 \cdot v_1)(Fv_3^\perp \cdot Fv_1) + t^2(v_3 \cdot v_1)^2 |Fv_3^\perp|^2 = 0.$$

In view of $v_3 \cdot v_1 = -\cos \theta \neq 0$ we see that two distinct real solutions exist provided that

$$(Fv_3^\perp \cdot Fv_1)^2 > |Fv_3^\perp|^2 (|Fv_1|^2 - 1).$$

In view of (5.40) and $\det F = 1$, this is equivalent to

$$|Fv_3^\perp|^2 > (-v_3 \cdot v_1)^2 = \cos^2 \theta,$$

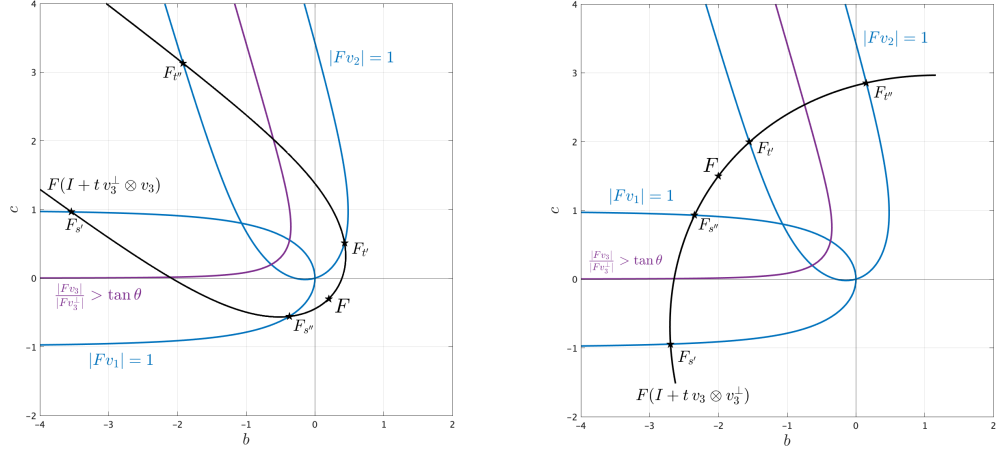


FIGURE 5.7: Rank-one lines F_t shown in the b, c -coordinate system as in Figure 5.6 for the two cases from Lemma 8: $F \in \mathcal{A}_\perp$ (left) and $F \in \mathcal{A}$ (right).

which immediately follows from the assumption $|Fv_3|/|Fv_3^\perp| < \sin \theta / \cos \theta$ since due to $\det F = 1$ we have $|Fv_3||Fv_3^\perp| \geq 1$ and thus

$$|Fv_3^\perp|^2 = \frac{|Fv_3^\perp|}{|Fv_3|} |Fv_3^\perp| |Fv_3| > \frac{\cos \theta}{\sin \theta} > \cos^2 \theta.$$

In addition, $|Fv_1|^2 - 1 > 0$ by assumption, and hence the solutions have the same sign. The proof for (5.47) is obtained by simply replacing v_1 by v_2 .

We next look at the sign of the solutions for (5.46) and (5.47), again only in the first case of (5.44). It suffices to show that the average of the two solutions of the first equation has a different sign than the average of the two solutions of the second one. The average of the two solutions of (5.46) is

$$-\frac{Fv_3^\perp \cdot Fv_1}{(v_3 \cdot v_1)|Fv_3^\perp|^2},$$

and, replacing v_1 by v_2 , the average of the two solutions of (5.47) is

$$-\frac{Fv_3^\perp \cdot Fv_2}{(v_3 \cdot v_2)|Fv_3^\perp|^2}.$$

Since $(v_3 \cdot v_1)(v_3 \cdot v_2) = \cos^2 \theta > 0$, it suffices to show that

$$(Fv_3^\perp \cdot Fv_1)(Fv_3^\perp \cdot Fv_2) < 0.$$

We observe that

$$v_1 = -(\cos \theta)v_3 + (\sin \theta)v_3^\perp \quad v_2 = -(\cos \theta)v_3 - (\sin \theta)v_3^\perp$$

and thus

$$(Fv_3^\perp \cdot Fv_1)(Fv_3^\perp \cdot Fv_2) = \cos^2 \theta |Fv_3^\perp \cdot Fv_3|^2 - \sin^2 \theta |Fv_3^\perp|^4.$$

The assertion follows in view of Cauchy-Schwarz inequality and the assumption $|Fv_3|/|Fv_3^\perp| < \sin \theta / \cos \theta$. $\square$

Motivated by the ideas of Kohn and Strang, 1986a; Kohn and Strang, 1986b on the construction of rank-one envelopes through repeated laminations, for $F \in \mathbb{R}^{2 \times 2}$ we define $W^{lc}(F)$ as follows:

$$W^{lc}(F) = \inf \left\{ \mu W(F_0) + (1 - \mu) W(F_1) : \right. \\ \left. \mu \in [0, 1], \mu F_0 + (1 - \mu) F_1 = F, \text{rank}(F_0 - F_1) \leq 1 \right\}. \quad (5.48)$$

For the case of orthogonal slip systems the first order lamination envelope W^{lc} is already identical to the rank-one envelope W^{rc} but this is not true for general angles between slip normals. The following two lemmas evaluate W^{lc} from above inside the sets $\mathcal{A} \cup \mathcal{A}_\perp$ and $\mathcal{N}_1 \cap \mathcal{N}_2$, respectively, in terms of the functions $h, h^\perp$. The proof will reveal that the corresponding optimal laminates connect matrices on $\mathcal{M}_1$ and $\mathcal{M}_2$ with equal energies and hence that W^{lc} is constant on these laminates.

Lemma 9. *If $F \in \mathcal{A}$ then*

$$W^{lc}(F) \leq h(|Fv_3|),$$

and if $F \in \mathcal{A}_\perp$ then

$$W^{lc}(F) \leq h^\perp(|Fv_3^\perp|).$$

Proof. If $F \in \mathcal{A}$, Lemma 8 yields four mutually different real numbers s', s'', t', t'' such that s', s'' and t', t'' have opposite signs, and satisfy

$$|F_{s'}v_1| = |F_{s''}v_1| = |F_{t'}v_2| = |F_{t''}v_2| = 1.$$

Here F_t is the rank-one line with direction $Fv_3 \otimes v_3^\perp$, i.e., $F_t = F(I + tv_3 \otimes v_3^\perp)$. Note that $z = |F_tv_3| = |Fv_3|$ is independent of t . Using the formula (5.40) with $a = v_i$, $i = 1, 2$ and $b = v_3$, we get for $t \in \{s', s'', t', t''\}$,

$$F_tv_i \cdot F_tv_3 = \pm \sqrt{|F_tv_i|^2 |F_tv_3|^2 - (v_i^\perp \cdot v_3)^2} = \pm \sqrt{z^2 - \sin^2 \theta}.$$

Hence by (5.41),

$$|F_t|^2 = \frac{1 + z^2 \pm 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta}, \quad t \in \{s', s'', t', t''\}. \quad (5.49)$$

We may assume that $s' < s'', t' < t''$. Then either $s' < s'' < 0 < t' < t''$ or $t' < t'' < 0 < s' < s''$ holds. Since $|F_t|^2$ is a quadratic function of t , it never takes the same value more than twice. Moreover, there are only two possible values of $|F_t|^2$ for $t = s', s'', t', t''$, and hence we conclude that $|F_{s'}| = |F_{t''}|$ and $|F_{s''}| = |F_{t'}|$. Because the quadratic term t^2 appearing in $|F_t|^2$ has positive coefficient, we also see that the value with minus sign in (5.49) is taken for $t = s'', t'$ (when $s' < s'' < 0 < t' < t''$) or for $t = t'', s'$ (when $t' < t'' < 0 < s' < s''$).

Therefore, we can choose $s_\star \in \{s', s''\}$ and $t_\star \in \{t', t''\}$ such that

$$|F_{s_\star}|^2 = |F_{t_\star}|^2 = \frac{1 + z^2 - 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta}.$$

Note that s_* and t_* have different signs. Since F is a rank-one convex combination of F_{s_*} and F_{t_*} and

$$W(F_{s_*}) = |F_{s_*}|^2 - 2 = |F_{t_*}|^2 - 2 = W(F_{t_*}),$$

we conclude

$$W^{lc}(F) \leq W(F_{s_*}) = |F_{s_*}|^2 - 2 = h^*(z) = h(z),$$

the last equality following from Lemma 7.

The case $F \in \mathcal{A}_\perp$ is shown analogously, interchanging the roles of v_3 and $v_3^\perp$ and the roles of $\cos \theta$ and $\sin \theta$. $\square$

Lemma 10. *If $F \in \mathcal{N}_1 \cap \mathcal{N}_2$ then*

$$W^{lc}(F) \leq h(|Fv_3|).$$

Proof. Consider the rank-one line given by $F_t = F(I + t v_3 \otimes v_3^\perp)$. Since $|F_t v_1| \rightarrow \infty$ as $t \rightarrow \pm\infty$ and $|F_0 v_1| < 1$, by continuity we find $t_{1,\pm}$ with $t_{1,-} < 0 < t_{1,+}$ and $|F_{t_{1,\pm}} v_1| = 1$. Since $F_t v_1 \cdot Fv_3 = Fv_1 \cdot Fv_3 + t|Fv_3|^2(v_3^\perp \cdot v_1)$ and $v_3^\perp \cdot v_1 = \sin \theta > 0$, it follows that $F_{t_{1,+}} v_1 \cdot Fv_3 > F_{t_{1,-}} v_1 \cdot Fv_3$. Combining this with (5.40), we have

$$F_{t_{1,+}} v_1 \cdot Fv_3 = \sqrt{z^2 - \sin^2 \theta}, \quad F_{t_{1,-}} v_1 \cdot Fv_3 = -\sqrt{z^2 - \sin^2 \theta},$$

where we set $z = |Fv_3|$. By (5.41) with $a = v_1, b = v_3$,

$$|F_{t_{1,-}}|^2 = \frac{1 + z^2 - 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta}.$$

Repeating the same argument with v_2 instead of v_1 leads to two values $t_{2,-} < 0 < t_{2,+}$, such that $|F_{t_{2,\pm}} v_2| = 1$. Since $v_3^\perp \cdot v_2 = -\sin \theta < 0$ we obtain as above,

$$|F_{t_{2,+}}|^2 = \frac{1 + z^2 - 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta}.$$

Employing a laminate between $F_{t_{1,-}}$ and $F_{t_{2,+}}$ as in the previous lemma and noting that $|Fv_3| \geq 1$ by Lemma 7 finishes the proof. $\square$

Remark 12. *Using the rank-one line $F(I + t v_3^\perp \otimes v_3)$ one can similarly show that $W^{rc}(F) \leq h^\perp(|Fv_3^\perp|)$ for $F \in \mathcal{N}_1 \cap \mathcal{N}_2$. However, this fact does not provide any additional information since for such F one has $h(|Fv_3|) \leq h^\perp(|Fv_3^\perp|)$.*

We next turn to estimating the envelopes of W from below in terms of h and $h^\perp$.

Lemma 11. *For any $F \in \mathbb{R}^{2 \times 2}$,*

$$\max\{h(|Fv_3|), h^\perp(|Fv_3^\perp|)\} \leq W^{pc}(F), W^{qc}(F), W^{rc}(F).$$

In addition, if $\det F \neq 1$ then

$$W^{pc}(F) = W^{rc}(F) = +\infty.$$

Proof. Since for $z \geq 0$, $h(z)$ and $h^\perp(z)$ are nondecreasing and convex functions taking finite values, the function $\max\{h(|Fv_3|), h^\perp(|Fv_3^\perp|)\}$ is convex and takes finite values for $F \in \mathbb{R}^{2 \times 2}$. Thus, this function is also polyconvex, quasiconvex and rank-one convex. Therefore, it suffices to show $h(|Fv_3|) \leq W(F)$ and $h^\perp(|Fv_3^\perp|) \leq W(F)$, where $F = F_\gamma^i = I + \gamma v_i \otimes v_i^\perp$, for any $\gamma \in \mathbb{R}$ and each $i = 1, 2$. In view of $\det F_\gamma^i =$

1 and the relation $W(F_\gamma^i) = \gamma^2$, it is sufficient to check that $h(|F_\gamma^i v_3|) \leq \gamma^2$ and $h^\perp(|F_\gamma^i v_3^\perp|) \leq \gamma^2$ for any $\gamma \in \mathbb{R}$ and each $i = 1, 2$. To begin with, we fix $i = 1$. Since $|F_\gamma^1 v_3|^2 = 1 + 2\gamma \cos \theta \sin \theta + \gamma^2 \sin^2 \theta$, we see that $|F_\gamma^1 v_3|^2 - \sin^2 \theta \geq 0$ and from (5.39)

$$h(|F_\gamma^1 v_3|) \leq h^*(|F_\gamma^1 v_3|) = \left(\frac{\cos \theta - \sin \theta \sqrt{(\gamma + \frac{\cos \theta}{\sin \theta})^2}}{\sin \theta} \right)^2.$$

Hence, if $\gamma \geq -(\cos \theta / \sin \theta)$, we obtain $h(|F_\gamma^1 v_3|) \leq h^*(|F_\gamma^1 v_3|) = \gamma^2$. This inequality holds even if $\gamma < -(\cos \theta / \sin \theta)$. Indeed,

$$h(|F_\gamma^1 v_3|) \leq h^*(|F_\gamma^1 v_3|) \leq \frac{1 + |F_\gamma^1 v_3|^2 + 2 \cos \theta \sqrt{|F_\gamma^1 v_3|^2 - \sin^2 \theta}}{\sin^2 \theta} - 2$$

and a simple calculation shows that the last expression is equal to γ^2 .

The same is true for $h^\perp$, where we find that $h^\perp(|F_\gamma^1 v_3^\perp|) \leq h^{\perp*}(|F_\gamma^1 v_3^\perp|) = \gamma^2$ for $\gamma \leq (\sin \theta / \cos \theta)$ and $h^\perp(|F_\gamma^1 v_3^\perp|) \leq h^{\perp*}(|F_\gamma^1 v_3^\perp|) \leq \gamma^2$ otherwise. The proof for $i = 2$ is analogous.

To show the statement about infinite values, we follow Conti, Dolzmann, and Kreisbeck, 2013b and define the function

$$I(F) = \begin{cases} 0 & \text{if } \det F = 1, \\ +\infty & \text{otherwise,} \end{cases}$$

and the functions

$$g(F) = h(|Fv_3|) + I(F), \quad g^\perp(F) = h^\perp(|Fv_3^\perp|) + I(F).$$

Since I is convex when viewed as a function of $\det A$ and thus polyconvex, the functions g and $g^\perp$, being sums of a convex and a polyconvex function, are polyconvex functions, and so is $\max\{g, g^\perp\}$. The function $\max\{g, g^\perp\}$ takes the value $+\infty$ for F such that $\det F \neq 1$, and the first part of this proof shows that $\max\{g, g^\perp\} \leq W$. Since, in general, $W^{pc} \leq W^{rc}$ even for extended-valued function W according to Müller, 1999, we deduce $\max\{g, g^\perp\} \leq W^{pc} \leq W^{rc}$. Therefore $W^{pc}(F) = W^{rc}(F) = +\infty$ if $\det F \neq 1$. $\square$

Remark 13. Based on the ideas in the proof of Lemma 11, we may extend the results of Lemmas 9 and 10 up to the boundaries of the involved sets, i.e., Lemma 9 holds with $\mathcal{A}, \mathcal{A}_\perp$ replaced by $\overline{\mathcal{A}}, \overline{\mathcal{A}_\perp}$, respectively, and Lemma 10 holds with $\mathcal{N}_1 \cap \mathcal{N}_2$ replaced by $\overline{\mathcal{N}_1 \cap \mathcal{N}_2}$. To see why, first note that, using the notation of Lemma 11, if a matrix F_γ^1 belongs to the boundary of the set $\mathcal{A} \cup (\mathcal{N}_1 \cap \mathcal{N}_2)$ then $\gamma \geq -(\cos \theta / \sin \theta)$. Assuming on the contrary $\gamma < -(\cos \theta / \sin \theta) (< 0)$ leads after some calculation to $F_\gamma^1 v_1 \cdot F_\gamma^1 v_2 = \cos 2\theta + \gamma \sin 2\theta < 0$, and thus $F_\gamma^1 \in \partial \mathcal{A}_\perp$. But the only matrices belonging to both $\partial(\mathcal{A} \cup (\mathcal{N}_1 \cap \mathcal{N}_2))$ and $\partial \mathcal{A}_\perp$ are pure rotations, which shows that for $F_\gamma^1 \in \partial(\mathcal{A} \cup (\mathcal{N}_1 \cap \mathcal{N}_2))$ we indeed have $\gamma \geq -(\cos \theta / \sin \theta)$, and the proof of Lemma 11 in turn implies $h^*(|F_\gamma^1 v_3|) = \gamma^2 = W(F_\gamma^1) = |F_\gamma^1 v_1^\perp|^2 - 1$. Moreover, since for such F_γ^1 Lemma 7 yields $|F_\gamma^1 v_3| \geq 1$, we also have $h^*(|F_\gamma^1 v_3|) = h(|F_\gamma^1 v_3|)$. The corresponding identity $h^\perp(|F_\gamma^1 v_3^\perp|) = h^{\perp*}(|F_\gamma^1 v_3^\perp|) = \gamma^2 = W(F_\gamma^1) = |F_\gamma^1 v_1^\perp|^2 - 1$ holds for $F_\gamma^1 \in \overline{\mathcal{A}_\perp}$, and a similar argument can be made about F_γ^2 . The statement in the beginning of this remark then follows from the relation $W^{lc} \leq W$.

Unfortunately, in the remaining set $(\mathcal{N}_1 \setminus \mathcal{N}_2) \cup (\mathcal{N}_2 \setminus \mathcal{N}_1)$, the estimate of the envelopes from above that is derived from optimal first order laminates does not match with the estimate from below obtained in Lemma 11. We include the result for completeness and because the boundedness of W^{lc} , which it implies, will be needed later.

Lemma 12. For each $i = 1, 2$ and $F \in \overline{\mathcal{N}_i}$,

$$W^{lc}(F) \leq |Fv_i^\perp|^2 - 1. \quad (5.50)$$

If additionally $F \in (\mathcal{N}_1 \setminus \mathcal{N}_2) \cup (\mathcal{N}_2 \setminus \mathcal{N}_1)$ then

$$W^{lc}(F) \leq \min \left\{ h_+(|Fv_3|), h_+^\perp(|Fv_3^\perp|) \right\}.$$

Here $h_+, h_+^\perp$ are given by

$$h_+(z) := \frac{1 + z^2 + 2 \cos \theta \sqrt{z^2 - \sin^2 \theta}}{\sin^2 \theta} - 2,$$

and

$$h_+^\perp(z) := \frac{1 + z^2 + 2 \sin \theta \sqrt{z^2 - \cos^2 \theta}}{\cos^2 \theta} - 2.$$

Proof. Let $F_t = F(I + t v_i^\perp \otimes v_i)$. Since $|F_0 v_i| = |F v_i| \leq 1$ and $|F_t v_i| \rightarrow \infty$ as $t \rightarrow \pm\infty$, there are two values $t_- \leq 0 \leq t_+$ such that $|F_{t_\pm} v_i| = 1$. We have

$$W(F_{t_-}) = |F_{t_-}|^2 - 2 = |F_{t_-} v_i^\perp|^2 + |F_{t_-} v_i|^2 - 2 = |F v_i^\perp|^2 - 1,$$

and similarly $W(F_{t_+}) = |F v_i^\perp|^2 - 1$. Since F is a rank-one convex combination of F_{t_-} and F_{t_+} , the relation (5.50) follows from the definition (5.48) of W^{lc} .

Let us next prove that $W^{lc}(F) \leq h_+(|Fv_3|)$ for $F \in \mathcal{N}_1 \setminus \mathcal{N}_2$. We take the rank-one line $F_t = F(I + t v_3 \otimes v_3^\perp)$ and construct laminates between its intersections with $\mathcal{M}_1, \mathcal{M}_2$. To do so, we consider the transformation $\xi : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}^{2 \times 2}$ given by $F \mapsto RFR^{-1} = RFR$, where R is the reflection about v_3 . In particular, taking v_3 and $v_3^\perp$ as the coordinate system, $R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Note that if $\det F = 1$, then $\det \xi(F) = 1$. Moreover, since $Rv_1 = v_2$ and $Rv_2 = v_1$, we have $\xi(\mathcal{N}_1) = \mathcal{N}_2$, $\xi(\mathcal{M}_1) = \mathcal{M}_2$ and $\xi(\mathcal{N}_2) = \mathcal{N}_1$, $\xi(\mathcal{M}_2) = \mathcal{M}_1$.

Since $|(UF)v_i| = |Fv_i|$ for any $U \in SO(2)$, we can assume that F is an upper triangular matrix $F = \begin{pmatrix} \alpha & \beta \\ 0 & \gamma \end{pmatrix}$ with $\alpha > 0$. Then $\{F_t : t \in \mathbb{R}\} = \{\xi(F_t) : t \in \mathbb{R}\}$ holds because

$$\xi(F_t) = RFR + tRFv_3 \otimes v_3^\perp R = \begin{pmatrix} \alpha & -\beta \\ 0 & \gamma \end{pmatrix} - t \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix} = F - (\alpha t + 2\beta)v_3 \otimes v_3^\perp.$$

From the proof of Lemma 10, we see that there exist $t_\pm$ such that $t_- < 0 < t_+$ and $|F_{t_\pm} v_1| = 1$. Since the line F_t is invariant with respect to the transformation ξ , one finds that it intersects the set $\mathcal{M}_1$ at F_{t_-} and F_{t_+} , and the set $\mathcal{M}_2$ at $\xi(F_{t_-})$ and $\xi(F_{t_+})$. Since the intersection of $\mathcal{M}_1$ and $\mathcal{M}_2$ is the set $SO(2)$ which is not crossed by the line F_t more than once, at least three of the matrices F_{t_-} , F_{t_+} , $\xi(F_{t_-})$, $\xi(F_{t_+})$ are different.

Regarding the intersections of F_t with $\mathcal{M}_2$, there exist t'_- and t'_+ such that $\xi(F_{t_-}) = F_{t'_-}$ and $\xi(F_{t_+}) = F_{t'_+}$. In view of the form of $\xi(F_t)$ and $\alpha > 0$, we can see that $t'_+ < t'_-$.

Furthermore, since $|F_t v_2|^2$ is a quadratic function with a positive coefficient of the second-order term and $|F v_2| \geq 1$, t'_- and t'_+ have the same sign.

We consider the two cases for the sign of t'_+ and t'_- separately, beginning with the case $0 < t'_+ < t'_-$. Note that then $t'_- \leq t_+$ cannot happen because then $t_- < 0 < t'_+ < t'_- \leq t_+$, while $|F_t|^2$ is a quadratic function of t and $|F_{t_+}|^2 = |\zeta(F_{t_+})|^2 = |F_{t'_+}|^2$ along with $|F_{t_-}|^2 = |\zeta(F_{t_-})|^2 = |F_{t'_-}|^2$. Therefore, $t_- < 0 < t'_+, t_+ < t'_-$, i.e., $t_- = \min\{t_-, t_+, t'_-, t'_+\}$ and $t'_- = \max\{t_-, t_+, t'_-, t'_+\}$. In this case we show that we can construct optimal laminates between F_{t_-} and $F_{t'_-}$. Let $F_\gamma^\pm = R_\pm(I + \gamma_\pm v_1 \otimes v_1^\perp)$ for $\gamma_\pm \in \mathbb{R}$ and $R_\pm \in SO(2)$ be the matrices on $\mathcal{M}_1$ such that $F_{t_\pm} = F_\gamma^\pm$. The calculations in Lemma 11 show that $W(F_{t_\pm}) = |F_{t_\pm}|^2 - 2 = W(F_\gamma^\pm) = |F_\gamma^\pm|^2 - 2 = \gamma_\pm^2$ is equal to either $h(|F_\gamma^\pm v_3|) = h(|F v_3|)$ or to $h_+(|F_\gamma^\pm v_3|) = h_+(|F v_3|)$, depending on the value of $\gamma_\pm$. We notice that $|F_{t_+}| \neq |F_{t_-}|$. Indeed, if it is not the case, the quadratic function $|F_t|^2$ would take the same value for more than two values of t , in view of $|F_{t_+}|^2 = |F_{t'_+}|^2$ and $|F_{t_-}|^2 = |F_{t'_-}|^2$. This implies $\{W(F_{t_+}), W(F_{t_-})\} = \{h(|F v_3|), h_+(|F v_3|)\}$. Now, $W(F_t) = |F_t|^2 - 2$ is a quadratic function of t with positive coefficient of the quadratic term, $W(F_{t_+}) = W(F_{t'_+})$, and $t_- < t'_+, t_+ < t'_-$. In view of $h(|F v_3|) \leq h_+(|F v_3|)$, we conclude that $W(F_{t_-}) = W(F_{t'_-}) = h_+(|F v_3|)$, which implies $W^{lc}(F) \leq h_+(|F v_3|)$. The second case $t'_+ < t'_- < 0$ is analogous, leading to laminates between F_{t_+} and $F_{t'_+}$.

This finishes the proof of $W^{lc}(F) \leq h_+(|F v_3|)$ for $F \in \mathcal{N}_1 \setminus \mathcal{N}_2$. The proof for $F \in \mathcal{N}_2 \setminus \mathcal{N}_1$ is identical except for exchanging the roles of v_1 and v_2 . Finally, the proof of $W^{rc}(F) \leq h_+^\perp(|F v_3^\perp|)$ is done in the same way replacing the rank-one line $F_t = F(I + t v_3 \otimes v_3^\perp)$ by $F_t = F(I + t v_3^\perp \otimes v_3)$. $\square$

Proof of Proposition 6

First we confirm that for $F \in \mathcal{N}$, i.e., for $F \in \mathbb{R}^{2 \times 2}$ with $\det F = 1$, we have $W^{rc}(F) \leq W^{lc}(F)$ and $W^{pc}(F) \leq W^{lc}(F)$. Indeed, the first inequality follows directly from the definition of the envelopes, while the second one follows by noting that we may restrict F_0, F_1 to matrices with determinant 1 in the definition (5.48) of W^{lc} . Then if $F = \mu F_0 + (1 - \mu)F_1$ for $\mu \in [0, 1]$ and $F_0, F_1 \in \mathcal{N}$, we have $\det F = \mu \det F_0 + (1 - \mu) \det F_1$. Since $W^{pc}(F)$ can be expressed as a convex function of F and $\det F$, we get $W^{pc}(F) \leq \mu W^{pc}(F_0) + (1 - \mu)W^{pc}(F_1) \leq \mu W(F_0) + (1 - \mu)W(F_1)$. Taking infimum over all rank-one connected F_0, F_1 proves the second inequality.

Lemmas 9, 10 and 11, together with Remark 13 imply

$$W^{pc}(F) = W^{rc}(F) = W^{lc}(F) = \max\{h(|F v_3|), h^\perp(|F v_3^\perp|)\}$$

for $F \in \overline{\mathcal{A}} \cup (\overline{\mathcal{N}_1 \cap \mathcal{N}_2}) \cup \overline{\mathcal{A}_\perp}$. Since, in view of Lemmas 9, 10 and 12, W^{lc} is finite on $\mathcal{N}$, Conti and Dolzmann, 2016, Thm. 3.1 implies that $W^{qc} \leq W^{lc}$ on $\mathcal{N}$. Hence, taking into account Lemma 11, all the envelopes $W^{pc}, W^{qc}, W^{rc}, W^{lc}$ are equal on $\overline{\mathcal{A}} \cup (\overline{\mathcal{N}_1 \cap \mathcal{N}_2}) \cup \overline{\mathcal{A}_\perp}$. The remaining claims of Proposition 6, pertaining to values of envelopes on the boundary and to their infinite values, follow from Lemma 11 and the accompanying Remark 13. $\square$

5.3 Homogenization for General Slips

In this section we assume that the angle between slip directions is arbitrary and, based on the analysis of convex envelopes in Section 5.2, prove the partial homogenization result in Theorem 8, assuming that the Γ -limit is an integral functional. This assumption is needed because the general theory (see, e.g., Braides and De-franceschi, 1998) that guarantees Γ -limits to be integral functionals applies only to

sequences of functionals with standard growth conditions, while in our setting we deal with functionals whose admissible functions form a nonconvex subset of the underlying Sobolev space (see, e.g., Davoli and Fonseca, 2016 for a problem with linear differential constraint).

First, we present generalized versions of the two pillars of the proof, namely, Proposition 5 and Corollary 1.

Corollary 2. *Let $\Omega = (0, l)^2$ for $l > 0$ be a cube, and let $(u_\epsilon)_\epsilon \subset W^{1,2}(\Omega; \mathbb{R}^2)$ be such that $E_\epsilon(u_\epsilon) \leq C$ for all $\epsilon > 0$ and $u_\epsilon \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ for $u \in W^{1,2}(\Omega; \mathbb{R}^2)$ with gradient of the form (5.5). If, in addition, u is piecewise affine, then*

$$\liminf_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) \geq \lambda \int_{\Omega} f \left(\frac{1}{\lambda} (\nabla u - (1 - \lambda)R) \right) dx, \quad (5.51)$$

where

$$f(F) = \max \left\{ h(|Fv_3|), h^\perp(|Fv_3^\perp|) \right\},$$

with v_3 defined by (5.36) and $h, h^\perp$ by (5.38). In particular, for $N := \frac{1}{\lambda}(\nabla u - (1 - \lambda)R)$,

$$\liminf_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) \geq \begin{cases} \lambda |\Omega| h(|Nv_3|) & \text{if } N \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1 \cap \mathcal{N}_2}, \\ \lambda |\Omega| h^\perp(|Nv_3^\perp|) & \text{if } N \in \overline{\mathcal{A}_\perp}. \end{cases}$$

Proof. As noted in the proof of Lemma 11, f is a convex function on $\mathbb{R}^{2 \times 2}$ and hence continuous. Moreover, Lemma 11 and Remark 13 show that $f(\nabla u_\epsilon) = W(\nabla u_\epsilon)$ a.e. in $\epsilon Y_{soft} \cap \Omega$ for u_ϵ with finite energy. For such u_ϵ we additionally have $f(\nabla u_\epsilon) = 0$ a.e. in $\epsilon Y_{rig} \cap \Omega$. Therefore, beginning with

$$\liminf_{\epsilon \rightarrow 0} E_\epsilon(u_\epsilon) = \liminf_{\epsilon \rightarrow 0} \int_{\epsilon Y_{soft} \cap \Omega} W(\nabla u_\epsilon) dx = \liminf_{\epsilon \rightarrow 0} \int_{\Omega} f(\nabla u_\epsilon) dx,$$

the arguments in the proof of Proposition 5 are still valid, yielding the relation (5.51).

Finally, the analysis in Section 5.2 implies

$$\lambda |\Omega| f(N) = \begin{cases} \lambda |\Omega| h(|Nv_3|) & \text{if } N \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1 \cap \mathcal{N}_2}, \\ \lambda |\Omega| h^\perp(|Nv_3^\perp|) & \text{if } N \in \overline{\mathcal{A}_\perp}. \end{cases}$$

□

Corollary 3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded domain. If $N \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1 \cap \mathcal{N}_2} \cup \overline{\mathcal{A}_\perp}$, let $F_+, F_- \in \mathcal{M}_1 \cup \mathcal{M}_2$ and $\mu \in (0, 1)$ be such that $N = \mu F_+ + (1 - \mu)F_-$ and $\mu \in (0, 1)$. Specifically,*

- (i) *if $N \in \mathcal{A}$ or $N \in \mathcal{A}_\perp$, let $F_- = F_{s,*}$, $F_+ = F_{t,*}$ with $F_{s,*}, F_{t,*}$ defined in the proof of Lemma 9, and a suitable μ ;*
- (ii) *if $N \in \mathcal{N}_1 \cap \mathcal{N}_2$, let $F_- = F_{t_{1,-}}$, $F_+ = F_{t_{2,+}}$ with $F_{t_{1,-}}, F_{t_{2,+}}$ defined in the proof of Lemma 10, and a suitable μ ;*
- (iii) *if $N \in \mathcal{M}_1 \cup \mathcal{M}_2$, let $F_+ = F_- = N$ and $\mu \in (0, 1)$ be arbitrary.*

Then for every $\delta > 0$ there exists $\Omega_\delta \subset \Omega$ with $|\Omega \setminus \Omega_\delta| < \delta$ and $u_\delta \in W^{1,\infty}(\Omega; \mathbb{R}^2)$ such that u_δ coincides with a simple laminate between F_+ and F_- with weights μ and $1 - \mu$ and period $h_\delta < \delta$ in Ω_δ , $\nabla u_\delta \in \mathcal{M}_1 \cup \mathcal{M}_2$ a.e. in Ω , and $u_\delta = Nx$ on $\partial\Omega$. Moreover, there is a constant c depending only on N , such that for any $\delta \in (0, 1)$,

$$|\nabla u_\delta| \leq c \quad \text{a.e. in } \Omega. \quad (5.52)$$

In particular, $\nabla u_\delta \rightharpoonup N$ in $L^2(\Omega; \mathbb{R}^{2 \times 2})$ as $\delta \rightarrow 0$.

Proof. The proof is almost the same as for Corollary 1. Let us only deal with the case that $N \in \mathcal{A}$, since a similar argument applies to the cases $N \in \mathcal{A}_\perp$ and $N \in (\mathcal{N}_1 \cap \mathcal{N}_2) \setminus (\mathcal{M}_1 \cup \mathcal{M}_2)$. For the given F_+, F_- and $\delta > 0$, Theorem 9 with p taken as v_3 yields a finitely piecewise affine function v_δ and a set Ω_δ . Since $\text{dist}(\nabla v_\delta, [F_+, F_-]) \leq \delta$ and $0 < \delta < 1$, there is $c > 0$ independent of δ , such that

$$|\nabla v_\delta q| < c \quad \text{for } q = v_3, v_3^\perp, v_1, v_2.$$

The desired function u_δ is obtained as a result of applying Theorem 10 to modify v_δ in the finitely many subsets of $\Omega \setminus \Omega_\delta$ where it is affine. Taking any such subset S , there are three possible cases for the value of ∇v_δ in S : either $\nabla v_\delta \in \mathcal{A}$ or $\nabla v_\delta \in \mathcal{A}_\perp$ or $\min\{|\nabla v_\delta v_1|, |\nabla v_\delta v_2|\} < 1$. We first investigate the case $\nabla v_\delta \in \mathcal{A}$ in S . Recalling that the condition $|Fv_3|/|Fv_3^\perp| > \sin \theta / \cos \theta$ is equivalent to $Fv_1 \cdot Fv_2 > 0$, we can apply Theorem 10 to the in-approximation $(U_i^\delta)_i$ of $\mathcal{K} := (\mathcal{M}_1 \cap \mathcal{M}_2) \cap \{F \in \mathbb{R}^{2 \times 2} : |Fv_3| \leq c, Fv_1 \cdot Fv_2 \geq 0\}$ defined as

$$U_i^\delta := \{F \in \mathbb{R}^{2 \times 2} : \det F = 1, |Fv_3| < c, |Fv_1| > 1, |Fv_2| > 1, Fv_1 \cdot Fv_2 > 0\} \\ \cap \left\{F \in \mathbb{R}^{2 \times 2} : |Fv_1| < 1 + 2^{-(i-1)} \text{ or } |Fv_2| < 1 + 2^{-(i-1)}\right\}, \quad i \in \mathbb{N}.$$

It can be shown similarly to Corollary 1 that $(U_i^\delta)_i$ is an in-approximation of $\mathcal{K}$. The case when $\nabla v_\delta \in \mathcal{A}_\perp$ in S is handled analogously. The case when $\min\{|\nabla v_\delta v_1|, |\nabla v_\delta v_2|\} < 1$ in a subset S can be reduced to the proof of Conti and Theil, 2005, Lem. 2. The estimate (5.52) can also be proved in the same way as in the proof of Corollary 1. $\square$

We now turn to the proof of Theorem 8. Noting that the set of admissible functions for the Γ -limit E coincides with the set of weak limits $\lim_{j \rightarrow \infty} u_j$ of admissible sequences $(u_j)_j$ for $(E_j)_j$, we deduce from Proposition 5.5 that E is finite exactly for functions u with the form (5.5).

Step 1: Liminf inequality for affine u

Let u be affine and let $(u_j)_j$ be a sequence converging to u in $L^2(\Omega; \mathbb{R}^2)$ such that $(E_{\epsilon_j}(u_j))_j$ is bounded, where $\epsilon_j \rightarrow 0+$ as $j \rightarrow \infty$. Then, taking a subsequence if necessary, $u_j \rightharpoonup u$ in $W^{1,2}(\Omega; \mathbb{R}^2)$ and $\nabla u = R(I + \gamma e_1 \otimes e_2)$ for some $R \in SO(2)$ and $\gamma \in L^2(\Omega)$. Similarly to the proof of Theorem 7, it is sufficient to consider the case when Ω is a square. If $N = (1/\lambda)(\nabla u - (1 - \lambda)R) \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1} \cap \overline{\mathcal{N}_2} \cup \overline{\mathcal{A}_\perp}$, we obtain from Corollary 2

$$\liminf_{j \rightarrow \infty} E_{\epsilon_j}(u_j) \geq \lambda |\Omega| f(N). \quad (5.53)$$

Step 2: Recovery sequence for affine u

Let u be affine. If $N = (1/\lambda)(\nabla u - (1 - \lambda)R) \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1} \cap \overline{\mathcal{N}_2} \cup \overline{\mathcal{A}_\perp}$, Corollary 3 can be used to perform convex integration as in the proof of Theorem 7 (see Section 5.1.3), and to show the existence of a recovery sequence $(u_j)_j$. Then

$$\lim_{j \rightarrow \infty} E_{\epsilon_j}(u_j) = \lambda |\Omega| f(N). \quad (5.54)$$

Step 3: $W_{hom} = f$ on $\overline{\mathcal{A}} \cup \overline{\mathcal{N}_1} \cap \overline{\mathcal{N}_2} \cup \overline{\mathcal{A}_\perp}$

Let $N \in \overline{\mathcal{A}} \cup \overline{\mathcal{N}_1} \cap \overline{\mathcal{N}_2} \cup \overline{\mathcal{A}_\perp}$ and take u affine such that $(1/\lambda)(\nabla u - (1 - \lambda)R) = N$. By the assumption that the Γ -limit is an integral functional with density λW_{hom} , there

is a recovery sequence $(u_j)_j$ such that $\lim_{j \rightarrow \infty} E_{\varepsilon_j}(u_j) = \lambda|\Omega|W_{\text{hom}}(N)$. From this and (5.53), $f(N) \leq W_{\text{hom}}(N)$. On the other hand, from Step 2, there is a recovery sequence $(u_j)_j$ such that (5.54) holds. Using the liminf inequality in the definition of Γ -limit, we get $\liminf_{j \rightarrow \infty} E_{\varepsilon_j}(u_j) \geq \lambda|\Omega|W_{\text{hom}}(N)$, and hence $W_{\text{hom}}(N) \leq f(N)$. These two inequalities yield $W_{\text{hom}}(N) = f(N)$. $\square$

5.4 Evolutionary Γ -Convergence in Layered Composites

We conclude this chapter by collecting results on evolutionary Γ -convergence in Davoli and Kreisbeck, 2022 for the class of settings treated in this chapter.

Building on the static homogenization of Christowiak and Kreisbeck, 2017, Davoli and Kreisbeck, 2022 extend the analysis to quasi-static time evolution (RIS) and, via evolutionary Γ -convergence, identify the limiting system in the specific case of a single slip system, where the slip direction is either parallel or orthogonal to the layers. We use the same notation as in the previous section: $Y = [0, 1]^2$, $Y_{\text{soft}} = [0, 1] \times [0, \lambda)$ with $\lambda \in (0, 1)$, and $Y_{\text{rig}} = Y \setminus Y_{\text{soft}}$. As in Christowiak and Kreisbeck, 2017, within finite-strain crystal plasticity, a slip system is given by a unit vector s with normal $m = s^\perp$. The admissible deformation gradients are

$$\nabla u = R(I + \gamma s \otimes m), \quad R \in SO(2),$$

with $\gamma \in L^2(\Omega)$ only on Y_{soft} and $\gamma = 0$ on Y_{rig} .

RIS Framework. Let the state space be $\mathcal{Q} = \mathcal{Y} \times \mathcal{Z}$ with

$$\mathcal{Y} = W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2), \quad \mathcal{Z} = L^2(\Omega),$$

and write $q = (u, \gamma)$. For a energy functional $\mathcal{E}_\varepsilon : \mathcal{Q} \rightarrow \mathbb{R} \cup \{\infty\}$ and a time-dependent load $g_\varepsilon \in W^{1,1}(0, T; L^2(\Omega; \mathbb{R}^2))$, set

$$\mathcal{E}_\varepsilon(t, q) = \mathcal{E}_\varepsilon(q) - \int_\Omega g_\varepsilon(t) \cdot u \, dx.$$

Dissipation distances (with $\delta > 0$) are

$$\mathcal{D}_1(q, \tilde{q}) = \delta \int_\Omega |\gamma - \tilde{\gamma}| \, dx, \quad \mathcal{D}_\geq(q, \tilde{q}) = \begin{cases} \mathcal{D}_1(q, \tilde{q}), & \gamma \geq \tilde{\gamma} \text{ a.e.}, \\ +\infty, & \text{otherwise.} \end{cases}$$

The admissible sets for a single slip system specified by a unit vector $s \in \mathbb{R}$ are given by

$$\mathcal{B}_\varepsilon^{(s)} = \left\{ (u, \gamma) \in \mathcal{Q} : \nabla u = R(I + \gamma s \otimes m), R \in L^\infty(\Omega; SO(2)), \gamma = 0 \text{ on } \varepsilon Y_{\text{rig}} \right\}.$$

If we define

$$\mathcal{A}_\varepsilon^{(s)} := \left\{ u \in W^{1,2} \cap L_0^2(\Omega; \mathbb{R}^2) : \nabla u = R(I + \gamma s \otimes m), R \in L^\infty(\Omega; SO(2)), \right. \\ \left. \gamma = 0 \text{ on } \varepsilon Y_{\text{rig}} \right\},$$

then we can write

$$\mathcal{B}_\varepsilon^{(s)} := \left\{ (u, \gamma) \in \mathcal{Q} : u \in \mathcal{A}_\varepsilon^{(s)} \right\}.$$

Thus, the admissible set of deformations coincides with that of the static model, but in the RIS model the slip coefficient is taken as an explicit degree of freedom directly entering the description, unlike in the static model (cf. (5.3)).

ε -Dependent Energies.

- $s = e_2$ (orthogonal to the layers):

$$\mathcal{E}_\varepsilon(q) = \begin{cases} \int_{\Omega} \gamma^2 dx, & q \in \mathcal{B}_\varepsilon^{(e_2)}, \\ +\infty, & \text{otherwise.} \end{cases}$$

- $s = e_1$ (parallel to the layers): two families are considered.

$$(\text{rigid rotations}) \quad \mathcal{E}_\varepsilon^{\text{rig}}(t, q) = \int_{\Omega} \gamma^2 dx - \int_{\Omega} g_\varepsilon(t) \cdot u dx,$$

$$u \in \mathcal{A}_\varepsilon^{(e_1), \text{rig}} := \{ u \in \mathcal{A}_\varepsilon^{(e_1)} : R = \text{const} \},$$

$$(\text{regularization}) \quad \mathcal{E}_{\varepsilon, \tau_\varepsilon}^{\text{reg}}(t, q) = \int_{\Omega} \gamma^2 dx - \int_{\Omega} g_\varepsilon(t) \cdot u dx + \tau_\varepsilon \|\partial_1 \gamma\|_{L^2(\Omega)}^2,$$

$$u \in \mathcal{A}_\varepsilon^{(e_1)}, \quad \tau_\varepsilon \rightarrow \infty.$$

Assumptions for Evolutionary Γ -convergence. In Davoli and Kreisbeck, 2022, the following hypotheses (H1)–(H5) (or (H1)–(H4) together with (H6), a sufficient condition for (H5)) are verified. They are a specialization of the general framework introduced in Section 4.2.

(H1) $\mathcal{D}_\varepsilon = \mathcal{D}$ for all $\varepsilon > 0$, where $\mathcal{D} : \mathcal{Z} \times \mathcal{Z} \rightarrow [0, +\infty]$ is a lower semicontinuous quasi-distance (definite and satisfying the triangle inequality).

(H2) $\mathcal{E}_\varepsilon : \mathcal{Q} \rightarrow \mathbb{R} \cup \{+\infty\}$ are lower semicontinuous for all $\varepsilon > 0$.

(H3) There exist $C, c > 0$ and $\alpha > 1$ such that $\mathcal{E}_\varepsilon(q) \geq C\|q\|_{\mathcal{Q}}^\alpha - c$ for all $q \in \mathcal{Q}$ and $\varepsilon > 0$, and $l_\varepsilon \rightarrow l_0$ in $W^{1,1}(0, T; \mathcal{Y}')$.

(H4) $(\mathcal{E}_\varepsilon)_\varepsilon$ Γ -converges to $\mathcal{E}_0$ with respect to the topology of $\mathcal{Q}$.

(H5) (Stability closedness) If $(t_\varepsilon, q_\varepsilon)_\varepsilon$ is a stable sequence, i.e., $q_\varepsilon \in \mathcal{S}_\varepsilon(t_\varepsilon)$ for all $\varepsilon > 0$ and $\sup_{\varepsilon > 0} \mathcal{E}_\varepsilon(t_\varepsilon, q_\varepsilon) < +\infty$, and $(t_\varepsilon, q_\varepsilon) \rightarrow (t, q)$ in $[0, T] \times \mathcal{Q}$, then $q \in \mathcal{S}_0(t)$.

As a corollary of Theorem 3 (or Mielke and Roubíček, 2015, Th. 2.4.10), if (H1)–(H5) hold and $(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon)$ admits energy solutions, then

$$(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{D}_\varepsilon) \xrightarrow{\text{ev-}\Gamma} (\mathcal{Q}, \mathcal{E}_0, \mathcal{D}).$$

Moreover, if (H1), (H3), (H4), and (H5) hold, then, by means of a relaxation in the form of a straightforward extension of Mielke and Roubíček, 2015, Th. 2.5.1, one obtains approximate evolutionary Γ -convergence. This result ensures the convergence of approximate solutions of the time-discrete problems (cf. Remark 7)

Theorem 11 (Evolutionary Γ -convergence for $s = e_2$). (Davoli and Kreisbeck, 2022, Th. 3.1)

$$(\mathcal{Q}, \mathcal{E}_\varepsilon, \mathcal{D}) \xrightarrow{\text{ev-}\Gamma_{\text{app}}} (\mathcal{Q}, \mathcal{E}_0, \mathcal{D}),$$

i.e., in the limit there is no dissipation along transitions between finite-energy states (“dissipation-free”). The stable set at time t is

$$\begin{aligned} \mathcal{S}_0(t) &= \{q \in \mathcal{Q} : \mathcal{E}_0(t, q) \leq \mathcal{E}_0(t, \tilde{q}) \forall \tilde{q}\} \\ &= \{(Rx, 0) : R \in \arg \min_{S \in SO(2)} (-\int_{\Omega} g_0(t) \cdot Sx \, dx)\}, \end{aligned}$$

i.e., pure rigid rotations (if the load vanishes, every rotation is stable).

The proof verifies (H6) via a mutual recovery argument and exploits the fact that the dissipation of any finite-energy state is zero, which yields limit stability. This case is formulated as an approximate ev- Γ -limit.

Theorem 12 (Evolutionary Γ -convergence for $s = e_1$). (Davoli and Kreisbeck, 2022, Th. 3.3)

- **Limit energy.**

$$\mathcal{E}_0(t, q) = \frac{1}{\lambda} \int_{\Omega} \gamma^2 \, dx - \int_{\Omega} g_0(t) \cdot u \, dx, \quad u \in \mathcal{A}^{(e_1)}.$$

- **Main assertions.**

1. Under the rigid-rotation (one-dimensional) reduction with monotone dissipation,

$$(\mathcal{Q}, \mathcal{E}_{\varepsilon}^{\text{rig}}, \mathcal{D}_{\geq}) \xrightarrow{\text{ev-}\Gamma} (\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_{\geq}).$$

Even without the monotonicity constraint one obtains the analogous result with symmetric dissipation:

$$(\mathcal{Q}, \mathcal{E}_{\varepsilon}^{\text{rig}}, \mathcal{D}_1) \xrightarrow{\text{ev-}\Gamma} (\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_1).$$

2. For the regularized family with an L^2 -penalty on the dislocation tensor $G(F_{\text{pl}}) = (\partial_1 \gamma) e_1$ (with $\tau_{\varepsilon} \rightarrow \infty$) and symmetric dissipation,

$$(\mathcal{Q}, \mathcal{E}_{\varepsilon, \tau_{\varepsilon}}^{\text{reg}}, \mathcal{D}_1) \xrightarrow{\text{ev-}\Gamma_{\text{app}}} (\mathcal{Q}, \mathcal{E}_0, \mathcal{D}_1).$$

In both limits the energy and dissipation do not interfere.

Rigidification/regularization suppresses fast micro-oscillations of rotations and shear, thereby avoiding weak $\times$ weak product limits. In general settings where both rotation and shear strongly oscillate, a new compensated compactness mechanism would be required; this remains open. The energy limit (H4) follows from the static Γ -limit of Christowiak and Kreisbeck, 2017, while stability (H6) is verified via mutual recovery.

In these evolutionary Γ -limit results, the energy part is inherited from the static Γ -limit. By contrast, although the dissipation terms formally agree with the static model, they are not obtained via Mosco convergence.

Chapter 6

Dynamic Extension

The purpose of this chapter is to provide an energetic formulation of RIS with inertia and viscosity for a small-strain, single-slip elasto-plastic body, and to establish the existence of inertial energetic solutions in the sense of Definition 6 (Theorem 13). The present formulation follows the standard framework of RIS due to Mielke and Roubíček (dissipation distance, stability, and energy balance), while moving beyond the quasi-static limit commonly assumed in the literature and addressing a dynamic setting that retains inertia and Kelvin–Voigt viscosity. See, e.g., Mielke and Roubíček, 2015.

Compression experiments on layered structures reported in Drozdenko et al., 2022 show that localized kink-bands nucleate and propagate in sequence, with load drops correlated to each event (see, for instance, Fig. 5 and the discussion on p. 12 in Drozdenko et al., 2022). This sequential character suggests that inertia may influence the timing and selection of deformation patterns. Motivated by this observation, one would ultimately like to analyze periodic materials within an energetic framework that incorporates inertia. As a first step toward that goal, this chapter develops the dynamic small-strain single-slip formulation in a nonperiodic setting with variable coefficients, providing a baseline analytical and mechanical framework for subsequent extensions to periodic microstructures and homogenization with inertia.

6.1 Specific Dynamic Model Analysis

We take the slip γ as an internal variable and prescribe the single-slip plastic strain $e^p = \gamma S$, where S is a fixed symmetric slip-system projector. The stored energy is given by a position-dependent quadratic elastic form via $\mathbb{C}(x)$, the viscous dissipation by a quadratic form via $\mathbb{D}(x)$, and the rate-independent dissipation for γ by $\int_{\Omega} \theta(x) |\dot{\gamma}| dx$. To secure compactness, we incorporate a strain-gradient regularization $\kappa \|\nabla \gamma\|_{L^\alpha(\Omega)}^\alpha$ with an exponent α chosen so that the compactness property required is guaranteed. Loads f and g are assumed to be H^1 -regular in time, and the momentum balance at small strain includes density $\rho > 0$, thereby combining inertia, viscosity, and elasticity in weak form.

Because inertia is present, the purely rate-independent notion of global stability together with an exact energy balance is not directly applicable. In this chapter, a solution is defined by the combination of the weak momentum balance, a local semistability condition for γ , and an energy inequality that accounts for kinetic energy, viscous dissipation, and rate-independent dissipation; see Definition 6. This setting is consistent with dynamic elasto-plastic models for viscous solids at small strains (Roubíček, 2009; Roubíček, 2010) and with abstract existence results for systems coupling a rate-independent process with a second-order (in time) evolution (Rossi and Thomas, 2017).

The existence theorem of this chapter specialize these results to a concrete material model of single-slip elasto-plasticity with gradient regularization, and they are carried out in a nonperiodic setting with variable coefficients.

6.1.1 Problem Setting

Geometric and kinematic setting. Let $\Omega \subset \mathbb{R}^d$ ($d \in \{2, 3\}$) be a bounded Lipschitz domain whose boundary is decomposed into a Dirichlet part Γ_0 and a Neumann part Γ_1 , with $\Gamma_0 \cap \Gamma_1 = \emptyset$ and

$$\mathcal{H}^{d-1}(\Gamma_0) > 0, \quad \partial\Omega = \overline{\Gamma_0} \cup \overline{\Gamma_1}.$$

A fixed final time $T > 0$ is prescribed.

We work in the infinitesimal-strain regime. The deformation is $y : [0, T] \times \Omega \rightarrow \mathbb{R}^d$, the displacement is $u := y - \text{id}$, and the symmetric strain is

$$e(u) := \frac{1}{2}(\nabla u + \nabla u^\top) \in \mathbb{R}_{\text{sym}}^{d \times d}.$$

Single-slip plasticity. The plastic distortion is assumed to take the additive single-slip form

$$e^p := \gamma \frac{1}{2}(s \otimes m + m \otimes s) =: \gamma S, \quad s, m \in \mathbb{R}^d, \quad s \perp m, \quad |s| = |m| = 1,$$

with scalar slip field $\gamma : [0, T] \times \Omega \rightarrow \mathbb{R}$, and $S := \frac{1}{2}(s \otimes m + m \otimes s)$ denotes the (constant) slip-system projector. The elastic strain therefore reads

$$e^e := e(u) - e^p = e(u) - \gamma S.$$

Stored energy. We define the elastic energy density by

$$W(x, e^e) := \frac{1}{2} \mathbb{C}(x)(e(u) - \gamma S) : (e(u) - \gamma S),$$

where $\mathbb{C}(x) : \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$ is a fourth-order elasticity tensor defined for a.e. $x \in \Omega$. We assume the positive definiteness and boundedness

$$\exists 0 < \lambda_0 \leq \Lambda_0 < \infty \text{ s.t. for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}_{\text{sym}}^{d \times d} : \quad \lambda_0 |\xi|^2 \leq \mathbb{C}(x)\xi : \xi \leq \Lambda_0 |\xi|^2,$$

and the symmetry on symmetric tensors

$$\mathbb{C}(x)\xi : \eta = \mathbb{C}(x)\eta : \xi \quad \text{for all } \xi, \eta \in \mathbb{R}_{\text{sym}}^{d \times d}, \text{ a.e. } x \in \Omega.$$

The corresponding elastic stress tensor is given by

$$P := \partial_{e^e} W = \mathbb{C}(e(u) - \gamma S).$$

Material coefficients and external data. The following list collects all material parameters, hardening moduli, and prescribed loadings that will enter the energetic formulation.

- **Mass density** $\rho \in L^\infty(\Omega)$ with $0 < \rho_- \leq \rho(x)$ for some constant ρ_- .

- **Viscous dissipation potential** We define the viscous dissipation potential as

$$V(\dot{u}) := \frac{1}{2} \int_{\Omega} \mathbb{D}(x) e(\dot{u}) : e(\dot{u}) dx,$$

where $\mathbb{D}(x) : \mathbb{R}_{\text{sym}}^{d \times d} \rightarrow \mathbb{R}_{\text{sym}}^{d \times d}$ is a fourth-order viscosity tensor for a.e. $x \in \Omega$. We assume the positive definiteness and boundedness

$$\begin{aligned} \exists 0 < \delta_0 \leq \Delta_0 < \infty \text{ s.t. for a.e. } x \in \Omega, \forall \xi \in \mathbb{R}_{\text{sym}}^{d \times d} : \\ \delta_0 |\xi|^2 \leq \mathbb{D}(x) \xi : \xi \leq \Delta_0 |\xi|^2, \end{aligned}$$

and the symmetry on symmetric tensors

$$\mathbb{D}(x) \xi : \eta = \mathbb{D}(x) \eta : \xi, \quad \text{for all } \xi, \eta \in \mathbb{R}_{\text{sym}}^{d \times d}, \text{ a.e. } x \in \Omega.$$

- **Slip resistance** $\theta \in C(\Omega)$ with $0 < \theta_0 \leq \theta(x)$ for some constant θ_0 .
- **Hardening potential** $w : \mathbb{R} \rightarrow \mathbb{R}_{\geq 0}$ is strictly convex and satisfies the q -growth condition

$$C_1 |\gamma|^q - c_0 \leq w(\gamma) \leq C_2 (1 + |\gamma|^q), \quad C_1, C_2 > 0, c_0 \geq 0, q \geq 2.$$

- **Strain-gradient regularization** We add the term $\kappa |\nabla \gamma|^\alpha$ to the free energy of the model, with exponent $\alpha > 0$ and coefficient $\kappa > 0$.

The compactness requirement is given by $\alpha \geq q$. This condition is related to the Gagliardo–Nirenberg inequality: Define $\eta \in [0, 1)$ by

$$\frac{1}{\alpha} = \frac{1-\eta}{q} + \eta \left(\frac{1}{\alpha} - \frac{1}{n} \right) \iff \eta = \frac{\frac{1}{q} - \frac{1}{\alpha}}{\frac{1}{n} + \frac{1}{q} - \frac{1}{\alpha}} = \frac{n \left(\frac{1}{q} - \frac{1}{\alpha} \right)}{1 + n \left(\frac{1}{q} - \frac{1}{\alpha} \right)}.$$

Then, by the assumption $\alpha \geq q$, we have $\eta \in [0, 1)$. By the Gagliardo–Nirenberg theorem, for a bounded domain $\Omega \subset \mathbb{R}^d$ it holds that

$$\|\gamma\|_{L^\alpha(\Omega)} \leq C \left(\|\nabla \gamma\|_{L^\alpha(\Omega)}^\eta \|\gamma\|_{L^q(\Omega)}^{1-\eta} + \|\gamma\|_{L^q(\Omega)} \right). \quad (6.1)$$

In particular, $\|\gamma\|_{L^\alpha(\Omega)}$ can be estimated from above in terms of $\|\nabla \gamma\|_{L^\alpha(\Omega)}$ and $\|\gamma\|_{L^q(\Omega)}$.

Moreover, when $\alpha \geq q$, by the Rellich–Kondrachov theorem, $W^{1,\alpha}(\Omega)$ is compactly embedded into $L^q(\Omega)$.

- **Boundary displacement** $u_D \in H^1(\Omega)$ is prescribed, satisfying

$$u_D(x) = 0 \quad \text{for almost every } x \in \Gamma_0.$$

- **External loads** As defined above, we impose the homogeneous Dirichlet condition $u = 0$ on Γ_0 . The external loads consist of a body force $f \in H^1(0, T; L^2(\Omega; \mathbb{R}^d))$ and a boundary traction $g \in H^1(0, T; L^2(\Gamma_1; \mathbb{R}^d))$.

Let

$$\sigma := \mathbb{C}(e(u) - \gamma S) + \mathbb{D}e(\dot{u})$$

denote the Cauchy stress and n the unit outward normal to $\partial\Omega$. The body force acts in the bulk through the balance

$$\rho \ddot{u} - \operatorname{div} \sigma = f(t) \quad \text{in } \Omega \text{ for a.e. } t \in (0, T),$$

while the traction is given on Γ_1 by the Neumann condition

$$\sigma n = g(t) \quad \text{on } \Gamma_1 \text{ for a.e. } t \in (0, T).$$

Energy functional and rate-independent dissipation. We now specify the kinetic energy, total free energy, and the rate-independent slip dissipation that drive the evolution.

$$\begin{aligned} K(\dot{u}) &= \int_{\Omega} \frac{\rho(x)}{2} |\dot{u}|^2 dx, \\ E(t, u, \gamma) &= \int_{\Omega} [W(e^e) + w(\gamma) + \kappa |\nabla \gamma|^\alpha] dx - \int_{\Omega} f(t) \cdot u dx - \int_{\Gamma_1} g(t) \cdot u dS, \\ R(\dot{\gamma}) &= \int_{\Omega} \theta(x) |\dot{\gamma}| dx, \quad D(\gamma_1, \gamma_2) = \int_{\Omega} \theta(x) |\gamma_1 - \gamma_2| dx. \end{aligned}$$

Admissible configurations. We work with homogeneous Dirichlet data on a designated part of the boundary. We define the closed subspace of $H^1(\Omega; \mathbb{R}^d)$ as

$$H_{\Gamma_0}^1(\Omega; \mathbb{R}^d) := \{ v \in H^1(\Omega; \mathbb{R}^d) \mid \operatorname{Tr} v = 0 \text{ on } \Gamma_0 \},$$

where Tr denotes the trace operator. At each time $t \in [0, T]$ the state variables (u, γ) are restricted to the set

$$\mathcal{Q}(t) := \left\{ (u, \gamma) \mid u(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d), \gamma(t) \in W^{1,\alpha}(\Omega) \right\}.$$

Remark 14. For simplicity, we have imposed the Dirichlet boundary condition so that solution always vanishes on Γ_0 . This assumption is not restrictive. We can generalize the boundary condition by introducing $b_D \in H^1(0, T; H^1(\Omega))$ satisfying

$$b_D(t, x) = 0 \quad \text{for almost every } (t, x) \in [0, T] \times (\partial\Omega \setminus \Gamma_0).$$

Using such a function b_D , we redefine the set of admissible configurations as

$$\mathcal{Q}(t) := \left\{ (u, \gamma) \mid u - b_D(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d), \gamma \in W^{1,\alpha}(\Omega) \right\}.$$

In this case, whenever the Poincaré or Korn inequality is used in the later proofs, it should be applied to $u - b_D$ instead of u . As a result, an additional term involving the H^1 -norm of b_D , namely $\|b_D\|_{H^1(\Omega)}$, appears in the upper bounds of the uniform estimates. However, since b_D is fixed, this modification does not affect the overall reasoning.

6.1.2 Energetic Formulation with Inertia

Definition 6 (Local solution with inertia and viscosity). A pair (u, γ) satisfying

$$\begin{aligned} u &\in H^1(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)), \\ \gamma &\in L^\alpha(0, T; W^{1,\alpha}(\Omega)) \end{aligned}$$

is called a local solution with inertia and viscosity if the following conditions hold:

1. **Weak momentum balance**

$$\begin{aligned} & \int_0^T \left[\int_{\Omega} \left(\mathbb{C}(e(u(t))) - \gamma(t)S + \mathbb{D}e(\dot{u}(t)) \right) : e(\phi(t)) \, dx - \int_{\Omega} \rho \dot{u}(t) \cdot \dot{\phi}(t) \, dx \right] dt \\ & + \int_{\Omega} \rho \dot{u}(T) \cdot \phi(T) \, dx \\ & = \int_{\Omega} \rho v_0 \cdot \phi(0) \, dx + \int_0^T \left(\int_{\Omega} f(t) \cdot \phi(t) \, dx + \int_{\Gamma_1} g(t) \cdot \phi(t) \, dS \right) dt, \end{aligned}$$

for all $\phi \in H^1(0, T; H_{\Gamma_0}^1(\Omega; \mathbb{R}^d))$.

2. **Local semi-stability**

For a.e. $t \in [0, T]$,

$$E(t, u(t), \gamma(t)) \leq E(t, u(t), \tilde{\gamma}) + D(\gamma(t), \tilde{\gamma}) \quad \forall \tilde{\gamma} \in W^{1,\alpha}(\Omega).$$

3. **Energy inequality**

For a.e. t_1, t_2 ($0 \leq t_1 \leq t_2 \leq T$),

$$\begin{aligned} & K(\dot{u}(t_2)) + E(t_2, u(t_2), \gamma(t_2)) + \int_{t_1}^{t_2} \left[R(\dot{\gamma}(s)) + \int_{\Omega} \mathbb{D}e(\dot{u}(s)) : e(\dot{u}(s)) \, dx \right] ds \\ & \leq K(\dot{u}(t_1)) + E(t_1, u(t_1), \gamma(t_1)) + \int_{t_1}^{t_2} \partial_s L(s, u(s)) \, ds, \end{aligned}$$

where

$$L(t, u) = - \int_{\Omega} f(t) \cdot u \, dx - \int_{\Gamma_1} g(t) \cdot u \, dS.$$

6.2 Existence Theorem

Theorem 13 (Existence of inertial energetic solutions). *Assume the data and constitutive functions stated in Section 6.1.1. Given initial conditions $u^0 \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$, $v^0 \in L^2(\Omega; \mathbb{R}^d)$, and $\gamma^0 \in L^q(\Omega) \cap W^{1,\alpha}(\Omega)$, there exists at least one energetic solution with inertia in the sense of Definition 6, satisfying $u(0) = u^0$, $\dot{u}(0) = v^0$, and $\gamma(0) = \gamma^0$.*

Although it is necessary to include the kinetic energy term and to establish the weak momentum balance, the overall structure of the proof is the same as that of the existence proof for the energetic solution discussed in Chapter 3. Namely, we first formulate the minimization problem at each discrete time step and prove the existence of its solution (Proposition 7). Next, we derive uniform estimates that are independent of the time discretization parameter (Proposition 8, Corollary 4). Finally, by exploiting appropriate compactness results deduced from these uniform estimates, we obtain various convergences and pass to the limit in the discrete weak momentum balance, the discrete stability condition, and the discrete energy inequality obtained from the time-discrete minimization problem.

Discrete minimisation at a fixed time-step

Fix a step-size $\tau = T/N > 0$ ($N \in \mathbb{N}$), and a previous state $(u^k, \gamma^k) \in \mathcal{Q}(t_k)$ with $t_k = k\tau$. Denote $t_{k+1} = t_k + \tau$. For $k = 0$ the second order difference term

$\frac{1}{\tau^2}(u - 2u^k + u^{k-1})$ involves a fictitious value u^{-1} ; we set $u^{-1} := u^0 - \tau v^0$, so that the discrete velocity $(u^0 - u^{-1})/\tau$ matches the prescribed initial velocity v^0 . We introduce the functional

$$\mathcal{J}_\tau^k(u, \gamma) := \frac{1}{2\tau^2} \int_\Omega \rho |u - 2u^k + u^{k-1}|^2 dx + \tau V\left(\frac{u - u^k}{\tau}\right) + E(t_{k+1}, u, \gamma). \quad (6.2)$$

Proposition 7 (Well-posedness of one time-step). *Under the hypotheses of Theorem 13 and for every $\tau > 0$, $N \in \mathbb{N}$, and $k = 0, \dots, N - 1$, consider the following two incremental minimization problems:*

$$\left. \begin{array}{l} \text{minimize } \mathcal{J}_\tau^k(u, \gamma^k) \\ \text{subject to } u \in H_{\Gamma_0}^1(\Omega), \end{array} \right\} \quad (6.3)$$

and, for any u^{k+1} chosen as a minimiser of the first problem,

$$\left. \begin{array}{l} \text{minimize } E(t_{k+1}, u^{k+1}, \gamma) + D(\gamma^k, \gamma) \\ \text{subject to } \gamma \in W^{1,\alpha}(\Omega). \end{array} \right\} \quad (6.4)$$

Then both problems admit a unique solution. In particular, there exists a unique $u^{k+1} \in H_{\Gamma_0}^1(\Omega)$ solving (6.3), and, for this u^{k+1} , there exists a unique $\gamma^{k+1} \in W^{1,\alpha}(\Omega)$ solving (6.4).

In what follows, whenever domain is omitted in a norm, it is understood to be Ω .

Step 1: Coercivity. Both minimization problems consist of nonnegative terms and include the functional E . Therefore, it is sufficient to establish the coercivity of $E(t, u, \gamma)$.

For the external loading terms, since $\mathcal{H}^{d-1}(\Gamma_0) > 0$ and $u \in H_{\Gamma_0}^1(\Omega)$, we can use the fact that Poincaré inequality on $H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$ holds: there exists $C_P = C_P(\Omega, \Gamma_0) > 0$ such that

$$\|u\|_{L^2(\Omega)} \leq C_P \|\nabla u\|_{L^2(\Omega)} \quad \text{for all } u \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d). \quad (6.5)$$

Moreover, by the trace theorem there exists $C_{\text{tr}} = C_{\text{tr}}(\Omega) > 0$ with

$$\|u\|_{L^2(\partial\Omega)} \leq C_{\text{tr}} \|u\|_{H^1(\Omega)} \quad \text{for all } u \in H^1(\Omega; \mathbb{R}^d). \quad (6.6)$$

Combining (6.5) and (6.6) we obtain, for the restriction to $\Gamma_1 \subset \partial\Omega$,

$$\|u\|_{L^2(\Gamma_1)} \leq C_{\text{tr}} (\|u\|_{L^2} + \|\nabla u\|_{L^2}) \leq C_{\text{tr}} (1 + C_P) \|\nabla u\|_{L^2} =: \tilde{C}_{\text{tr}} \|\nabla u\|_{L^2}. \quad (6.7)$$

Hence, by Young's inequality, for any $\varepsilon_1 > 0$,

$$\left| \int_\Omega f \cdot u dx + \int_{\Gamma_1} g \cdot u dS \right| \leq \varepsilon_1 \|\nabla u\|_{L^2}^2 + C_{\varepsilon_1} (\|f\|_{L^2}^2 + \|g\|_{L^2(\Gamma_1)}^2). \quad (6.8)$$

For the positive contributions of elasticity, hardening, and strain-gradient regularization, the coercivity of $\mathbb{C}$ with constant λ_0 implies

$$\begin{aligned} \int_\Omega \left[\frac{1}{2} \mathbb{C}(e^e(u)) : (e^e(u)) + w(\gamma) + \kappa |\nabla \gamma|^\alpha \right] dx \\ \geq \frac{\lambda_0}{2} \|e^e(u)\|_{L^2}^2 + C_1 \|\gamma\|_{L^q}^q + \kappa \|\nabla \gamma\|_{L^\alpha}^\alpha - c_0 |\Omega|. \end{aligned} \quad (6.9)$$

By the triangle inequality and Young's inequality, starting from

$$e(u) = e^e(u) + \gamma S,$$

we obtain

$$\|e(u)\|_{L^2}^2 \leq 2\|e^e(u)\|_{L^2}^2 + 2|S|^2\|\gamma\|_{L^2}^2 = 2\|e^e(u)\|_{L^2}^2 + \|\gamma\|_{L^2}^2. \quad (6.10)$$

Since the domain Ω is bounded and $q \geq 2$, we have the embedding $L^q(\Omega) \hookrightarrow L^2(\Omega)$. First, consider the case where $q > 2$. In this case, for any $\varepsilon_2 > 0$, by choosing C_{ε_2} sufficiently large, we obtain

$$\|\gamma\|_{L^2}^2 \leq C_\Omega \|\gamma\|_{L^q}^2 \leq C_{\varepsilon_2} + \varepsilon_2 \|\gamma\|_{L^q}^q. \quad (6.11)$$

Combining (6.10) and (6.11) yields

$$\|e(u)\|_{L^2}^2 \leq 2\|e^e(u)\|_{L^2}^2 + C_{\varepsilon_2} + \varepsilon_2 \|\gamma\|_{L^q}^q. \quad (6.12)$$

Combining (6.8), (6.9), Korn's inequality ($\|\nabla u\|_{L^2} \leq C_K \|e(u)\|_{L^2}$), and (6.12) which estimates $\|e(u)\|_{L^2}$ in terms of $\|e^e(u)\|_{L^2}$ and $\|\gamma\|_{L^q}$, we obtain that, by taking $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ sufficiently small, the negative terms can be absorbed. Therefore, there exist constants $c, C_0 > 0$ such that

$$E(t, u, \gamma) \geq c \left(\|\nabla u\|_{L^2}^2 + \|\gamma\|_{L^q}^q + \|\nabla \gamma\|_{L^q}^\alpha \right) - C_0. \quad (6.13)$$

By the Gagliardo–Nirenberg inequality, the $W^{1,\alpha}$ -norm of γ can also be bounded from above in terms of E .

In the case $q = 2$, we take λ_1 such that $0 < \lambda_1 < \min\{\lambda_0, 4C_1\}$. The estimate (6.9) can be relaxed as follows:

$$\begin{aligned} \int_\Omega \left[\frac{1}{2} C(e^e(u)) : (e^e(u)) + w(\gamma) + \kappa |\nabla \gamma|^\alpha \right] dx \\ \geq \frac{\lambda_1}{2} \|e^e(u)\|_{L^2}^2 + C_1 \|\gamma\|_{L^q}^q + \kappa \|\nabla \gamma\|_{L^q}^\alpha - c_0 |\Omega|. \end{aligned}$$

With this relaxation, from (6.10), we can again absorb the negative term with respect to γ , and thus the required coercivity is obtained.

Step 2: Compactness. Let $\{u_n\}_{n \in \mathbb{N}}$ be a minimizing sequence for (6.3). By the coercivity established in Step 1, we may extract a subsequence (not relabelled) such that

$$u_n \rightharpoonup u^* \quad \text{in } H^1(\Omega; \mathbb{R}^d).$$

The boundary space $H_{\Gamma_0}^1(\Omega)$ is weakly closed; thus $u^* \in H_{\Gamma_0}^1(\Omega)$.

Similarly, let $\{\gamma_m\}$ be a minimizing sequence for (6.4). Then, after extracting a subsequence,

$$\gamma_m \rightharpoonup \gamma^* \quad \text{in } W^{1,\alpha}(\Omega).$$

Because the embedding $W^{1,\alpha}(\Omega) \hookrightarrow L^q(\Omega)$ is compact, $\gamma_m \rightarrow \gamma^*$ strongly in $L^q(\Omega)$.

Step 3: Lower semicontinuity.

- The inertial term $u \mapsto \int_\Omega \rho |u - 2u^k + u^{k-1}|^2 dx = \|\sqrt{\rho}(u - (2u^k - u^{k-1}))\|_{L^2(\Omega)}^2$ is the square of a norm in a weighted L^2 -space. Since $0 < \rho_- \leq \rho(x) \leq \rho^+ < \infty$

a.e. in Ω , the weighted norm

$$\|r\|_{L^2_\rho(\Omega)} := \left(\int_\Omega \rho(x) |r(x)|^2 dx \right)^{1/2}$$

is equivalent to the standard $L^2(\Omega)$ -norm. Consequently, the functional

$$u \mapsto \|\sqrt{\rho}(u - (2u^k - u^{k-1}))\|_{L^2(\Omega)}^2$$

is convex and lower semicontinuous with respect to the weak topology of $L^2(\Omega)$.

- The viscous part is $V\left(\frac{u-u^k}{\tau}\right) = \frac{1}{2} \int_\Omega \mathbb{D} e\left(\frac{u-u^k}{\tau}\right) : e\left(\frac{u-u^k}{\tau}\right) dx$. The integrand is a quadratic form in e , and because $\mathbb{D}$ is positive definite it is therefore convex and weakly lower semicontinuous with respect to $e(\cdot)$ in $L^2(\Omega)$. If $u_n \rightharpoonup u^*$ in $H^1(\Omega; \mathbb{R}^d)$, then $e\left(\frac{u_n-u^k}{\tau}\right) \rightharpoonup e\left(\frac{u^*-u^k}{\tau}\right)$ in $L^2(\Omega; \mathbb{R}^{d \times d})$.
- The plastic distance $D(\gamma, \gamma^k) = \int_\Omega \theta |\gamma - \gamma^k| dx$ is continuous with respect to strong L^q -convergence; hence $D(\gamma_n, \gamma^k) \rightarrow D(\gamma, \gamma^k)$ whenever $\gamma_n \rightarrow \gamma$ in $L^q(\Omega)$.
- For $E(t_{k+1}, u, \gamma)$,
 1. Similarly to the viscous part, the elastic part $\int_\Omega W(e(u) - \gamma S) dx$ is weakly l.s.c in $L^2(\Omega)$ with respect to $e(\cdot)$,
 2. the gradient-hardening term $\|\nabla \gamma\|_{L^\alpha}^\alpha$ is weakly l.s.c in $L^\alpha(\Omega)$ with respect to $\nabla \gamma$,
 3. the hardening potential $\int_\Omega w(\gamma) dx$ is continuous under the established strong L^q -convergence of $\gamma_m \rightarrow \gamma^*$, and
 4. the loading terms are linear and therefore weakly continuous in $L^2(\Omega)$ with respect to u .

u^* and γ^* , by lower semicontinuity, actually attain the lower limit of the minimizing sequence and therefore realize the minimum. Moreover, all the functionals appearing above, except for the dissipation distance D and the linear loading terms, are not only convex but strictly convex. Since D and the linear loading terms are convex and added to strictly convex functionals, both mappings

$$u \mapsto \mathcal{J}_\tau^k(u, \gamma^k) \quad \text{and} \quad \gamma \mapsto E(t_{k+1}, u^{k+1}, \gamma) + D(\gamma^k, \gamma)$$

are strictly convex on their respective domains. Therefore, each of the minimization problems (6.3) and (6.4) admits a unique minimizer.

Consequently, $u^{k+1} := u^*$ and $\gamma^{k+1} := \gamma^*$ are unique minimizers of (6.3) and (6.4), respectively.

Time interpolation and uniform τ -estimates

Let $\{(u^k, \gamma^k)\}_{k=0}^N$ be a sequence of discrete solutions obtained by successive minimisation (Proposition 7) with time grid $t_k = k\tau$ ($k = 0, \dots, N$) and set $t_{-1} = -\tau$.

Define the piecewise affine/constant interpolants on the interval $[-\tau, T]$ as follows: For $k = 0, \dots, N$ and $t \in [t_{k-1}, t_k]$, set

$$\begin{aligned} u_\tau(t) &:= u^{k-1} + \frac{t - t_{k-1}}{\tau} (u^k - u^{k-1}), \\ \bar{u}_\tau(t) &:= u^{k-1}, \\ \gamma_\tau(t) &:= \gamma^{k-1} + \frac{t - t_{k-1}}{\tau} (\gamma^k - \gamma^{k-1}), \\ \bar{\gamma}_\tau(t) &:= \gamma^{k-1}. \end{aligned}$$

At the endpoint $t = T$, we set

$$u_\tau(T) := u^N, \quad \bar{u}_\tau(T) := u^N, \quad \gamma_\tau(T) := \gamma^N, \quad \bar{\gamma}_\tau(T) := \gamma^N.$$

For the difference quotients, define for $k = 1, \dots, N$

$$v^k := \frac{u^k - u^{k-1}}{\tau}, \quad \beta^k := \frac{\gamma^k - \gamma^{k-1}}{\tau}.$$

In addition, we define the piecewise constant functions K_τ and E_τ on $[0, T]$ by for $k = 0, \dots, N-1$

$$\begin{aligned} K_\tau(t) &:= K^k := K(v^k) = \frac{1}{2} \int_{\Omega} \rho |v^k|^2 dx, & t \in [t_k, t_{k+1}), \\ E_\tau(t) &:= E^k := E(t_k, u^k, \gamma^k), & t \in [t_k, t_{k+1}). \end{aligned}$$

At the endpoint $t = T$, we set

$$K_\tau(T) := K^N := \frac{1}{2} \int_{\Omega} \rho |v^N|^2 dx, \quad E_\tau(T) := E^N := E(T, u^N, \gamma^N).$$

The initial values are $K_\tau(0) := K^0 = K(v^0)$, $E_\tau(0) := E^0 = E(0, u^0, \gamma^0)$, where the triplet (u^0, v^0, γ^0) is prescribed in Theorem 13. v^k and β^k are the velocity vectors on each time interval, and the following holds:

$$\begin{aligned} \dot{u}_\tau(t) &= v^k, & t \in [t_{k-1}, t_k), \\ \dot{\gamma}_\tau(t) &= \beta^k, & t \in [t_{k-1}, t_k), \end{aligned}$$

and at the endpoint $t = T$, $\dot{u}_\tau(T) = v^N$, $\dot{\gamma}_\tau(T) = \beta^N$.

Lemma 13 (Discrete energy inequality). *For every $t \in \{0, \tau, \dots, N\tau\}$,*

$$K_\tau(t) + E_\tau(t) + \int_0^t \left[2V(\dot{u}_\tau(s)) + R(\dot{\gamma}_\tau(s)) \right] ds \leq K(0) + E(0) + \int_0^t \partial_s L(s, \bar{u}_\tau(s)). \quad (6.14)$$

Proof. Step 1: Minimality. Following the approach in Mielke and Roubiřek, 2015 (see p. 468-469), the minimality of u^{k+1} for $\mathcal{J}_\tau^k$ gives

$$K^{k+1} + 2\tau V(v^{k+1}) + \tau R(\beta^{k+1}) + E^{k+1} \leq K^k + E(t_{k+1}, u^k, \gamma^k). \quad (6.15)$$

In more detail, from the Euler–Lagrange equation obtained from (6.3), testing with $(u^{k+1} - u^k)/\tau$ yields

$$\begin{aligned}
& \frac{1}{\tau^2} \int_{\Omega} \rho (u^{k+1} - 2u^k + u^{k-1}) \cdot \frac{u^{k+1} - u^k}{\tau} dx \\
& + \frac{1}{\tau} \int_{\Omega} \mathbb{D} e(u^{k+1} - u^k) : e\left(\frac{u^{k+1} - u^k}{\tau}\right) dx \\
& + \int_{\Omega} \mathbb{C}(x) (e(u^{k+1}) - \gamma^k S) : e\left(\frac{u^{k+1} - u^k}{\tau}\right) dx \\
& = \int_{\Omega} f(t_{k+1}) \cdot \frac{u^{k+1} - u^k}{\tau} dx + \int_{\Gamma_1} g(t_{k+1}) \cdot \frac{u^{k+1} - u^k}{\tau} dS.
\end{aligned} \tag{6.16}$$

Using the inequalities

$$\begin{aligned}
& \int_{\Omega} \rho (u^{k+1} - 2u^k + u^{k-1}) \cdot (u^{k+1} - u^k) dx \geq \\
& \quad \frac{1}{2} \int_{\Omega} \rho |u^{k+1} - u^k|^2 dx - \frac{1}{2} \int_{\Omega} \rho |u^k - u^{k-1}|^2 dx, \\
& \int_{\Omega} \mathbb{C}(x) (e(u^{k+1}) - \gamma^k S) : e(u^{k+1} - u^k) dx \geq \\
& \quad \frac{1}{2} \int_{\Omega} \mathbb{C}(x) (e(u^{k+1}) - \gamma^k S) : (e(u^{k+1}) - \gamma^k S) dx \\
& \quad - \frac{1}{2} \int_{\Omega} \mathbb{C}(x) (e(u^k) - \gamma^k S) : (e(u^k) - \gamma^k S) dx,
\end{aligned}$$

and multiplying both sides of (6.16) by τ , we obtain

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \rho \left| \frac{u^{k+1} - u^k}{\tau} \right|^2 dx - \frac{1}{2} \int_{\Omega} \rho \left| \frac{u^k - u^{k-1}}{\tau} \right|^2 dx \\
& + \tau \int_{\Omega} \mathbb{D} e\left(\frac{u^{k+1} - u^k}{\tau}\right) : e\left(\frac{u^{k+1} - u^k}{\tau}\right) dx \\
& + \frac{1}{2} \int_{\Omega} \mathbb{C}(x) (e(u^{k+1}) - \gamma^k S) : (e(u^{k+1}) - \gamma^k S) dx \\
& - \frac{1}{2} \int_{\Omega} \mathbb{C}(x) (e(u^k) - \gamma^k S) : (e(u^k) - \gamma^k S) dx \\
& \leq \int_{\Omega} f(t_{k+1}) \cdot (u^{k+1} - u^k) dx + \int_{\Gamma_1} g(t_{k+1}) \cdot (u^{k+1} - u^k) dS.
\end{aligned}$$

Rearranging terms gives the inequality

$$K^{k+1} + 2\tau V(v^{k+1}) + E(t_{k+1}, u^{k+1}, \gamma^k) \leq K^k + E(t_{k+1}, u^k, \gamma^k). \tag{6.17}$$

Furthermore, by testing (6.4) with γ^k , we obtain

$$E(t_{k+1}, u^{k+1}, \gamma^{k+1}) + D(\gamma^k, \gamma^{k+1}) \leq E(t_{k+1}, u^{k+1}, \gamma^k). \tag{6.18}$$

Combining (6.18) with (6.17), we finally arrive at (6.15).

Step 2: Isolating the explicit time derivative. Because $E(t, u, \gamma)$ depends on t only through L , we have

$$E(t_k, u^k, \gamma^k) = E(t_{k+1}, u^k, \gamma^k) - \int_{t_k}^{t_{k+1}} \partial_s L(s, u^k) ds. \quad (6.19)$$

Subtracting (6.19) from (6.15) and rearranging yields

$$[K^{k+1} + E^{k+1}] - [K^k + E^k] + 2\tau V(v^{k+1}) + \tau R(\beta^{k+1}) \leq \int_{t_k}^{t_{k+1}} \partial_s L(s, u^k) ds. \quad (6.20)$$

Step 3: Telescopic sum. For n with $n \in \{1, \dots, N\}$ sum (6.20) over $k = 0, \dots, n-1$:

$$K^n + E^n + 2 \sum_{k=0}^{n-1} \tau V(v^{k+1}) + \sum_{k=0}^{n-1} \tau R(\beta^{k+1}) \leq K^0 + E^0 + \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \partial_s L(s, u^k) ds. \quad (6.21)$$

Step 4: Identification with the interpolants. Since $\dot{u}_\tau$ and $\dot{\gamma}_\tau$ are interval-wise constants v^{k+1} and β^{k+1} , the Riemann sums equal the integrals for $t = n\tau$:

$$\sum_{k=0}^{n-1} \tau V(v^{k+1}) = \int_0^t V(\dot{u}_\tau) ds, \quad \sum_{k=0}^{n-1} \tau R(\beta^{k+1}) = \int_0^t R(\dot{\gamma}_\tau) ds.$$

Moreover $K^n = K_\tau(n\tau)$, $E^n = E_\tau(n\tau)$, and $\bar{u}_\tau(s) = u^k$ for $s \in [t_k, t_{k+1})$. Thus (6.21) becomes (6.14).

It is straightforward to extend this result to, for every $t_1, t_2 \in \{0, \tau, \dots, N\tau\}$, $t_1 \leq t_2$,

$$\begin{aligned} K_\tau(t_2) + E_\tau(t_2) + \int_{t_1}^{t_2} [2V(\dot{u}_\tau(s)) + R(\dot{\gamma}_\tau(s))] ds \\ \leq K_\tau(t_1) + E_\tau(t_1) + \int_{t_1}^{t_2} \partial_s L(s, \bar{u}_\tau(s)) ds. \end{aligned} \quad (6.22)$$

□

Proposition 8 (Uniform τ -bounds). *There exists a constant $C > 0$, independent of τ , such that*

$$\|\dot{u}_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad \|\nabla u_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad (6.23)$$

$$\|\gamma_\tau\|_{L^\infty([0,T];L^q)} \leq C, \quad \|\nabla \gamma_\tau\|_{L^\infty([0,T];L^q)} \leq C, \quad (6.24)$$

$$\|e(\dot{u}_\tau)\|_{L^2([0,T];L^2)} \leq C, \quad \int_0^T R(\dot{\gamma}_\tau) ds \leq C. \quad (6.25)$$

Proof. **Step 1: Evaluation of the discrete energy inequality at each time.** Lemma 13 applied with $t = n\tau$ ($1 \leq n \leq N$), yields

$$K^n + E^n + 2 \sum_{k=0}^{n-1} \tau V(v^{k+1}) + \sum_{k=0}^{n-1} \tau R(\beta^{k+1}) \leq K^0 + E^0 + \int_0^{n\tau} \partial_s L(s, \bar{u}_\tau(s)) ds. \quad (6.26)$$

Step 2: Forcing term estimate. For $t = n\tau$ with $0 \leq n \leq N$, since $\bar{u}_\tau(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$ vanishes on Γ_0 , the Poincaré inequality yields

$$\|\bar{u}_\tau(t)\|_{H^1(\Omega)} \leq C_P \|\nabla \bar{u}_\tau(t)\|_{L^2(\Omega)}.$$

The trace estimate yields

$$\|\bar{u}_\tau(t)\|_{L^2(\Gamma_1)} \leq C_{\text{tr}} \|\bar{u}_\tau(t)\|_{H^1(\Omega)}.$$

For the loading potential we compute

$$\partial_t L(t, \bar{u}_\tau(t)) = - \int_{\Omega} \dot{f}(t) \cdot \bar{u}_\tau(t) dx - \int_{\Gamma_1} \dot{g}(t) \cdot \bar{u}_\tau(t) dS,$$

so that by Cauchy–Schwarz, Poincaré, and the trace inequality,

$$|\partial_t L(t, \bar{u}_\tau(t))| \leq C(\|\dot{f}(t)\|_{L^2(\Omega)} + \|\dot{g}(t)\|_{L^2(\Gamma_1)}) \cdot \|\nabla \bar{u}_\tau(t)\|_{L^2(\Omega)}.$$

Setting $a(t) := \|\dot{f}(t)\|_{L^2(\Omega)} + \|\dot{g}(t)\|_{L^2(\Gamma_1)}$, $X := \|\nabla \bar{u}_\tau\|_{L^2([0,T];L^2(\Omega))}$, Hölder's inequality with $a \in L^2([0,T];L^2(\Omega))$ gives

$$\int_0^{n\tau} |\partial_t L(t, \bar{u}_\tau(t))| dt \leq AX, \quad A := C\|a\|_{L^2([0,T];L^2(\Omega))}.$$

Young's inequality implies for any $\delta > 0$

$$AX \leq \frac{\delta^2}{2} X^2 + \frac{1}{2} \delta^{-2} A^2,$$

hence

$$\int_0^{n\tau} |\partial_t L(t, \bar{u}_\tau(t))| dt \leq \frac{\delta^2}{2} X^2 + \frac{1}{2} \delta^{-2} A^2 =: \frac{\delta^2}{2} X^2 + C_L.$$

Step 3: Lower estimate of the internal energy. As (6.13) in the proof of Proposition 7, there exist constants $c, C_0 > 0$ such that, for any $(u, \gamma) \in \mathcal{Q}(t)$, the following estimate holds:

$$E(t, u, \gamma) \geq c(\|\nabla u\|_{L^2}^2 + \|\gamma\|_{L^q}^q + \|\nabla \gamma\|_{L^\alpha}^\alpha) - C_0.$$

Set $t = n\tau$ and substitute $(u, \gamma) = (u^n, \gamma^n)$ to obtain

$$E^n \geq c(\|\nabla u^n\|_{L^2}^2 + \|\gamma^n\|_{L^q}^q + \|\nabla \gamma^n\|_{L^\alpha}^\alpha) - C_0. \quad (6.27)$$

Similarly, $K^n \geq \frac{\rho_-}{2} \|v^n\|_{L^2}^2$.

Step 4: Comparison and absorption. Let $\|\nabla u^m\|_{L^2} = \max_n \{\|\nabla u^n\|_{L^2}\}$, then,

$$\|\nabla \bar{u}_\tau\|_{L^2(0,T;L^2)}^2 \leq T \|\nabla u^m\|_{L^2}^2.$$

Discarding the non-negative viscous and dissipation sums in (6.26), combining with (6.27), and gathering constants gives for any n ($1 \leq n \leq N$)

$$\begin{aligned} \frac{\rho_-}{2} \|v^n\|_{L^2}^2 + c(\|\nabla u^n\|_{L^2}^2 + \|\gamma^n\|_{L^q}^q + \|\nabla \gamma^n\|_{L^\alpha}^\alpha) \\ \leq K^0 + E^0 + C_0 + \frac{\delta^2}{2} \|\nabla \bar{u}_\tau\|_{L^2(0,T;L^2)}^2 + C_L \\ \leq K^0 + E^0 + C_0 + \frac{\delta^2 T}{2} \|\nabla u^m\|_{L^2}^2 + C_L. \end{aligned}$$

By taking $n = m$ on the left-hand side, we obtain

$$c\|\nabla u^m\|_{L^2}^2 \leq K^0 + E^0 + C_0 + \frac{\delta^2 T}{2} \|\nabla u^m\|_{L^2}^2 + C_L.$$

Choosing

$$\frac{\delta^2 T}{2} = \frac{c}{2} \quad (\text{i.e. } \delta = \sqrt{c/T}),$$

the gradient contribution $(\delta^2 T/2) \|\nabla u^m\|_{L^2}^2$ is absorbed into the left-hand side $c \|\nabla u^m\|_{L^2}^2$, leaving a positive remainder $(c/2) \|\nabla u^m\|_{L^2}^2$. Defining

$$C_{\text{in}} := \frac{2}{c} (K^0 + E^0 + C_0 + C_L),$$

we conclude

$$\|\nabla u^m\|_{L^2}^2 \leq C_{\text{in}}$$

where C_{in} is independent of τ . So, $\|\nabla u^n\|_{L^2} \leq \|\nabla u^m\|_{L^2} \leq C_{\text{in}}$, and

$$\frac{\rho_-}{2} \|v^n\|_{L^2}^2 + c (\|\gamma^n\|_{L^q}^q + \|\nabla \gamma^n\|_{L^\alpha}^\alpha) \leq C.$$

Here, C does not depend on τ , again. From these, (6.23) and (6.24) follow. Returning to (6.26), by uniformly estimating the right-hand side, we obtain (6.25). $\square$

Corollary 4. Let $\mathcal{V} = H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$. Then we obtain

$$\|\ddot{u}_\tau\|_{\mathcal{M}([0,T]; \mathcal{V}^*)} \leq C,$$

where C is independent of τ . Here, $\mathcal{M}([0,T]; X)$ denotes the Banach space of X -valued finite Radon measures on $[0,T]$, where X is a Banach space, endowed with the norm (defined by duality)

$$\|\mu\|_{\mathcal{M}([0,T]; X)} = \sup \left\{ \int_{[0,T]} \langle \phi(t), d\mu(t) \rangle \mid \phi \in C([0,T]; X^*), \sup_{t \in [0,T]} \|\phi(t)\|_{X^*} \leq 1 \right\}.$$

This corresponds to (5.1.33d) in Mielke and Roubíček, 2015. For details, see p. 471, (5.1.36).

Proof. In the sense of distributions, we have

$$\ddot{u}_\tau = \sum_{k=1}^{N-1} \frac{u^{k+1} - 2u^k + u^{k-1}}{\tau} \delta_{t_k} \quad \text{in } \mathcal{M}([0,T]; \mathcal{V}^*).$$

Therefore,

$$\|\rho \ddot{u}_\tau\|_{\mathcal{M}([0,T]; \mathcal{V}^*)} = \sum_{k=1}^{N-1} \left\| \rho \frac{u^{k+1} - 2u^k + u^{k-1}}{\tau} \right\|_{\mathcal{V}^*}.$$

For each $k = 0, \dots, N-1$, multiplying the Euler–Lagrange equation of (6.3) by τ and testing with arbitrary $\phi \in \mathcal{V} := H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$, we obtain

$$\begin{aligned} \left\langle \rho \frac{u^{k+1} - 2u^k + u^{k-1}}{\tau}, \phi \right\rangle &= - \int_{\Omega} \mathbb{D} e(u^{k+1} - u^k) : e(\phi) dx \\ &\quad - \tau \int_{\Omega} \mathbb{C}(e(u^{k+1}) - \gamma^k S) : e(\phi) dx \\ &\quad + \tau \left[\int_{\Omega} f(t_{k+1}) \cdot \phi dx + \int_{\Gamma_1} g(t_{k+1}) \cdot \phi dS \right]. \end{aligned} \quad (6.28)$$

By the boundedness of $\mathbb{C}$ and $\mathbb{D}$, evaluating (6.28) over $\|\phi\|_{\mathcal{V}} \leq 1$, we get for some $c > 0$

$$\left\| \rho \frac{u^{k+1} - 2u^k + u^{k-1}}{\tau} \right\|_{\mathcal{V}^*} \leq c \|e(u^{k+1} - u^k)\|_{L^2} + c\tau \left(\|e(u^{k+1})\|_{L^2} + \|\gamma^k\|_{L^2} + \|f(t_{k+1})\|_{L^2} + \|g(t_{k+1})\|_{L^2(\Gamma_1)} \right).$$

By Cauchy–Schwarz inequality,

$$\begin{aligned} \sum_{k=0}^{N-1} \|e(u^{k+1} - u^k)\|_{L^2} &\leq \sqrt{T} \left(\sum_{k=0}^{N-1} \tau \cdot \frac{\|e(u^{k+1} - u^k)\|_{L^2}^2}{\tau^2} \right)^{1/2} \\ &= \sqrt{T} \|e(\dot{u}_\tau)\|_{L^2([0,T];L^2)} \leq \sqrt{T}C. \end{aligned}$$

Moreover, the other terms are bounded by TC , and thus

$$\|\rho \ddot{u}_\tau\|_{\mathcal{M}([0,T];\mathcal{V}^*)} \leq C \quad (\text{independent of } \tau).$$

From $\rho_- \leq \rho$, the claim of Corollary follows. $\square$

Proof. (of Theorem 13) With the above preparations, we now prove the existence of (local) energetic solutions in accordance with Definition 6. Since the boundary Γ_0 has positive measure, by Korn’s inequality with (6.23) and (6.25), we have

$$\exists C_K > 0 : \quad \|\dot{u}_\tau(t)\|_{H^1(\Omega)} \leq C_K \left(\|e(\dot{u}_\tau(t))\|_{L^2(\Omega)} + \|\dot{u}_\tau(t)\|_{L^2(\Omega)} \right) \leq C,$$

while (6.5) with (6.23) yield

$$\|u_\tau\|_{H^1([0,T];H_{\Gamma_0}^1(\Omega))} \leq C, \quad (6.29)$$

so that $\{u_\tau\}$ is bounded in $H^1([0,T];H_{\Gamma_0}^1)$. By Banach–Alaoglu theorem, there exist u and a subsequence (not relabeled) such that

$$u_\tau \rightharpoonup u \quad H^1([0,T];H_{\Gamma_0}^1(\Omega)).$$

On the other hand, by (6.1) with (6.24),

$$\|\gamma_\tau\|_{L^\infty([0,T];W^{1,\alpha}(\Omega))} < C.$$

So, there exist γ and a subsequence such that

$$\gamma_\tau \xrightarrow{*} \gamma \quad \text{in } L^\infty([0,T];W^{1,\alpha}(\Omega)).$$

In the following, we may extract further subsequences without further mention. Let $\bar{f}_\tau(t)$ and $\bar{g}_\tau(t)$ denote the piecewise constant interpolants of f and g respect to the same time partition as u_τ , respectively. Because $f, g \in H^1([0,T];L^2(\Omega))$ we have

$$\bar{f}_\tau \rightarrow f, \quad \bar{g}_\tau \rightarrow g \quad \text{strongly in } L^2([0,T];L^2(\Omega)). \quad (6.30)$$

Step 1: Compactness

Let us show that Proposition 8 and Aubin–Lions compactness theory give

$$u_\tau \rightarrow u \quad \text{in } C([0, T]; L^2(\Omega)), \quad (6.31a)$$

$$u_\tau(t) \rightarrow u(t) \quad \text{for a.e. } t \in [0, T] \text{ in } H^1(\Omega), \quad (6.31b)$$

$$\dot{u}_\tau \rightharpoonup \dot{u} \quad \text{in } L^2([0, T]; H^1(\Omega)), \quad (6.31c)$$

$$\dot{u}_\tau \rightarrow \dot{u} \quad \text{in } L^2([0, T]; L^2(\Omega)), \quad (6.31d)$$

$$\gamma_\tau \rightarrow \gamma \quad \text{in } L^q([0, T]; L^q(\Omega)), \quad (6.31e)$$

$$\dot{\gamma}_\tau \xrightarrow{*} \dot{\gamma} \quad \text{in } \mathcal{M}((0, T) \times \Omega). \quad (6.31f)$$

Strong compactness of u_τ in $C([0, T]; L^2(\Omega))$ (6.31a) :

Since u_τ is bounded in $L^\infty([0, T]; H^1(\Omega))$ and $\dot{u}_\tau$ is bounded in $L^\infty([0, T]; L^2(\Omega)) \subset L^2([0, T]; H^{-1}(\Omega))$, we can apply the Simon's version of the Aubin–Lions theorem (see Theorem 14 below) with

$$H^1(\Omega; \mathbb{R}^d) \hookrightarrow L^2(\Omega; \mathbb{R}^d) \hookrightarrow H^{-1}(\Omega; \mathbb{R}^d) \quad (6.32)$$

to obtain

$$u_\tau \rightarrow u \quad \text{in } C([0, T]; L^2(\Omega; \mathbb{R}^d)).$$

Theorem 14. (Simon, 1986, Aubin–Lions/Simon Theorem, Thm. 5 and Cor. 4)

Let $X \hookrightarrow B \hookrightarrow Y$ be Banach spaces with compact embedding $X \hookrightarrow B$ and continuous embedding $B \hookrightarrow Y$.

1. Let $1 < p < \infty$. If a set $\mathcal{F} \subset L^p([0, T]; X)$ is bounded and

$$\lim_{h \rightarrow 0} \sup_{f \in \mathcal{F}} \|f(\cdot + h) - f(\cdot)\|_{L^p([0, T-h]; Y)} = 0,$$

then $\mathcal{F}$ is relatively compact in $L^p([0, T]; B)$.

2. More strongly, if $\mathcal{F} \subset L^\infty([0, T]; X)$ is bounded and $\{\partial_t f\}_{f \in \mathcal{F}}$ is bounded in $L^r([0, T]; Y)$ for some $r > 1$, then $\mathcal{F}$ is relatively compact in $C([0, T]; B)$.

Strong compactness of $\dot{u}_\tau$ in $L^2([0, T]; L^2(\Omega))$ (6.31d) :

In particular, from (6.29), $\dot{u}_\tau$ is bounded in $L^2([0, T]; H^1(\Omega))$. Since we already know that $\ddot{u}_\tau \in \mathcal{M}([0, T]; H^{-1}(\Omega))$ is bounded as a measure, we obtain that $\dot{u}_\tau$ is uniformly bounded in $BV([0, T]; H^{-1}(\Omega))$. This uniform (in τ) BV-in-time bound implies, by a standard BV computation, the time-translation estimate

$$\|\dot{u}_\tau(\cdot + h) - \dot{u}_\tau(\cdot)\|_{L^2([0, T-h]; H^{-1}(\Omega))} \leq C h^{1/2} \quad \text{for all } h \in (0, T),$$

hence Simon's time-translation condition in Theorem 14 (Case 1) holds; see Ambrosio, Gigli, and Savaré, 2008, Chap. 1. By applying Theorem 14 with (6.32), we obtain

$$\dot{u}_\tau \rightarrow \dot{u} \quad \text{in } L^2([0, T]; L^2(\Omega)).$$

Strong compactness of γ_τ in $L^q([0, T]; L^q(\Omega))$ (6.31e) :

From (6.24) and the condition $q \geq \alpha$, we deduce that

$$\|\gamma_\tau\|_{L^\infty([0, T]; W^{1, \alpha}(\Omega))} \leq C.$$

Furthermore, combining (6.25) with $\theta \geq \theta_0$, we obtain

$$\int_0^T \|\dot{\gamma}_\tau(s)\|_{L^1(\Omega)} ds \leq \theta_0^{-1} \int_0^T R(\dot{\gamma}_\tau) ds \leq C,$$

which shows that $\{\gamma_\tau\}$ is bounded in $BV([0, T]; L^1(\Omega))$. Applying the same compactness principle as above with the embedding

$$W^{1,\alpha}(\Omega; \mathbb{R}^d) \hookrightarrow L^q(\Omega; \mathbb{R}^d) \hookrightarrow L^1(\Omega; \mathbb{R}^d),$$

we conclude that

$$\gamma_\tau \rightarrow \gamma \quad \text{in } L^q([0, T]; L^q(\Omega)).$$

Weak-* convergence of $\dot{\gamma}_\tau$ in $\mathcal{M}((0, T) \times \Omega)$ (6.31f):

Moreover, $\dot{\gamma}_\tau(t, x)$ is uniformly bounded in $L^1((0, T) \times \Omega)$. Identifying a function $\varphi \in L^1((0, T) \times \Omega)$ with the measure $\varphi dt dx$, the sequence $\{\dot{\gamma}_\tau\}$ is bounded in total variation in

$$\mathcal{M}((0, T) \times \Omega) = (C_0((0, T) \times \Omega))^*.$$

By the Banach–Alaoglu theorem, there exists a subsequence (not relabeled) and $\mu \in \mathcal{M}((0, T) \times \Omega)$ such that

$$\dot{\gamma}_\tau \xrightarrow{*} \mu \quad \text{in } \mathcal{M}((0, T) \times \Omega).$$

For any $\varphi \in C_c^\infty((0, T) \times \Omega)$ we have

$$\iint \varphi d(\dot{\gamma}_\tau) = - \iint \gamma_\tau \dot{\varphi} dx dt \rightarrow - \iint \gamma \dot{\varphi} dx dt,$$

where the convergence on the right-hand side follows from $\gamma_\tau \rightarrow \gamma$ in $L^q([0, T]; L^q(\Omega))$ and $\dot{\varphi} \in L^\infty((0, T) \times \Omega)$. On the other hand, the left-hand side converges to $\iint \varphi d\mu$ due to $\dot{\gamma}_\tau \xrightarrow{*} \mu$ in $\mathcal{M}((0, T) \times \Omega)$. Hence,

$$\iint \varphi d\mu = - \iint \gamma \partial_t \varphi dx dt \quad \forall \varphi \in C_c^\infty((0, T) \times \Omega),$$

which means that $\mu = \dot{\gamma}$ in the sense of distributions.

Pointwise-in-time strong convergence of u_τ in $H^1(\Omega)$ (6.31b):

The gradient operator with respect to u ,

$$A_\gamma := \partial_u E(t, \cdot, \gamma): H_{\Gamma_0}^1(\Omega; \mathbb{R}^d) \rightarrow H_{\Gamma_0}^{-1}(\Omega; \mathbb{R}^d),$$

satisfies the so-called property of type (S^+) . As mentioned in Mielke and Roubíček, 2015 (p. 152), the condition (S^+) takes the following form in our setting:

$$\left. \begin{aligned} &u_\tau(t) \rightharpoonup u(t) \text{ in } H^1(\Omega), \quad \gamma_\tau(t) \rightarrow \gamma(t) \text{ in } L^q(\Omega), \\ &\limsup_{\tau \rightarrow 0} \langle \partial_u E(t, u_\tau, \gamma_\tau) - \partial_u E(t, u, \gamma), u_\tau - u \rangle \leq 0 \end{aligned} \right\} \Rightarrow u_\tau(t) \rightarrow u(t) \text{ in } H^1(\Omega).$$

As shown below, in our problem this implication can be obtained in an integral-in-time form of the second condition, which yields the strong convergence of u_τ in the Bochner space $L^2([0, T]; H^1(\Omega))$. From this, the required almost everywhere strong convergence follows. However, this argument requires the strong convergence of γ_τ in $L^2([0, T]; L^2(\Omega))$ (see (6.31e)).

Indeed, the operator is expressed in weak form for $p, q \in \mathcal{V}$ as

$$\langle A_\gamma(p), q \rangle = \int_{\Omega} \mathbb{C}(x)(e(p) - \gamma S) : e(q) dx - \int_{\Omega} f(t) \cdot q dx - \int_{\Gamma_1} g(t) \cdot q dS.$$

Hence, for all $p, q \in \mathcal{V}$,

$$\langle A_\gamma(p) - A_\gamma(q), p - q \rangle = \int_{\Omega} \mathbb{C}(x)e(p - q) : e(p - q) dx \geq \lambda_0 \|e(p - q)\|_{L^2}^2.$$

By Korn's inequality, $\|p - q\|_{H^1} \leq C_K \|e(p - q)\|_{L^2}$,

$$\langle A_\gamma(p) - A_\gamma(q), p - q \rangle \geq c \|p - q\|_{H^1}^2, \quad c := \lambda_0 / C_K^2 > 0.$$

This estimate does not depend on γ . Therefore, if

$$\limsup_{\tau \rightarrow 0} \int_0^T \langle A_\gamma(\bar{u}_\tau) - A_\gamma(u), \bar{u}_\tau - u \rangle dt \leq 0, \quad (6.33)$$

then it follows that

$$0 \leq \int_0^T c \|\bar{u}_\tau - u\|_{H^1(\Omega)}^2 dt \xrightarrow{\limsup} 0.$$

Since the left-hand side equals $c \|\bar{u}_\tau - u\|_{L^2([0, T]; H^1(\Omega))}^2$, we have $\bar{u}_\tau \rightarrow u$ strongly in $L^2([0, T]; H^1(\Omega))$. Hence, after extracting a subsequence (not relabeled),

$$\bar{u}_\tau(t) \rightarrow u(t) \quad \text{in } H^1(\Omega) \quad \text{for a.e. } t \in [0, T].$$

Since the convergence in (6.33) does not depend on γ , if we establish (6.33) for

$$A := \partial_u E(t, \cdot, \bar{\gamma}_\tau(t - \tau)), \quad (6.34)$$

then the desired strong pointwise convergence is obtained. This choice is made in order to make use of the (discrete) weak momentum balance.

Let $\phi \in H^1([0, T]; H_{\Gamma_0}^1(\Omega; \mathbb{R}^d))$, and let ϕ_τ denote the linear interpolation with respect to the same time partition as u_τ , while $\bar{\phi}_\tau$ denotes the piecewise constant interpolation. We simply denote the value of $\bar{\phi}_\tau$ on each interval $[t_k, t_{k+1})$ by $\phi^k \in H_{\Gamma_0}^1$. Testing (6.3) with ϕ^{k+1} yields

$$\begin{aligned} & \frac{1}{\tau^2} \int_{\Omega} \rho (u^{k+1} - 2u^k + u^{k-1}) \cdot \phi^{k+1} dx + \int_{\Omega} \mathbb{C}(e(u^{k+1}) - \gamma^k S) : e(\phi^{k+1}) dx \\ & + \frac{1}{\tau} \int_{\Omega} \mathbb{D} e(u^{k+1} - u^k) : e(\phi^{k+1}) dx \\ & = \int_{\Omega} f(t_{k+1}) \cdot \phi^{k+1} dx + \int_{\Gamma_1} g(t_{k+1}) \cdot \phi^{k+1} dS. \end{aligned} \quad (6.35)$$

Summing from $k = 0$ to $k = N - 1$ and making use of the identity

$$\begin{aligned} & \sum_{k=0}^{N-1} \left(\int_{\Omega} \rho \left(u^{k+1} - 2u^k + u^{k-1} \right) \cdot \phi^{k+1} dx \right) \\ &= - \sum_{k=1}^N \left(\int_{\Omega} \rho \left(u^{k-1} - u^{k-2} \right) \cdot \left(\phi^k - \phi^{k-1} \right) dx \right) \\ & \quad + \int_{\Omega} \rho \left(u^N - u^{N-1} \right) \cdot \phi^N dx - \int_{\Omega} \rho \left(u^0 - u^{-1} \right) \cdot \phi^0 dx, \end{aligned}$$

we then arrive at the discrete equation of motion

$$\begin{aligned} & \int_0^T \left[\int_{\Omega} \left(\mathbb{C}(e(\bar{u}_{\tau}(t)) - \bar{\gamma}_{\tau}(t - \tau)S) + \mathbb{D} e(\dot{u}_{\tau}(t)) \right) : e(\bar{\phi}_{\tau}(t)) dx \right. \\ & \quad \left. - \int_{\Omega} \rho \dot{u}_{\tau}(t - \tau) \cdot \dot{\phi}_{\tau}(t) dx \right] dt + \int_{\Omega} \rho \dot{u}_{\tau}(T) \cdot \phi_{\tau}(T) dx \\ &= \int_{\Omega} \rho v^0 \cdot \phi_{\tau}(0) dx + \int_0^T \left(\int_{\Omega} \bar{f}_{\tau}(t) \cdot \phi_{\tau}(t) dx + \int_{\Gamma_1} \bar{g}_{\tau}(t) \cdot \phi_{\tau}(t) dS \right) dt. \end{aligned} \quad (6.36)$$

Take v_{τ} and $\bar{v}_{\tau}$ a piecewise affine and piecewise constant approximation of u , such that

$$v_{\tau} \rightarrow u \quad \text{in } H^1([0, T]; H_{\Gamma_0}^1(\Omega)).$$

Concerning A in (6.34), from the boundedness of $\mathbb{C}$ and Proposition 8, we obtain the uniform boundedness of $A(\bar{u}_{\tau})$ as well as $A(\bar{v}_{\tau}) - A(u) \rightarrow 0$ in $L^2([0, T]; H_{\Gamma_0}^{-1}(\Omega))$. Since $\bar{v}_{\tau} \rightarrow u$ and $\bar{u}_{\tau} \rightharpoonup u$ in $L^2([0, T]; H_{\Gamma_0}^1(\Omega))$, it follows that

$$\begin{aligned} & \lim_{\tau \rightarrow 0} \int_0^T \langle A(\bar{u}_{\tau}) - A(u), \bar{u}_{\tau} - u \rangle - \langle A(\bar{u}_{\tau}) - A(\bar{v}_{\tau}), \bar{u}_{\tau} - \bar{v}_{\tau} \rangle dt \\ &= \lim_{\tau \rightarrow 0} \int_0^T \langle A(\bar{u}_{\tau}), \bar{v}_{\tau} - u \rangle + \langle A(\bar{v}_{\tau}) - A(u), \bar{u}_{\tau} \rangle + \langle A(u), u \rangle - \langle A(\bar{v}_{\tau}), \bar{v}_{\tau} \rangle dt \\ &= 0. \end{aligned}$$

Therefore, it suffices to show that

$$\limsup_{\tau \rightarrow 0} \int_0^T \langle A(\bar{u}_{\tau}) - A(\bar{v}_{\tau}), \bar{u}_{\tau} - \bar{v}_{\tau} \rangle dt \leq 0.$$

Here,

$$\begin{aligned} \langle A(\bar{u}_{\tau}), \bar{u}_{\tau} - \bar{v}_{\tau} \rangle &= \int_{\Omega} \mathbb{C}(x) \left(e(\bar{u}_{\tau}) - \bar{\gamma}_{\tau}(t - \tau)S \right) : e(\bar{u}_{\tau} - \bar{v}_{\tau}) dx \\ & \quad - \int_{\Omega} f(t) \cdot (\bar{u}_{\tau} - \bar{v}_{\tau}) dx - \int_{\Gamma_1} g(t) \cdot (\bar{u}_{\tau} - \bar{v}_{\tau}) dS. \end{aligned}$$

By employing (6.36) with the choice $\phi = v_\tau - u_\tau$ which means that $\phi_\tau = v_\tau - u_\tau$ and $\bar{\phi}_\tau = \bar{v}_\tau - \bar{u}_\tau$, we obtain

$$\begin{aligned}
& \limsup_{\tau \rightarrow 0} \int_0^T \left\langle A(\bar{u}_\tau) - A(\bar{v}_\tau), \bar{u}_\tau - \bar{v}_\tau \right\rangle dt \\
&= \limsup_{\tau \rightarrow 0} \int_0^T \left\langle A(\bar{v}_\tau) - A(\bar{u}_\tau), \bar{v}_\tau - \bar{u}_\tau \right\rangle dt \\
&= \limsup_{\tau \rightarrow 0} \int_0^T \left[\left\langle A(\bar{v}_\tau), \bar{v}_\tau - \bar{u}_\tau \right\rangle - \int_\Omega \rho \dot{u}_\tau(t - \tau) \cdot (\dot{v}_\tau - \dot{u}_\tau) dx \right. \\
&\quad \left. - \int_\Omega \mathbb{D} e(\dot{u}_\tau) : e(\bar{u}_\tau - \bar{v}_\tau) dx \right] dt \\
&\quad - \int_0^T \left(\int_\Omega \bar{f}_\tau \cdot (\bar{v}_\tau - \bar{u}_\tau) dx + \int_{\Gamma_1} \bar{g}_\tau \cdot (\bar{v}_\tau - \bar{u}_\tau) dS \right) dt \\
&\quad + \int_\Omega \rho \dot{u}_\tau(T) \cdot (v_\tau(T) - u_\tau(T)) dx - \int_\Omega \rho v^0 \cdot (v_\tau(0) - u_0) dx.
\end{aligned} \tag{6.37}$$

Regarding the first term on the right-hand side, due to the strong convergence of $\mathbb{C}(x) \left(e(\bar{v}_\tau) - \bar{\gamma}_\tau(t - \tau) S \right)$ in $L^2([0, T]; H^{-1}(\Omega))$ as implied by (6.31e), and the weak convergence of $\bar{v}_\tau - \bar{u}_\tau$ to zero in $L^2([0, T]; H^1(\Omega))$, it follows that this term converges to zero.

For the second term, we first estimate the difference between the shifted and unshifted inertial terms for any $\phi \in H^1([0, T]; H_{\Gamma_0}^1(\Omega))$ as follows:

$$\begin{aligned}
& \left| \int_0^T \langle \rho \dot{u}_\tau(t - \tau) - \rho \dot{u}_\tau(t), \dot{\phi}(t) \rangle dt \right| = \left| \int_0^T \langle \rho \dot{u}_\tau(s), \dot{\phi}(s + \tau) - \dot{\phi}(s) \rangle ds \right| \\
& \leq \|\rho \dot{u}_\tau\|_{L^\infty([0, T]; L^2)} \|\dot{\phi}(\cdot + \tau) - \dot{\phi}\|_{L^1([0, T]; L^2)}.
\end{aligned}$$

Here, $\dot{\phi}$ are extended by 0 outside $[0, T]$ so that the shift $\dot{\phi}(\cdot + \tau)$ is well-defined and the above estimate holds. It is a standard property that for any $p \in [1, \infty)$ and $v \in L^p([0, T]; X)$, where X is a Banach space, the translation operator is continuous in L^p , that is,

$$\|v(\cdot + h) - v\|_{L^p([0, T]; X)} \xrightarrow{h \rightarrow 0} 0.$$

Therefore, we obtain from (6.23)

$$\int_0^T \langle \rho \dot{u}_\tau(t - \tau), \dot{\phi}(t) \rangle dt = \int_0^T \langle \rho \dot{u}_\tau(t), \dot{\phi}(t) \rangle dt + o(1) \quad \text{as } \tau \rightarrow 0.$$

By setting $\phi = v_\tau - u_\tau$, and noting that $\dot{v}_\tau - \dot{u}_\tau \rightarrow 0$ in $L^2([0, T]; L^2(\Omega))$ by (6.31d), it follows that the second term converges to zero.

For the third term, by summing

$$\int_\Omega \mathbb{D} e(u^k - u^{k-1}) : e(u^k) dx \geq V(u^k) - V(u^{k-1})$$

for $k = 1, \dots, N$, we obtain

$$\begin{aligned} \sum_{k=1}^N \tau \int_{\Omega} \mathbb{D} e\left(\frac{u^k - u^{k-1}}{\tau}\right) : e(u^k) dx &= \int_0^T \int_{\Omega} \mathbb{D} e(\dot{u}_{\tau}) : e(\bar{u}_{\tau}) dx dt \\ &\geq V(u_{\tau}(T)) - V(u^0). \end{aligned}$$

Furthermore, by the weak convergence of $\mathbb{D} e(\dot{u}_{\tau})$ and the strong convergence of $e(\bar{v}_{\tau})$ in $L^2([0, T]; L^2(\Omega))$, we have

$$\begin{aligned} \limsup_{\tau \rightarrow 0} \left(- \int_0^T \int_{\Omega} \mathbb{D} e(\dot{u}_{\tau}) : e(\bar{u}_{\tau} - \bar{v}_{\tau}) dx dt \right) \\ \leq \limsup_{\tau \rightarrow 0} \left(V(u_0) - V(u_{\tau}(T)) + \int_0^T \int_{\Omega} \mathbb{D} e(\dot{u}_{\tau}) : e(\bar{v}_{\tau}) dx dt \right) \\ = V(u_0) - \liminf_{\tau \rightarrow 0} V(u_{\tau}(T)) + \int_0^T \int_{\Omega} \mathbb{D} e(\dot{u}) : e(u) dx dt \\ \leq V(u_0) - V(u(T)) + \int_0^T \int_{\Omega} \mathbb{D} e(\dot{u}) : e(u) dx dt. \end{aligned} \tag{6.38}$$

Here we used $u_{\tau}(T) \rightharpoonup u(T)$ in $H^1(\Omega)$ and the weak lower semicontinuity of V . The pointwise weak convergence of u_{τ} in $H^1(\Omega)$ can be shown as follows. Fix an arbitrary $t \in [0, T]$. By Proposition 8, the set $\{u_{\tau}(t)\}$ is bounded in $H^1(\Omega)$. Hence, by Banach–Alaoglu, every sequence (τ_n) admits a subsequence (not relabeled) and a function $v \in H^1(\Omega)$ with

$$u_{\tau_n}(t) \rightharpoonup v \quad \text{in } H^1(\Omega).$$

Because the embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is continuous, the same subsequence converges weakly in $L^2(\Omega)$ to v . On the other hand, by (6.31a), for the same (arbitrary) t we have the strong convergence

$$u_{\tau_n}(t) \rightarrow u(t) \quad \text{in } L^2(\Omega).$$

Therefore $v = u(t)$. Since every subsequence admits a further subsequence with weak limit $u(t)$, the whole net $\{u_{\tau}(t)\}_{\tau}$ converges weakly to $u(t)$ in $H^1(\Omega)$.

By applying the integration-by-parts formula (Lemma B.5.7 in Mielke and Roubíček, 2015), the rightmost term in (6.38) becomes zero, and hence the lim sup of the third term in (6.37) is estimated from above by zero.

For the external force term in (6.37), due to the weak convergence of $v_{\tau} - u_{\tau}$ to zero in $L^2([0, T]; L^2(\Omega))$ and its weak convergence to zero in the trace space, it follows that this term converges to zero. From (6.23), we have $\|\dot{u}_{\tau}(T)\|_{L^2} \leq C$, and since $v_{\tau}(T) - u_{\tau}(T) \rightarrow 0$ in $L^2(\Omega)$ from (6.31a), the convergence of the second but last term in (6.37) follows. The last term is zero since $v_{\tau}(0) - u_0 = 0$.

Step 2: Weak momentum balance

We consider the passage to the limit as $\tau \rightarrow 0$ in (6.36). Since all the remaining terms can be handled by standard limiting arguments, we focus on the limit of the inertial term $\int_{\Omega} \rho \dot{u}_{\tau}(T) \cdot \phi_{\tau}(T) dx$. The uniform estimate (6.23) ensures that, possibly after taking a subsequence, we have the weak convergence $\dot{u}_{\tau}(T) \rightharpoonup \dot{u}(T)$ in $L^2(\Omega)$. Here we note that the estimate given by (6.23) is, in fact, a supremum rather than an essential supremum. Since we have the inner product of a weakly convergent sequence and a constant function in $L^2(\Omega)$, the desired convergence follows.

Step 3: Local Semistability

By testing (6.4) with a general $\tilde{\gamma} \in W^{1,\alpha}(\Omega)$ and applying the triangle inequality, we obtain

$$E(t_{k+1}, u^{k+1}, \gamma^{k+1}) \leq E(t_{k+1}, u^{k+1}, \tilde{\gamma}) + D(\gamma^{k+1}, \tilde{\gamma}).$$

Similarly to (6.19), for $t \in [t_{k+1}, t_{k+2})$ we have

$$\begin{aligned} E(t, u^{k+1}, \gamma^{k+1}) &= E(t_{k+1}, u^{k+1}, \gamma^{k+1}) + \int_{t_{k+1}}^t \partial_s L(s, u^{k+1}) ds, \\ E(t, u^{k+1}, \tilde{\gamma}) &= E(t_{k+1}, u^{k+1}, \tilde{\gamma}) + \int_{t_{k+1}}^t \partial_s L(s, u^{k+1}) ds. \end{aligned}$$

Hence, it follows immediately that

$$E(t, \bar{u}_\tau, \bar{\gamma}_\tau) \leq E(t, \bar{u}_\tau, \tilde{\gamma}) + D(\bar{\gamma}_\tau, \tilde{\gamma}). \quad (6.39)$$

The pointwise convergence of the left-hand side of (6.39) is shown in the same way as in Step 4 (p. 68) of the proof of Theorem 3.2 in Francfort and Mielke, 2006b as follows. From this discrete semistability together with (6.31a), (6.31b) and (6.31e), we have, for a.e. $t \in [0, T]$,

$$\begin{aligned} E(t, u(t), \gamma(t)) &= \lim_{\tau \rightarrow 0} E(t, u(t), \gamma(t)) + D(\bar{\gamma}_\tau(t), \gamma(t)) \\ &= \lim_{\tau \rightarrow 0} E(t, \bar{u}_\tau(t), \gamma(t)) + D(\bar{\gamma}_\tau(t), \gamma(t)) \\ &\geq \limsup_{\tau \rightarrow 0} E(t, \bar{u}_\tau(t), \bar{\gamma}_\tau(t)). \end{aligned}$$

Since the weak lower semicontinuity has already been discussed in Proposition 7, it follows from Proposition 8 that, possibly after taking a subsequence, the opposite liminf inequality holds. Therefore, the pointwise convergence is obtained.

As for the pointwise convergence of the right-hand side of (6.39) for almost every $t \in [0, T]$, the convergence of the first term follows from (6.31a) and (6.31b), while the convergence of the second term follows from (6.31e).

Step 4: Energy Imbalance

From (6.22), for any $t_1, t_2 \in [0, T]$ with $t_1 \leq t_2$, we can gain

$$\begin{aligned} &K(\dot{u}_\tau(\lfloor \tau^{-1} t_2 \rfloor \tau - \tau)) + E(\lfloor \tau^{-1} t_2 \rfloor \tau, \bar{u}_\tau, \bar{\gamma}_\tau) + \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} [2V(\dot{u}_\tau(s)) + R(\dot{\gamma}_\tau(s))] ds \\ &\leq K(\dot{u}_\tau(\lfloor \tau^{-1} t_1 \rfloor \tau - \tau)) + E(\lfloor \tau^{-1} t_1 \rfloor \tau, \bar{u}_\tau, \bar{\gamma}_\tau) + \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} \partial_s L(s, \bar{u}_\tau(s)) ds. \end{aligned}$$

By the similar reasoning as in Step 3 and as in the proof of Proposition 8 (Step 2), it follows immediately that

$$\begin{aligned} &K(\dot{u}_\tau(t_2 - \tau)) + E(t_2, \bar{u}_\tau, \bar{\gamma}_\tau) + \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} [2V(\dot{u}_\tau(s)) + R(\dot{\gamma}_\tau(s))] ds \\ &\leq K(\dot{u}_\tau(t_1 - \tau)) + E(t_1, \bar{u}_\tau, \bar{\gamma}_\tau) + \int_{t_1}^{t_2} \partial_s L(s, \bar{u}_\tau(s)) ds + \tau C \quad (6.40) \end{aligned}$$

If we restrict the interval to $[\tau, T]$,

$$\begin{aligned} \|\dot{u}_\tau(\cdot - \tau) - \dot{u}(\cdot)\|_{L^2([\tau, T]; L^2)} \\ \leq \|\dot{u}_\tau(\cdot - \tau) - \dot{u}(\cdot - \tau)\|_{L^2([\tau, T]; L^2)} + \|\dot{u}(\cdot - \tau) - \dot{u}(\cdot)\|_{L^2([\tau, T]; L^2)} \end{aligned}$$

Since both terms on the right-hand side converge to zero considering the continuity of time translation, the left-hand side also converges to zero. Moreover,

$$\|\dot{u}_\tau(\cdot - \tau) - \dot{u}(\cdot)\|_{L^2([0, \tau]; L^2)} \rightarrow 0,$$

so the convergence extends to the entire interval $[0, T]$. Then (6.31d) implies the convergence of the kinetic energy terms for a.e. $t_1, t_2 \in [0, T]$.

Fix $\varepsilon_0 > 0$, and define the positive numbers $\varepsilon_1(\tau)$ and $\varepsilon_2(\tau)$ by

$$\varepsilon_1(\tau) := t_1 - \lfloor \tau^{-1} t_1 \rfloor \tau, \quad \varepsilon_2(\tau) := t_2 - \lfloor \tau^{-1} t_2 \rfloor \tau.$$

As in the proof of Proposition 7, the viscosity term is weakly lower semicontinuous with respect to $e(\cdot)$ in $L^2([t_1, t_2 - \varepsilon_0]; L^2(\Omega))$. Since we are taking the limit as $\tau \rightarrow 0$, it suffices to consider sufficiently small τ such that $\varepsilon_2(\tau) < \varepsilon_0$; therefore, by (6.31c),

$$\int_{t_1}^{t_2 - \varepsilon_0} V(\dot{u}) \, ds \leq \liminf_{\tau \rightarrow 0} \int_{t_1}^{t_2 - \varepsilon_0} V(\dot{u}_\tau) \, ds \leq \liminf_{\tau \rightarrow 0} \int_{t_1 - \varepsilon_1(\tau)}^{t_2 - \varepsilon_2(\tau)} V(\dot{u}_\tau) \, ds$$

From this, we obtain

$$\int_{t_1}^{t_2} V(\dot{u}) \, ds \leq \liminf_{\tau \rightarrow 0} \int_{t_1 - \varepsilon_1(\tau)}^{t_2 - \varepsilon_2(\tau)} V(\dot{u}_\tau) \, ds + O(\varepsilon_0).$$

By letting $\varepsilon_0 \rightarrow 0$ in this expression, we obtain

$$\int_{t_1}^{t_2} V(\dot{u}) \, ds \leq \liminf_{\tau \rightarrow 0} \int_{t_1 - \varepsilon_1(\tau)}^{t_2 - \varepsilon_2(\tau)} V(\dot{u}_\tau) \, ds = \liminf_{\tau \rightarrow 0} \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} V(\dot{u}_\tau) \, ds.$$

In addition to an argument analogous to the one above, by applying Reshetnyak's theorem (see below) to (6.31f) with $U = (t_1, t_2) \times \Omega$ and $H((s, x), \xi) = \theta(x)|\xi|$, and using that R is convex one-homogeneous and θ is continuous, we obtain

$$\int_{t_1}^{t_2} R(\dot{\gamma}) \, ds \leq \liminf_{\tau \rightarrow 0} \int_{\lfloor \tau^{-1} t_1 \rfloor \tau}^{\lfloor \tau^{-1} t_2 \rfloor \tau} R(\dot{\gamma}_\tau) \, ds.$$

Theorem 15. (Ambrosio, Fusco, and Pallara, 2000, Reshetnyak's lower semicontinuity theorem, Thm. 2.38)

Let $U \subset \mathbb{R}^N$ be an open set, and let $\mu_k, \mu \in \mathcal{M}(U; \mathbb{R}^m)$ be finite Radon measures such that $\mu_k \xrightarrow{*} \mu$ weakly* in $\mathcal{M}(U; \mathbb{R}^m)$ (that is, in the dual of $C(U; \mathbb{R}^m)$). Suppose that $H : U \times \mathbb{R}^m \rightarrow [0, \infty]$ is lower semicontinuous in (x, ξ) , convex and positively 1-homogeneous in the second variable. Then

$$\int_U H\left(x, \frac{d\mu}{d|\mu|}(x)\right) d|\mu|(x) \leq \liminf_{k \rightarrow \infty} \int_U H\left(x, \frac{d\mu_k}{d|\mu_k|}(x)\right) d|\mu_k|(x).$$

By (6.30) and (6.31a) we have $\partial_s L(\cdot, \bar{u}_\tau) \rightarrow \partial_s L(\cdot, u)$ in $L^1(0, T)$, hence

$$\int_0^t \partial_s L(\bar{u}_\tau) \, ds \rightarrow \int_0^t \partial_s L(u) \, ds.$$

What remains to show in order to finish the limit process as $\tau \rightarrow 0$ in (6.40), is the pointwise convergence of E for a.e. $t_1, t_2 \in [0, T]$, which has already been proved in Step 3 above.

Consequently, the limit (u, γ) is a local solution with inertia in the sense of Definition 6. $\square$

6.3 Uniqueness of Energetic Solutions in the Quadratic Case

In the previous section, we proved the existence of an inertial local solution (a viscous energetic solution) in the sense of Definition 6. However, the energetic formulation of RIS, based on (local or global) stability and an energy inequality, provides a variational solution concept that may in general lead to multi-valued evolutions. Indeed, Mainik and Mielke, 2005 points out that, without additional regularity or convexity assumptions on the stored energy and the dissipation distance—in particular, without uniform convexity—uniqueness of solutions cannot in general be expected. Furthermore, Mielke, 2005 emphasizes that RIS frequently appear as multi-valued evolutions unless strong structural assumptions are imposed.

On the other hand, if the incremental minimization potential

$$(u, \gamma) \mapsto E(t_k, u, \gamma) + D(\gamma, \gamma_{k-1})$$

is strictly (or strongly) convex on the state space, then the minimizer at each time step is unique, and uniqueness of the corresponding continuous-time energetic solution follows from the limit passage of the time discretization. In fact, Mielke and Theil, 2004 establishes well-posedness (existence and uniqueness) under suitable uniform convexity assumptions, and also demonstrates that the loss of strict convexity may lead to non-uniqueness.

The present dynamic model contains both an inertia term $K(\dot{u})$ and a viscous dissipation $V(\dot{u})$, but these ingredients do not automatically eliminate the intrinsic multi-valued nature of RIS (cf. Mielke, 2005). Nevertheless, in the context of dynamic plasticity, the introduction of hardening and viscosity is known to enhance monotonicity and coercivity, thereby contributing to regularity and uniqueness of solutions (see, e.g., Showalter and Shi, 1998).

With the aim of keeping the analysis technically transparent and supporting the reasonableness of our solution concept, we restrict in this section to the case of a quadratic hardening potential,

$$w(\gamma) = \frac{\beta}{2} \gamma^2, \quad \beta > 0,$$

and a quadratic gradient regularization,

$$\kappa |\nabla \gamma|^2, \quad \alpha = 2.$$

Theorem 16. *Under the above assumptions, the energetic solution in Theorem 13 is unique.*

Proof. Let

$$\mathcal{V} := H_{\Gamma_0}^1(\Omega; \mathbb{R}^d), \quad \mathcal{W} := W^{1,2}(\Omega).$$

By the assumptions, the stored energy $\mathcal{E}$,

$$\mathcal{E}(u, \gamma) := \frac{1}{2} \int_{\Omega} \mathbb{C}(e(u) - \gamma S) : (e(u) - \gamma S) dx + \frac{\beta}{2} \int_{\Omega} \gamma^2 dx + \kappa \int_{\Omega} |\nabla \gamma|^2 dx,$$

is a quadratic form. In particular, $\partial\mathcal{E} := (\partial_u\mathcal{E}, \partial_\gamma\mathcal{E})$ is a bounded linear symmetric operator $\partial\mathcal{E} \in \text{Lin}(\mathcal{V} \times \mathcal{W}; \mathcal{V}^* \times \mathcal{W}^*)$. Each component is expressed as follows: For any test functions $\varphi \in \mathcal{V}$ and $\psi \in \mathcal{W}$,

$$\begin{aligned}\langle \partial_u\mathcal{E}(t, u, \gamma), \varphi \rangle &= \int_{\Omega} \mathbb{C}(e(u) - \gamma S) : e(\varphi) dx, \\ \langle \partial_\gamma\mathcal{E}(t, u, \gamma), \psi \rangle &= \int_{\Omega} \left(-\mathbb{C}(e(u) - \gamma S) : (S\psi) + \beta\gamma\psi + 2\kappa \nabla\gamma \cdot \nabla\psi \right) dx.\end{aligned}$$

Assume the gentle-start compatibility at $t = 0$:

$$\begin{aligned}\int_{\Omega} \mathbb{D}e(v^0) : e(\psi) dx + \int_{\Omega} \mathbb{C}(e(u^0) - \gamma^0 S) : e(\psi) dx &= \int_{\Omega} f(0) \cdot \psi dx + \int_{\Gamma_1} g(0) \cdot \psi dS \\ &\quad \forall \psi \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d).\end{aligned}\tag{6.41}$$

We use the standard backward differences for $w \in \mathbb{R}^n$:

$$\begin{aligned}D_\tau^k w &:= \frac{w^k - w^{k-1}}{\tau}, \quad [D_\tau^k]^2 w := \frac{w^k - 2w^{k-1} + w^{k-2}}{\tau^2}, \\ [D_\tau^{k+1}]^3 w &:= \frac{[D_\tau^{k+1}]^2 w - [D_\tau^k]^2 w}{\tau}.\end{aligned}$$

And also we use the formal displacement and slip:

$$u_\tau^{-2} = u_0 - 2\tau v_0 \quad \text{and} \quad z_\tau^{-1} = z_0.$$

This setting justifies the evaluation of the initial values required in the above high-order differentials and in the following discussion using the Gronwall inequality.

From the discrete minimization for γ (6.4) and the convexity of $\zeta \mapsto \tau R(\frac{\zeta - \gamma^{k-1}}{\tau}) + \mathcal{E}(u^k, \zeta)$ we obtain the first-order condition

$$0 \in \partial R(D_\tau^k \gamma_\tau) + \partial_\gamma \mathcal{E}(u^k, \gamma^k).$$

Equivalently, by the subgradient inequality, for all $\tilde{\zeta} \in \mathcal{W}$,

$$R(\tilde{\zeta}) + \langle \partial_\gamma \mathcal{E}(u^k, \gamma^k), \tilde{\zeta} - D_\tau^k \gamma_\tau \rangle \geq R(D_\tau^k \gamma_\tau).\tag{6.42}$$

Choosing in (6.42) $\tilde{\zeta} = D_\tau^{k+1} \gamma_\tau$ at level k and $\tilde{\zeta} = D_\tau^k \gamma_\tau$ at level $k+1$, adding the two inequalities, and using the linearity of $\partial_\gamma \mathcal{E}$ yield

$$\langle \partial_\gamma \mathcal{E}(D_\tau^{k+1} u_\tau, D_\tau^{k+1} \gamma_\tau), [D_\tau^{k+1}]^2 \gamma_\tau \rangle \leq 0.\tag{6.43}$$

Considering the discrete momentum balance for u_τ (6.35) for $\phi \in \mathcal{V}$, at levels $k+1$

and k , yields

$$\begin{aligned} & \int_{\Omega} \rho \left([D_{\tau}^{k+1}]^2 u_{\tau} - [D_{\tau}^k]^2 u_{\tau} \right) \cdot \phi \, dx + \int_{\Omega} \mathbb{C} \left(e(u^{k+1} - u^k) - (\gamma^k - \gamma^{k-1})S \right) : e(\phi) \, dx \\ & + \int_{\Omega} \mathbb{D} e \left([D_{\tau}^{k+1}] u_{\tau} - [D_{\tau}^k] u_{\tau} \right) : e(\phi) \, dx \\ & = \int_{\Omega} (f(t_{k+1}) - f(t_k)) \cdot \phi \, dx + \int_{\Gamma_1} (g(t_{k+1}) - g(t_k)) \cdot \phi \, dS. \end{aligned}$$

If we set $\phi = [D_{\tau}^{k+1}]^2 u$ and divide by τ ,

$$\begin{aligned} & \int_{\Omega} \rho \left([D_{\tau}^{k+1}]^3 u_{\tau} \right) \cdot \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) \, dx + \int_{\Omega} \partial_u \mathcal{E} \left(D_{\tau}^{k+1} u_{\tau}, D_{\tau}^k \gamma_{\tau} \right) : e([D_{\tau}^{k+1}]^2 u_{\tau}) \, dx \\ & + \int_{\Omega} \mathbb{D} e \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) : e \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) \, dx \\ & = \int_{\Omega} \left(D_{\tau}^{k+1} f \right) \cdot \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) \, dx + \int_{\Gamma_1} \left(D_{\tau}^{k+1} g \right) \cdot \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) \, dS. \end{aligned} \quad (6.44)$$

Let $U^{\ell} := D_{\tau}^{\ell} u_{\tau}$ and $Z^{\ell} := D_{\tau}^{\ell} \gamma_{\tau}$ for $\ell \in \{k, k+1\}$, and note that for $\ell \in \{k, k+1\}$ we have

$$\begin{aligned} [D_{\tau}^{k+1}]^2 u_{\tau} &= \frac{D_{\tau}^{k+1} u_{\tau} - D_{\tau}^k u_{\tau}}{\tau} = \frac{U^{k+1} - U^k}{\tau}, \\ [D_{\tau}^{k+1}]^2 \gamma_{\tau} &= \frac{D_{\tau}^{k+1} \gamma_{\tau} - D_{\tau}^k \gamma_{\tau}}{\tau} = \frac{Z^{k+1} - Z^k}{\tau}. \end{aligned}$$

Since $\mathcal{E}$ is a convex functional with respect to u and γ , the following holds:

$$\langle \partial_u \mathcal{E}(U^{k+1}, Z^k), [D_{\tau}^{k+1}]^2 u_{\tau} \rangle \geq \frac{1}{\tau} \left(\mathcal{E}(U^{k+1}, Z^k) - \mathcal{E}(U^k, Z^k) \right), \quad (6.45)$$

$$\langle \partial_{\gamma} \mathcal{E}(U^{k+1}, Z^{k+1}), [D_{\tau}^{k+1}]^2 \gamma_{\tau} \rangle \geq \frac{1}{\tau} \left(\mathcal{E}(U^{k+1}, Z^{k+1}) - \mathcal{E}(U^{k+1}, Z^k) \right). \quad (6.46)$$

Adding (6.45) and (6.46), we obtain

$$\langle \partial_u \mathcal{E}(U^{k+1}, Z^k), [D_{\tau}^{k+1}]^2 u_{\tau} \rangle + \langle \partial_{\gamma} \mathcal{E}(U^{k+1}, Z^{k+1}), [D_{\tau}^{k+1}]^2 \gamma_{\tau} \rangle \geq D_{\tau}^{k+1} \mathcal{E}(U, Z), \quad (6.47)$$

where $D_{\tau}^{k+1} \mathcal{E}(U, Z) = \tau^{-1} (\mathcal{E}(U^{k+1}, Z^{k+1}) - \mathcal{E}(U^k, Z^k))$.

If we write $a^k := [D_{\tau}^k]^2 u_{\tau} \in \mathcal{V}$, then we have

$$[D_{\tau}^{k+1}]^3 u_{\tau} = \frac{a^{k+1} - a^k}{\tau},$$

and moreover,

$$\int_{\Omega} \rho \left([D_{\tau}^{k+1}]^3 u_{\tau} \right) \cdot \left([D_{\tau}^{k+1}]^2 u_{\tau} \right) \, dx = \frac{1}{\tau} \int_{\Omega} (a^{k+1} - a^k) \cdot a^{k+1} \, dx.$$

In addition, using the fact that for any $x, y \in L^2(\Omega; \mathbb{R}^d)$ we have

$$\int_{\Omega} \rho(x - y) \cdot x \, dx = K(x) - K(y) + K(x - y),$$

it follows that

$$\begin{aligned} \frac{1}{\tau} \int_{\Omega} (a^{k+1} - a^k) \cdot a^{k+1} dx &= \frac{1}{\tau} \left(K(a^{k+1}) - K(a^k) + K(a^{k+1} - a^k) \right) \\ &= \frac{1}{\tau} \left(\tau D_{\tau}^{k+1} K(a) + K(a^{k+1} - a^k) \right) \\ &\geq D_{\tau}^{k+1} K([D_{\tau}^{k+1}]^2 u_{\tau}). \end{aligned}$$

Summing (6.43) and (6.44), and using (6.47)

$$D_{\tau}^{k+1} \left(K([D_{\tau}^{k+1}]^2 u_{\tau}) + \mathcal{E}(D_{\tau}^{k+1} u_{\tau}, D_{\tau}^{k+1} \gamma_{\tau}) \right) + V([D_{\tau}^{k+1}]^2 u_{\tau}) \leq \langle D_{\tau}^{k+1} L, [D_{\tau}^{k+1}]^2 u_{\tau} \rangle. \quad (6.48)$$

Let

$$\Phi^k := K([D_{\tau}^k]^2 u) + \mathcal{E}(D_{\tau}^k u, D_{\tau}^k \gamma), \quad W^k := V([D_{\tau}^k]^2 u).$$

Since $K, \mathcal{E}, V$ are positive definite,

$$\Phi^k \geq 0, \quad W^k \geq 0. \quad (6.49)$$

Using this notation, (6.48) becomes

$$D_{\tau}^{k+1} \Phi^k + W^{k+1} \leq \langle D_{\tau}^{k+1} L, [D_{\tau}^{k+1}]^2 u \rangle. \quad (6.50)$$

Using the Cauchy–Schwarz inequality, the Poincaré inequality, and the trace theorem, we obtain

$$\begin{aligned} \langle D_{\tau}^{k+1} L, [D_{\tau}^{k+1}]^2 u_{\tau} \rangle &\leq \|D_{\tau}^{k+1} f\|_{L^2(\Omega)} \| [D_{\tau}^{k+1}]^2 u_{\tau} \|_{L^2(\Omega)} + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)} \| [D_{\tau}^{k+1}]^2 u_{\tau} \|_{L^2(\Gamma_1)} \\ &\leq \frac{1}{2\varepsilon} \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right) + \frac{\varepsilon(C_P + \tilde{C}_{\text{tr}})}{2} \left\| \nabla([D_{\tau}^{k+1}]^2 u_{\tau}) \right\|_{L^2(\Omega)}^2 \\ &\leq \frac{1}{2\varepsilon} \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right) + \varepsilon C_* W^{k+1}. \end{aligned}$$

Choosing ε such that $0 < \varepsilon \leq \frac{1}{2C_*}$, we obtain

$$D_{\tau}^{k+1} \Phi^k + \frac{1}{2} W^{k+1} \leq C \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right).$$

Using (6.49), we conclude that

$$D_{\tau}^{k+1} \Phi^k \leq C \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right).$$

Summing this for $k = 0, \dots, m-1$ and multiplying both sides by τ gives

$$\Phi^m - \Phi^0 \leq C \sum_{k=0}^{m-1} \tau \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right).$$

By (6.41), we have $\Phi^0 = 0$, hence

$$\Phi^m \leq C \sum_{k=0}^{m-1} \tau \left(\|D_{\tau}^{k+1} f\|_{L^2(\Omega)}^2 + \|D_{\tau}^{k+1} g\|_{L^2(\Gamma_1)}^2 \right).$$

The right-hand side is uniformly bounded with respect to τ by the assumptions on f and g . Similarly,

$$W^k \leq C.$$

Then, we obtain the following uniform estimate, as in Proposition 8.

Proposition 9 (Enhanced uniform τ -bounds). *There exists a constant $C > 0$, independent of τ , such that*

$$\|\ddot{u}_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad \|\nabla \dot{u}_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad (6.51)$$

$$\|\dot{\gamma}_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad \|\nabla \dot{\gamma}_\tau\|_{L^\infty([0,T];L^2)} \leq C, \quad (6.52)$$

$$\|e(\ddot{u}_\tau)\|_{L^2([0,T];L^2)} \leq C. \quad (6.53)$$

From this, in particular, we may conclude that the solution u in the Theorem 13 satisfies $\ddot{u} \in L^\infty([0, T]; L^2)$.

The weak form of the momentum holds in the strong sense with respect to time; that is, the following holds.

Lemma 14. *Let*

$$u \in H^1([0, T]; H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)), \quad \gamma \in L^2([0, T]; W^{1,2}(\Omega)),$$

satisfy the weak-in-time momentum balance for all $\phi \in H^1([0, T]; H_{\Gamma_0}^1(\Omega; \mathbb{R}^d))$.

Then, for almost every $t \in (0, T)$,

$$\begin{aligned} \int_{\Omega} \rho \ddot{u}(t) \cdot \psi \, dx + \int_{\Omega} \mathbb{D} e(\dot{u}(t)) : e(\psi) \, dx + \int_{\Omega} \mathbb{C}(e(u(t)) - \gamma(t)S) : e(\psi) \, dx \\ = \int_{\Omega} f(t) \cdot \psi \, dx + \int_{\Gamma_1} g(t) \cdot \psi \, dS, \end{aligned} \quad (6.54)$$

for all $\psi \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$.

Equivalently,

$$\rho \ddot{u}(t) - \operatorname{div} \sigma = f(t) \quad \text{in } \mathcal{V}^*,$$

with natural boundary condition

$$\sigma n = g(t) \quad \text{in } H^{-\frac{1}{2}}(\Gamma_1; \mathbb{R}^d), \quad u(t) = 0 \text{ on } \Gamma_0.$$

Proof. If we define $F(t)$ for $t \in [0, T]$ by

$$\begin{aligned} F(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)^*, \\ \langle F(t), \psi \rangle := \int_{\Omega} \mathbb{D} e(\dot{u}(t)) : e(\psi) \, dx + \int_{\Omega} \mathbb{C}(e(u(t)) - \gamma(t)S) : e(\psi) \, dx \\ - \int_{\Omega} f(t) \cdot \psi \, dx - \int_{\Gamma_1} g(t) \cdot \psi \, dS, \end{aligned} \quad (6.55)$$

then $F(t)$ belongs to $L^1([0, T]; \mathcal{V}^*)$.

Insert test functions of the form $\phi(t, x) = \eta(t)\psi(x)$ with $\eta \in C_c^1([0, T])$ and $\psi \in H_{\Gamma_0}^1$ into the weak balance. The definition yields

$$- \int_0^T \left(\int_{\Omega} \rho \dot{u}(t) \cdot \psi \, dx \right) \dot{\eta}(t) \, dt + \int_0^T \langle F(t), \psi \rangle \eta(t) \, dt = 0.$$

Define $m_\psi(t) := \int_\Omega \rho \dot{u}(t) \cdot \psi \, dx \in L^2([0, T])$. Since $\langle F(\cdot), \psi \rangle \in L^1([0, T])$, this distributional identity shows that m_ψ is absolutely continuous and

$$\dot{m}_\psi(t) = \langle F(t), \psi \rangle \quad \text{for a.e. } t \in (0, T).$$

Hence, for every $\psi \in H_{\Gamma_0}^1$,

$$\int_\Omega \rho \ddot{u}(t) \cdot \psi \, dx = \langle F(t), \psi \rangle,$$

which is exactly (6.54). $\square$

Fix t such that local semi-stability holds. Using the 1-homogeneity and convexity of R , this is equivalent to

$$\langle -\partial_\gamma \mathcal{E}(t, u(t), \gamma(t)), \varphi \rangle \leq R(\varphi) \quad \forall \varphi \in Z,$$

i.e.

$$-\partial_\gamma \mathcal{E}(t, u(t), \gamma(t)) \in \partial R(0) \subset Z^*. \quad (6.56)$$

This is the standard subgradient optimality condition for the minimization of $E(\cdot) + R(\cdot - \gamma(t))$ at $\gamma(t)$.

For a.e. t we can take $\psi = \dot{u}(t) \in H_{\Gamma_0}^1(\Omega; \mathbb{R}^d)$ in (6.54) to obtain the following identity for every $0 \leq t_1 \leq t_2 \leq T$:

$$\begin{aligned} & K(\dot{u}(t_2)) - K(\dot{u}(t_1)) + \int_{t_1}^{t_2} \int_\Omega \mathbb{D} e(\dot{u}) : e(\dot{u}) \, dx \, dt \\ & + \left[\int_\Omega \frac{1}{2} \mathbb{C}(e(u) - \gamma S) : (e(u) - \gamma S) \, dx \right]_{t_1}^{t_2} \\ & = \int_{t_1}^{t_2} \left(\int_\Omega f \cdot \dot{u} \, dx + \int_{\Gamma_1} g \cdot \dot{u} \, dS \right) dt - \int_{t_1}^{t_2} \int_\Omega (\mathbb{C}(e(u) - \gamma S) : S) \dot{\gamma} \, dx \, dt. \end{aligned} \quad (6.57)$$

Adding the identity

$$\int_{t_1}^{t_2} \langle \beta \gamma - 2\kappa \Delta \gamma, \dot{\gamma} \rangle \, dt = \int_\Omega \left(\frac{\beta}{2} \gamma(t_2)^2 + \kappa |\nabla \gamma(t_2)|^2 - \frac{\beta}{2} \gamma(t_1)^2 - \kappa |\nabla \gamma(t_1)|^2 \right) dx$$

to (6.57) yields the energy–power balance

$$\begin{aligned} & K(\dot{u}(t_2)) + E(t_2, u(t_2), \gamma(t_2)) - \left(K(\dot{u}(t_1)) + E(t_1, u(t_1), \gamma(t_1)) \right) \\ & + \int_{t_1}^{t_2} \int_\Omega \mathbb{D} e(\dot{u}) : e(\dot{u}) \, dx \, dt \\ & = \int_{t_1}^{t_2} \partial_t L(s, u(s)) \, ds + \int_{t_1}^{t_2} \langle \partial_\gamma E(s, u(s), \gamma(s)), \dot{\gamma}(s) \rangle \, ds. \end{aligned} \quad (6.58)$$

Subtract (6.58) from the energy inequality. For all $0 \leq t_1 \leq t_2 \leq T$ we obtain

$$\int_{t_1}^{t_2} \left(R(\dot{\gamma}(s)) + \langle \partial_\gamma E(s, u(s), \gamma(s)), \dot{\gamma}(s) \rangle \right) ds \leq 0. \quad (6.59)$$

On the other hand, for a.e. t ,

$$R(\dot{\gamma}(t)) + \langle \partial_\gamma E(t, u(t), \gamma(t)), \dot{\gamma}(t) \rangle \geq 0.$$

Combining this pointwise nonnegativity with (6.59) forces

$$R(\dot{\gamma}(t)) + \langle \partial_{\gamma} E(t, u(t), \gamma(t)), \dot{\gamma}(t) \rangle = 0 \quad \text{for a.e. } t \in (0, T).$$

By the 1-homogeneity of R and the Fenchel–Young equality, the conjunction of (6.56) and the last equality is equivalent to

$$-\partial_{\gamma} \mathcal{E}(t, u(t), \gamma(t)) = -\partial_{\gamma} E(t, u(t), \gamma(t)) \in \partial R(\dot{\gamma}(t)) \quad \text{for a.e. } t. \quad (6.60)$$

Let (u_i, γ_i) , $i = 1, 2$, be two energetic solutions corresponding to the same data, which in particular satisfy, for a.e. $t \in (0, T)$,

$$\text{(momentum balance in } \mathcal{V}^*) \quad \rho \ddot{u}_i(t) + V \dot{u}_i(t) + \partial_u \mathcal{E}(u_i(t), \gamma_i(t)) = f(t) \quad \text{in } \mathcal{V}^*, \quad (6.61)$$

$$\text{(flow rule in } \mathcal{W}^*) \quad 0 \in \partial R(\dot{\gamma}_i(t)) + \partial_{\gamma} \mathcal{E}(u_i(t), \gamma_i(t)) \quad \text{in } \mathcal{W}^*. \quad (6.62)$$

Set

$$u_{12} := u_1 - u_2, \quad \gamma_{12} := \gamma_1 - \gamma_2.$$

And more, take $\xi_i(t) \in \partial R(\dot{\gamma}_i(t))$ such that

$$0 = \xi_i(t) + \partial_{\gamma} \mathcal{E}(u_i(t), \gamma_i(t)), \quad (6.63)$$

and set

$$\xi_{12} := \xi_1 - \xi_2.$$

Subtract the momentum balances for $i = 1$ and 2 :

$$\rho \ddot{u}_{12} + \mathbb{D} e(\dot{u}_{12}) + \partial_u \mathcal{E}(u_{12}, \gamma_{12}) = 0 \quad \text{in } \mathcal{V}^* \quad \text{a.e.}$$

Testing by $\dot{u}_{12} \in \mathcal{V}$ and using symmetry of $\mathbb{D}$,

$$\frac{d}{dt} K(\dot{u}_{12}) + \int_{\Omega} \mathbb{D} e(\dot{u}_{12}) : e(\dot{u}_{12}) dx + \langle \partial_u \mathcal{E}(u_{12}, \gamma_{12}), \dot{u}_{12} \rangle = 0 \quad \text{a.e.} \quad (6.64)$$

Subtracting (6.63) for $i = 1$ and $i = 2$, and using that $\partial_{\gamma} \mathcal{E}$ is linear in (u, γ) , we obtain for a.e. t :

$$\xi_{12}(t) + \partial_{\gamma} \mathcal{E}(u_{12}(t), \gamma_{12}(t)) = 0 \quad \text{in } \mathcal{W}^*. \quad (6.65)$$

Test (6.63) by $\dot{\gamma}_{12} \in L^2(\Omega)$:

$$\langle \partial_{\gamma} \mathcal{E}(u_{12}, \gamma_{12}), \dot{\gamma}_{12} \rangle = -\langle \xi_{12}, \dot{\gamma}_{12} \rangle \quad \text{a.e.} \quad (6.66)$$

Since $\mathcal{E}$ is quadratic, the chain rule holds:

$$\frac{d}{dt} \mathcal{E}(u_{12}, \gamma_{12}) = \langle \partial_u \mathcal{E}(u_{12}, \gamma_{12}), \dot{u}_{12} \rangle + \langle \partial_{\gamma} \mathcal{E}(u_{12}, \gamma_{12}), \dot{\gamma}_{12} \rangle.$$

Adding (6.64) and (6.66) and invoking the chain rule yields:

$$\frac{d}{dt} \left(K(\dot{u}_{12}) + \mathcal{E}(u_{12}, \gamma_{12}) \right) + \int_{\Omega} \mathbb{D} e(\dot{u}_{12}) : e(\dot{u}_{12}) dx = -\langle \xi_{12}, \dot{\gamma}_{12} \rangle \quad \text{a.e.} \quad (6.67)$$

By monotonicity of ∂R one has $\langle \xi_{12}, \dot{\gamma}_{12} \rangle \geq 0$ a.e.

Integrating (6.67) from 0 to t , using the same initial data we obtain

$$K(\dot{u}_{12}(t)) + \mathcal{E}(u_{12}(t), \gamma_{12}(t)) + \int_0^t \int_{\Omega} \mathbb{D} e(\dot{u}_{12}) : e(\dot{u}_{12}) \, dx \, ds \leq 0.$$

Since all terms on the left are nonnegative, they must vanish identically. In particular, $\dot{u}_{12} \equiv 0$ and $\mathcal{E}(u_{12}, \gamma_{12}) \equiv 0$ a.e. in $[0, T]$. By using that the two solutions share the same initial data $u_1(0) = u_2(0)$, $\gamma_1(0) = \gamma_2(0)$, and by the positive definiteness of the hardening terms $\beta > 0$, it follows that $u_{12} \equiv 0$ and $\gamma_{12} \equiv 0$. $\square$

Chapter 7

Conclusions and Outlook

We have analyzed a finite-strain crystal plasticity model for periodically layered composites. We have considered two-slip systems in the soft layers (with a single-slip constraint at each point) and assumed completely rigid behavior in the stiff layers. Using a Γ -convergence framework, we derived explicit homogenized energies for the case of two orthogonal slip systems and obtained partial characterizations for general angles. The analysis combines a refined laminate structure configuration with an envelope argument adapted to a nonconvex admissible set and the multiplicative structure of finite-strain plasticity.

However, there are some important limitations that suggest natural directions for future development. First, the static Γ -limit obtained in this study is purely energetic and includes only a term that penalizes slips quadratically, without a linear dissipative contribution. Nor do we provide a Γ -limit for dissipation functionals alone. Both elements are essential for describing the quasi-static evolution in elastoplasticity, and remain open questions in this study.

A first direction is evolutionary Γ -convergence. Specifically, we formulate incremental problems for the ε -models (including stored energy and rate-independent dissipation), investigate their evolutionary Γ -limits, and aim to identify a pair of limit functionals $(\mathcal{E}_{\text{hom}}, \mathcal{D}_{\text{hom}})$ that govern the quasi-static evolution based on the concept of energetic solution. This program naturally separates into two tasks: (i) identifying the Mosco limit of $\mathcal{D}_\varepsilon$ under the same microstructural scaling, and (ii) proving evolutionary Γ -convergence for the energy-dissipation functional that treats energy and dissipation jointly. While the results for single-slip suggest useful compactness mechanisms (e.g., controlled rotation), extending these to systems with two or more slip systems will require new ideas.

A second direction is to complete the homogenization of the two-slip problem for general angles. While our partial result from this work identifies the homogenized density on large subregions of $F \in \mathbb{R}^{2 \times 2} : \det F = 1$, the "mixed" region, where one of the values $|Fv_1|, |Fv_2|$ is above 1 and the other below 1, remains unexplored. Progress in this area may require the introduction of higher-order laminated structures, a more refined description of rank-one connections compatible with the single-slip constraint, or a different relaxation framework.

A third line of research concerns structural generalization. A complete angle-independent formulation has yet to be obtained, even for two-slip systems. Extension to a finite number of slip systems (still assuming a single-slip at each point) remains a completely unexplored area. Further advances would include (i) an extension to three dimensions, (ii) relaxing the "rigid-layer" assumption to very stiff (possibly anisotropic) elasticity, and (iii) incorporating linear hardening and softening. All of these extensions would improve the predictive ability of the homogenized model and more precisely explain experiments on kink-band formation in layered alloys.

In conclusion, this work clarifies the static, energy-level picture for two-slip layered composites and identifies the main analytical obstacles—dissipation, mixed slip regimes, and multiplicity of systems—that must be overcome to establish a complete evolutionary Γ -limit for realistic multilayered crystals.

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