MODULATION SPACES AND THEIR APPLICATIONS

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In this article, we consider the modulation spaces $M^{p,q}(\mathbf{R}^d)$ for the range of indexes $0 < p, q \leqq \infty$ and their basic properties, their multipliers and their recent applications to partial differential equations. First in section 1, we briefly review the basic facts on $M^{p,q}(\mathbf{R}^d)$. Next in section 2, we treat the topic concerning the multipliers on $M^{p,q}(\mathbf{R}^d)$. And then in section 3, we describe recent applications to partial differential equations of $M^{p,q}(\mathbf{R}^d)$.

We begin with the notations to be used here before beginning the main topic. Let $\mathcal{S}(\mathbf{R}^d)$ be the Schwartz space of all complex-valued rapidly decreasing infinitely differentiable functions on $\mathbf{R}^d$ with the topology defined by the semi-norms

$$p_M(\varphi) = \sup_{t\in\mathbf{R}^d} (1+|t|)^M \sum_{|\alpha|\leqq M} |\partial^\alpha \varphi(t)|, \quad M = 1, 2, \cdots$$

for $\varphi \in \mathcal{S}(\mathbf{R}^d)$. And let $\mathcal{S}'(\mathbf{R}^d)$ be the topological dual of $\mathcal{S}(\mathbf{R}^d)$. The Fourier transform is $\widehat{f}(\omega) = \int f(t) e^{-2\pi i\omega\cdot t} dt$, and the inverse Fourier transform is $f^\vee(t) = \widehat{f}(-t)$. We define for $0 < p < \infty$

$$||f||_{L^p} = \Big(\int_{\mathbf{R}^d} |f(t)|^p dt \Big)^{\frac{1}{p}}$$

and $||f||_{L^\infty} = \operatorname{ess.sup}_{t\in\mathbf{R}^d} |f(t)|$. For a function f on $\mathbf{R}^d$, the translation and the modulation operators are defined by

$$T_x f(t) = f(t-x), \quad and \quad M_\omega f(t) = e^{2\pi i\omega\cdot t} f(t) \quad (x, \omega \in \mathbf{R}^d),$$

respectively. Note that we have

$$(T_x f)^\wedge = M_{-x}\widehat{f} \quad and \quad (M_\omega f)^\wedge = T_\omega \widehat{f}.$$

1. Basic facts on $M^{p,q}(\mathbf{R}^d)$

We review the definition of the modulation spaces and their basic properties, following [7].

1.1. **Definition of modulation spaces ([7]).** First for $\alpha > 0$ we define $\Phi^\alpha(\mathbf{R}^d)$ to be the set of all $g \in \mathcal{S}(\mathbf{R}^d)$ satisfying supp $\widehat{g} \subset \{\xi \mid |\xi| \leqq 1\}$, *and* $\sum_{k\in\mathbf{Z}^d} \widehat{g}(\xi - \alpha k) = 1$, $\forall \xi \in \mathbf{R}^d$. (In the following, we choose a sufficiently small $\alpha > 0$ so that $\Phi^\alpha(\mathbf{R}^d)$ is not empty.) With this, we define the modulation spaces as follows:

Given a $g \in \Phi^\alpha(\mathbf{R}^d)$, and $0 < p, q \leqq \infty$, we define the modulation space $M^{p,q}(\mathbf{R}^d)$ to be the space of all tempered distributions $f \in \mathcal{S}'(\mathbf{R}^d)$ such that the quasi-norm

$$||f||_{M^{p,q}} := \Big(\int_{\mathbf{R}^d} \big(\int_{\mathbf{R}^d} \big| f * \big(M_\omega g\big)(x)\big|^p dx\Big)^{\frac{q}{p}} d\omega\Big)^{\frac{1}{q}}$$

is finite, with obvious modifications if p or $q = \infty$.

Our definition is close to the original definition of modulation spaces by Feichtinger [5], which is restricted to the case $1 \leqq p, q \leqq \infty$.

The quasi-norm $||f||_{M^{p,q}}$ of f is considered as follows if we use the notion of the Fourier transform: At first we do the Fourier transform of f, and then we cut off $\widehat{f}(\xi)$ by the window $\widehat{g}(\xi - \omega)$ any information other than the neighborhood of ω and do the inverse Fourier transform of $\widehat{f}(\xi) \cdot \widehat{g}(\xi - \omega)$. And we take the L^p-quasi-norm with respect to x of $\big(\widehat{f}(\cdot)\widehat{g}(\cdot - \omega)\big)^{\vee}(x)$ and take the L^q-quasi-norm with respect to ω of $\big\|\big(\widehat{f}(\cdot)\widehat{g}(\cdot - \omega)\big)^{\vee}(x)\big\|_{L^p_x}$.

Remark 1.1. In general, a real-valued function $|| \cdot ||$ on a vector space X over $\mathbf{C}$ is called a quasi-norm if it satisfies the following conditions:

$$\text{(i)} \qquad ||x|| \geqq 0, \ ||x|| = 0 \text{ iff } x = 0,$$

$$\text{(ii)} \qquad ||\alpha x|| = |\alpha| \ ||x||, \ \forall \alpha \in \mathbf{C}, \forall x \in X,$$

$$\text{(iii)} \qquad ||x + y|| \leqq \kappa(||x|| + ||y||), \ \forall x, y \in X,$$

where κ is a constant independent of x and y. Especially, if $X = L^p(\mathbf{R}^d)$ $(0 < p < 1)$, then condition (iii) is given by

$$||f + g||_{L^p} \leqq 2^{\frac{1}{p}-1}(||f||_{L^p} + ||g||_{L^p}),$$

and if $X = M^{p,q}(\mathbf{R}^d)$ $(0 < p < 1$ or $0 < q < 1)$, then (iii) is given by

$$||f_1 + f_2||_{M^{p,q}} \leqq \kappa_p \cdot \kappa_q\big(||f_1||_{M^{p,q}} + ||f_2||_{M^{p,q}}\big),$$

where,

$$\kappa_p = \begin{cases} 1, & 1 \leq p \leq \infty, \\ 2^{\frac{1}{p}-1}, & 0 < p < 1. \end{cases}$$

1.2. **Basic properties of modulation spaces ([7]).** Let $0 < p, q \leqq \infty$ and $g \in \Phi^\alpha(\mathbf{R}^d)$. Then
(1) The definition of $M^{p,q}(\mathbf{R}^d)$ is independent of the window $g \in \Phi^\alpha(\mathbf{R}^d)$. That is, different windows yield equivalent norms.

$$\text{(2)} \qquad \Big(\sum_{k \in \mathbf{Z}^d} \Big(\int_{\mathbf{R}^d} \big| f * \big(M_{\alpha k} g\big)(x)\big|^p dx\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M^{p,q}(\mathbf{R}^d)$ with modifications if p or $q = \infty$.
(3) Let $0 < p_0 \leqq p_1 \leqq \infty$ and $0 < q_0 \leqq q_1 \leqq \infty$. Then

$$M^{p_0,q_0}(\mathbf{R}^d) \subset M^{p_1,q_1}(\mathbf{R}^d).$$

(4) $(M^{p,q}(\mathbf{R}^d), || \ ||_{M^{p,q}})$ is a quasi-Banach space.
(5) We have continuous embeddings

$$\mathcal{S}(\mathbf{R}^d) \subset M^{p,q}(\mathbf{R}^d) \subset \mathcal{S}'(\mathbf{R}^d).$$

(6) If $0 < p, q < \infty$, then $\mathcal{S}(\mathbf{R}^d)$ is dense in $M^{p,q}(\mathbf{R}^d)$.
(7) If $\beta > 0$ is sufficiently small, then

$$\Big(\sum_{k\in\mathbf{Z}^d}\Big(\sum_{l\in\mathbf{Z}^d}\big|f*\big(M_{\alpha k}g\big)(\beta l)\big|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}}$$

is an equivalent quasi-norm on $M^{p,q}(\mathbf{R}^d)$.

Remark 1.2. Modulation space $M^{p,q}(\mathbf{R}^d)$ is also independent of the choice of α $(0 < \alpha < 2)$.

Remark 1.3. We note that quasi-normed spaces are metric spaces. Actually, let $(X, \|\cdot\|)$ be a quasi-normed space having a constant $\kappa \geqq 1$ in Remark 1.1 (iii). If we define $0 < \rho \leqq 1$ by $(2\kappa)^\rho = 2$ and define $||\cdot||^*$ by

$$||x||^* = \inf\Big\{\sum_{j=1}^n ||x_j||^\rho : \ \sum_{j=1}^n x_j = x, \ n \geqq 1\Big\},$$

then $d(x,y) := ||y-x||^*$ satisfies the conditions of metric on X and satisfies $d(x,y) \leqq ||y-x||^\rho \leqq 2d(x,y)$.

2. Multipliers on modulation spaces

2.1. Multipliers on $M^{p,q}(\mathbf{R}^d)$ and symbol classes.

Definition 2.1. Let $0 < p, q < \infty$ and $\sigma(\xi)$ be a function on $\mathbf{R}^d$. Then we say that σ is a multiplier on $M^{p,q}(\mathbf{R}^d)$, if the operator $\sigma(D)$ defined by

$$\sigma(D)f = \int_{\mathbf{R}^d} e^{2\pi i x\cdot\xi}\sigma(\xi)\widehat{f}(\xi)d\xi, \quad f \in \mathcal{S}(\mathbf{R}^d)$$

has a unique bounded extension to $M^{p,q}(\mathbf{R}^d)$.

Definition 2.2. For $g \in \Phi^\alpha(\mathbf{R}^d)$ and $0 < p < \infty$, we define $S(p)$ to be the space of all tempered distributions $\sigma \in \mathcal{S}'(\mathbf{R}^d)$ such that

$$||\sigma||_{S(p)} := ||\check{\sigma}||_{M^{p,\infty}} = \sup_{k\in\mathbf{Z}^d}\Big(\int_{\mathbf{R}^d}|(\sigma\cdot T_{\alpha k}\widehat{g})^\vee(x)|^p dx\Big)^{\frac{1}{p}} < \infty. \tag{2.1}$$

2.2. **Multiplier theorem.** As for the multiplier on modulation spaces, we have the following theorem (see [2], [9]):

Theorem 2.3.
(i) Let $1 \leqq p < \infty$, $0 < q < \infty$ and $\sigma \in S(1)$. Then $\sigma(D)$ is a multiplier operator on $M^{p,q}(\mathbf{R}^d)$ and we have

$$||\sigma(D)f||_{M^{p,q}} \leqq C||\sigma||_{S(1)}||f||_{M^{p,q}}, \quad f \in M^{p,q}(\mathbf{R}^d).$$

(ii) Let $0 < p < 1$, $0 < q < \infty$ and $\sigma \in S(p)$. Then $\sigma(D)$ is a multiplier operator on $M^{p,q}(\mathbf{R}^d)$ and we have

$$||\sigma(D)f||_{M^{p,q}} \leqq C||\sigma||_{S(p)}||f||_{M^{p,q}}, \quad f \in M^{p,q}(\mathbf{R}^d).$$

2.3. **Concrete examples of multipliers.**

Example 2.4 ([9])**.** Let $0 < p < \infty$ and K be a positive integer. If $K > \frac{d}{2p}$ then

$$\mathcal{B}^K := \{ f \in C^K(\mathbf{R}^d) \mid \sum_{|\alpha| \leqq K} ||\partial^\alpha f||_{L^\infty} < \infty \}$$

belongs to $S(p)$. Especially, in the case $0 < p \leqq 1$ and $0 < q < \infty$, elements of $\mathcal{B}^K$ $(K > \frac{d}{2p})$ are multipliers on $M^{p,q}(\mathbf{R}^d)$.

Example 2.5 ([9])**.** Let $0 < p \leqq 1$ and δ_{x_n} be the Dirac measure at a point $x_n \in \mathbf{R}^d$. Then, for a sequence of complex numbers $\{c_n\}_{n=-\infty}^{\infty} \in l^p(\mathbf{Z})$,

$$\sigma = \Big(\sum_{n=-\infty}^{\infty} c_n \delta_{x_n} \Big)\widehat{}$$

belongs to $S(p)$.

Remark 2.6. Oberlin [10] has proved that every bounded linear operator T on $L^p(\mathbf{R}^d)$ $(0 < p < 1)$ which commutes with translations is represented by $Tf = \sigma(D)f$ with $\sigma = (\sum c_n \delta_{x_n})\widehat{}$, where $\{c_n\} \in l^p(\mathbf{Z})$.

To our regret, the following example shows that Theorem 2.3 (i) is not the best result for $1 < p < \infty, 1 \leqq q < \infty$.

Example 2.7 ([1])**.** Let $1 < p < \infty$ and $1 \leqq q < \infty$. Then $\sigma(\xi) = -i \operatorname{sgn}(\xi)$ is a multiplier on $M^{p,q}(\mathbf{R})$. However, $\sigma(\xi)$ is not a multiplier on $M^{1,1}(\mathbf{R})$. That is, the Hilbert transform is not bounded on $M^{1,1}(\mathbf{R}^d)$.

Multipliers on $M^{p,q}(\mathbf{R}^d)$ related to partial differential equations are as follows:

Example 2.8 ([2] Theorem 1)**.** Put $\sigma_\alpha(\xi) = e^{i|\xi|^\alpha}$. Then for $1 \leqq p, q < \infty$ and $0 \leqq \alpha \leqq 2$, we have $\sigma_\alpha(\xi) \in S(1)$.

We can also say the following as an application of Example 2.8:

Example 2.9 ([2] Theorem 6)**.** Put $\sigma_2^t(\xi) = e^{\pi i t|\xi|^2}$. Then $\sigma_2^t \in S(1)$ and

$$||\sigma_2^t||_{S(1)} \leqq C(1+t^2)^{\frac{d}{4}}.$$

This shows the solution $u(x,t)$ of the Schrödinger equation

$$\begin{cases} i\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \quad x \in \mathbf{R}^d, t \geqq 0, \\ u(x,0) = u_0(x), \quad x \in \mathbf{R}^d, \end{cases} \tag{2.2}$$

given by

$$u(x,t) = \big(U(t)u_0\big)(x) = \int_{\mathbf{R}^d} e^{4\pi i t|\xi|^2} \widehat{u_0}(\xi) e^{2\pi i x\cdot\xi} d\xi,$$

satisfies that if $u_0 \in M^{p,q}(\mathbf{R}^d)$ then $u(\cdot,t) \in M^{p,q}(\mathbf{R}^d)$. Moreover,

$$||u(\cdot,t)||_{M^{p,q}} \leqq C(1+t)^{\frac{d}{2}} ||u_0||_{M^{p,q}}.$$

Remark 2.10. From the above-mentioned, we can consider modulation spaces $M^{p,q}(\mathbf{R}^d)$ are function spaces which are more suitable than the Lebesgue spaces $L^p(\mathbf{R}^d)$ for the studies of Schrödinger equations because it is known that only for $p = 2$ the solution operator $U(t)$ of (2.2) is bounded on $L^p(\mathbf{R}^d)$ but is bounded on $M^{p,q}(\mathbf{R}^d)$ for each $1 \leqq p, q < \infty$.

3. Recent applications to partial differential equations

In addition, Bényi-Okoudjou [3] apply the multiplier theorem to study the local-well posedness of the non-linear Schrödinger equation

$$\textbf{(NLS)} \qquad \begin{cases} i\frac{\partial u}{\partial t}(x,t) + \Delta u(x,t) = \lambda |u|^{2k} u, \quad \lambda \in \mathbf{R}, k \in \mathbf{N} \\ u(x,0) = u_0(x) \end{cases}$$

and they prove the following: Let $\frac{d}{d+1} < p \leqq \infty$. Then for any $u_0 \in M^{p,1}(\mathbf{R}^d)$, there exists $T^* = T^*(||u_0||_{M^{p,1}})$ such that (NLS) has a unique solution

$$u \in C\big([0,T^*], M^{p,1}(\mathbf{R}^d)\big).$$

Moreover, if $T^* < \infty$, then $\limsup_{t \to T^*} ||u(\cdot,t)||_{M^{p,1}} = \infty$.

Moreover, Cordero-Nicola [4] and Wang-Huang [12] also apply the theory of modulation spaces to partial differential equations.

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