

Construction of C^* -algebras

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1 Crossed products

In this note, we discuss several generalizations of dynamical systems and their crossed products. Throughout this note, A denotes a C^* -algebra.

Let G be a locally compact group. An *action* of G on A is a strongly continuous homomorphism $\alpha: G \rightarrow \text{Aut}(A)$. The triple (A, G, α) is called a *C^* -dynamical system*. From a C^* -dynamical system (A, G, α) , we get a C^* -algebra $A \rtimes_\alpha G$ which is called the *crossed product*[†] (see [Pe], for the detail).

When $G = \mathbb{Z}$, an action $\alpha: \mathbb{Z} \rightarrow \text{Aut}(A)$ is determined by $\alpha_1 \in \text{Aut}(A)$. By an abuse of notation, we denote α_1 by α , and identify actions of $\mathbb{Z}$ and automorphisms. The C^* -algebra $A \rtimes_\alpha \mathbb{Z}$ is sometimes called the *crossed product by the automorphism α* .

Definition 1.1 The crossed product $A \rtimes_\alpha \mathbb{Z}$ is the universal C^* -algebra generated by the images of the $*$ -homomorphism $\pi: A \rightarrow A \rtimes_\alpha \mathbb{Z}$ and the linear map $t: A \rightarrow A \rtimes_\alpha \mathbb{Z}$ satisfying

- (i) $t(x)\pi(a) = t(xa)$,
- (ii) $t(x)^*t(y) = \pi(x^*y)$,
- (iii) $\pi(a)t(x) = t(\alpha(a)x)$,
- (iv) $t(x)t(y)^* = \pi(\alpha^{-1}(xy^*))$

for $a, x, y \in A$.

In the definition above, “universal” means that for any C^* -algebra B , any $*$ -homomorphism $\pi': A \rightarrow B$ and any linear map $t': A \rightarrow B$ satisfying (i) – (iv) above, there exists a $*$ -homomorphism $\rho: A \rtimes_\alpha \mathbb{Z} \rightarrow B$ such that $\pi' = \rho \circ \pi$ and $t' = \rho \circ t$. We can show that there exists a unitary u in the multiplier algebra of $A \rtimes_\alpha \mathbb{Z}$ such that $t(x) = u\pi(x)$ for $x \in A$. This unitary u satisfies

$$u\pi(a)u^* = \pi(\alpha^{-1}(a)) \quad \text{for } a \in A. \quad (*)$$

[†]There are two types of crossed products, namely the reduced ones and the full ones. We do not go to the detail because we are only interested in the case $G = \mathbb{Z}$ where the two types of C^* -algebras coincide.

Conversely, if a $*$ -homomorphism $\pi': A \rightarrow B$ and a unitary u' in the multiplier algebra of B satisfies $(*)$, then the pair of the $*$ -homomorphism $\pi i'$ and the linear map $t': A \rightarrow B$ defined by $t'(x) = u\pi(x)$ for $x \in A$ satisfies (i) – (iv). Thus the above definition coincides with the ordinal one using the covariant condition $(*)$ (see for example [Pe]). There are many generalizations of this construction. One of them is a crossed product by a Hilbert C^* -bimodule [AEE].

Definition 1.2 ([BMS]) A Hilbert A -bimodule X is a Banach space which is an A -bimodule and has A -valued left and right inner products $(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ such that

- (i) $(\xi, \xi) \geq 0, \quad \langle \xi, \xi \rangle \geq 0,$
- (ii) $\|\xi\| = \|(\xi, \xi)\|^{1/2} = \|\langle \xi, \xi \rangle\|^{1/2},$
- (iii) $(a\xi, \eta) = a(\xi, \eta), \quad \langle \xi, \eta a \rangle = \langle \xi, \eta \rangle a,$
- (iv) $(\xi, \eta)\zeta = \xi(\eta, \zeta)$

for $\xi, \eta, \zeta \in X, a \in A$.

For $\xi, \eta \in X$ and $a \in A$, we can show $(\eta, \xi) = (\xi, \eta)^*, \langle \eta, \xi \rangle = \langle \xi, \eta \rangle^*$ from (i), and

$$(\xi a, \eta) = (\xi, \eta a^*), \quad \langle \xi, a\eta \rangle = \langle a^* \xi, \eta \rangle$$

from (iv). An automorphism $\alpha \in \text{Aut}(A)$ determines a Hilbert A -bimodule X_α as follows: As Banach spaces, X_α is isomorphic to A via the map $A \ni x \mapsto \xi_x \in X_\alpha$. The bimodule structure and inner products are defined as

$$a\xi_x b := \xi_{\alpha(a)xb}, \quad (\xi_x, \xi_y) := \alpha^{-1}(xy^*), \quad \langle \xi_x, \xi_y \rangle := x^*y$$

for $a, x, y \in A$. By this construction, we think that Hilbert C^* -bimodules generalize automorphisms. The compositions of automorphisms correspond to the tensor products of Hilbert C^* -bimodules[†], and the inverses correspond to the dual Hilbert C^* -bimodules.

Definition 1.3 ([AEE, Definition 2.1]) The crossed product $A \rtimes_X \mathbb{Z}$ of a C^* -algebra A by a Hilbert A -bimodule X is the universal C^* -algebra generated by the images of the $*$ -homomorphism $\pi: A \rightarrow A \rtimes_X \mathbb{Z}$ and the linear map $t: X \rightarrow A \rtimes_X \mathbb{Z}$ satisfying

- (i) $t(\xi)\pi(a) = t(\xi a),$
- (ii) $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle),$
- (iii) $\pi(a)t(\xi) = t(a\xi),$
- (iv) $t(\xi)t(\eta)^* = \pi((\xi, \eta)),$

for $a \in A$ and $\xi, \eta \in X$.

[†]With our convention, we have $X_\alpha \otimes X_\beta \cong X_{\beta \circ \alpha}$.

The conditions (i) and (iii) hold automatically from the conditions (ii) and (iv), respectively. It is straightforward to see $A \rtimes_{X_\alpha} \mathbb{Z} \cong A \rtimes_\alpha \mathbb{Z}$ for $\alpha \in \text{Aut}(A)$.

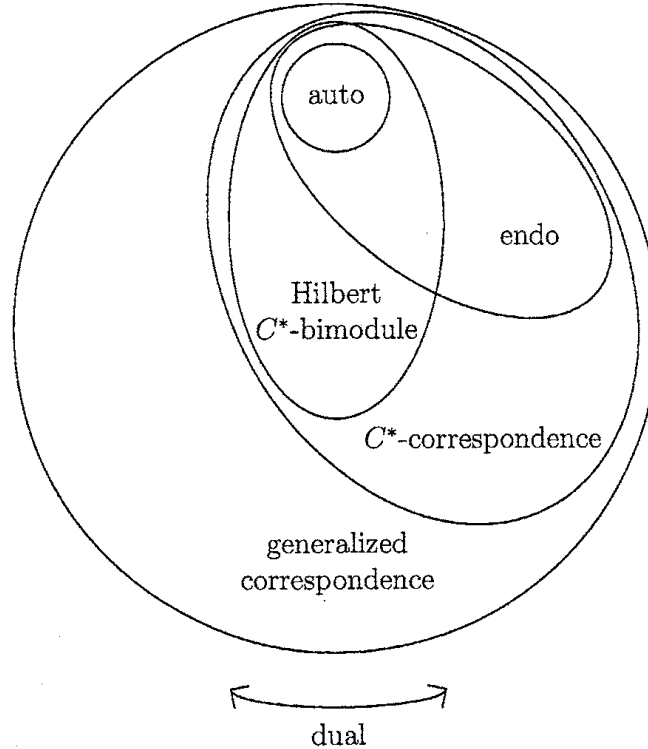
Another generalization of the crossed products by automorphisms is crossed products by endomorphisms [M, St]. These two generalizations can be unified to the construction of the *Pimsner algebra* $\mathcal{O}_X$ from a C^* -correspondence[†] X , which is defined in [Pi] and modified in [Ka5].

Definition 1.4 If a Banach space X satisfies all the conditions for Hilbert A -bimodules except the existence of a left inner product but instead satisfies $\langle a\xi, \eta \rangle = \langle \xi, a^*\eta \rangle$ for $\xi, \eta \in X$ and $a \in A$, then it is called a C^* -correspondence over A .

For a definition and properties of the Pimsner algebra, see the next section. Recently, Exel defines generalized correspondences and gives a method to construct C^* -algebras from them ([E]). A *ternary ring of operators* (TRO) is a Banach space X with a *ternary operation* $[\cdot, \cdot, \cdot]: X \times X \times X \rightarrow X$ which satisfies the conditions that the map $(x, y, z) \mapsto xy^*z$ satisfies ([Z]). A *generalized correspondence* over A is an A -bimodule which is a TRO such that the ternary operation satisfies

$$[\xi, a\eta, \zeta] = [\xi, \eta, a^*\zeta], \quad [\xi, \eta a, \zeta] = [\xi a^*, \eta, \zeta]$$

for $\xi, \eta, \zeta \in X$ and $a \in A$. A C^* -correspondence is a generalized correspondence by setting $[\xi, \eta, \zeta] := \xi \langle \eta, \zeta \rangle$.



[†]Pimsner called it a Hilbert-bimodule, and he assumed that its left action is faithful.

The class of generalized correspondences is a natural class which contains C^* -correspondences and is invariant under “taking duals”. In [E], Exel suggests one way to construct a C^* -algebra $C^*(A, X)$ from a generalized correspondence X over A , which generalizes the construction of Pimsner algebras. There are several things remained which have to be checked. For example, we do not know whether the natural embedding map $A \rightarrow C^*(A, X)$ is injective or not.

So far, we only consider the generalization of actions and crossed products for the case that the group is $\mathbb{Z}$ (or the semigroup $\mathbb{N}$). There is a generalization of actions by general groups using C^* -correspondences, which is called a *product system*.

Definition 1.5 Let Γ be a cone of a group. A product system over Γ is a family $\{X_\gamma\}_{\gamma \in \Gamma}$ of C^* -correspondences over A together with the isomorphisms as C^* -correspondences

$$w_{\gamma,\mu}: X_\gamma \otimes X_\mu \rightarrow X_{\gamma\mu},$$

satisfying the associative law

$$w_{\gamma\mu,\nu} \circ (w_{\gamma,\mu} \otimes \text{id}_{X_\nu}) = w_{\gamma,\mu\nu} \circ (\text{id}_{X_\gamma} \otimes w_{\mu,\nu}).$$

We should be careful of X_e where $e \in \Gamma$ is the identity (see [F]). If Γ has a topology (e.g. $\Gamma = \mathbb{R}_+$), then we have to take care of the “continuity” (or “measurability”) of the map $\gamma \rightarrow X_\gamma$ (see [H]). Product systems over the positive real line $\mathbb{R}_+$ are related to E_0 -semigroup (see [H, Sk]). A *higher rank graph* introduced in [KP] gives an example of product systems over the semigroup $\mathbb{N}^k$ (see [F, RSY]).

There is a natural construction of a C^* -algebra from a product system, which is analogue of Toeplitz algebra $\mathcal{T}_X$ defined below. However, except for special cases, we do not know how to define analogues of crossed products or Pimsner algebras of product systems.

2 Pimsner algebras

Let A be a C^* -algebra, and X be a C^* -correspondence over A .

Definition 2.1 A representation of X on a C^* -algebra B is a pair (π, t) consisting of a $*$ -homomorphism $\pi: A \rightarrow B$ and a linear map $t: X \rightarrow B$ satisfying

- (i) $t(\xi)\pi(a) = t(\xi a)$,
- (ii) $t(\xi)^*t(\eta) = \pi(\langle \xi, \eta \rangle)$,
- (iii) $\pi(a)t(\xi) = t(a\xi)$

for $a \in A$ and $\xi, \eta \in X$. We denote by $C^*(\pi, t)$ the C^* -algebra generated by the images of π and t in B .

Definition 2.2 We denote the universal representation by $(\bar{\pi}_X, \bar{t}_X)$. The C^* -algebra $C^*(\bar{\pi}_X, \bar{t}_X)$ is called the *Toeplitz algebra* of X , and denoted by $\mathcal{T}_X$.

The Toeplitz algebra $\mathcal{T}_X$ is not an analogue of crossed products. We need the condition corresponding (iv) in Definition 1.1 or Definition 1.3. To express this condition, we introduce some notations.

Definition 2.3 A map $T: X \rightarrow X$ is said to be *adjointable* if there exists $T^*: X \rightarrow X$ such that $\langle \xi, T\eta \rangle = \langle T^*\xi, \eta \rangle$ for $\xi, \eta \in X$.

We denote by $\mathcal{L}(X)$ the set of all adjointable operators on X .

It is routine to check that $\mathcal{L}(X)$ is a C^* -algebra, and the left action defines the $*$ -homomorphism $\varphi: A \rightarrow \mathcal{L}(X)$ by $\varphi(a)\xi = a\xi$.

Definition 2.4 For $\xi, \eta \in X$, the operator $\theta_{\xi, \eta} \in \mathcal{L}(X)$ is defined by $\theta_{\xi, \eta}(\zeta) = \xi \langle \eta, \zeta \rangle$ for $\zeta \in X$. We define $\mathcal{K}(X) \subset \mathcal{L}(X)$ by

$$\mathcal{K}(X) = \overline{\text{span}}\{\theta_{\xi, \eta} \mid \xi, \eta \in X\},$$

which is an ideal of $\mathcal{L}(X)$.

For the proof of the next lemma see [KPW, Lemma 2.2] or [FR, Remark 1.7].

Lemma 2.5 For a representation (π, t) of X , there exists a unique $*$ -homomorphism $\psi_t: \mathcal{K}(X) \rightarrow C^*(\pi, t)$ such that $\psi_t(\theta_{\xi, \eta}) = t(\xi)t(\eta)^*$ for $\xi, \eta \in X$.

Definition 2.6 For a C^* -correspondence X , we define an ideal J_X of A by

$$J_X := \{a \in A \mid \varphi(a) \in \mathcal{K}(X) \text{ and } ab = 0 \text{ for all } b \in \ker \varphi\}.$$

Definition 2.7 A representation (π, t) of X is said to be *covariant* if $\psi_t(\varphi(a)) = \pi(a)$ for all $a \in J_X$.

Definition 2.8 Let (π_X, t_X) be the universal covariant representation, and set $\mathcal{O}_X := C^*(\pi_X, t_X)$ which is called the *Pimsner algebra* of X .

One can check that this construction generalizes the crossed products by endomorphisms and the ones by Hilbert C^* -bimodules as well as other classes of C^* -algebras (see Section 3). We will give several characterizations of the representation (π_X, t_X) and the Pimsner algebra $\mathcal{O}_X$.

Definition 2.9 For two representations (π_1, t_1) and (π_2, t_2) of X , we write $(\pi_1, t_1) \succeq (\pi_2, t_2)$ if there exists a $*$ -homomorphism $\rho: C^*(\pi_1, t_1) \rightarrow C^*(\pi_2, t_2)$ such that $\pi_2 = \rho \circ \pi_1$ and $t_2 = \rho \circ t_1$.

Such a $*$ -homomorphism ρ is, if it exists, unique and surjective. We will say that two representations (π_1, t_1) and (π_2, t_2) are *equivalent* if $(\pi_1, t_1) \succeq (\pi_2, t_2)$ and $(\pi_2, t_2) \succeq (\pi_1, t_1)$. This is the same as the existence of an isomorphism $\rho: C^*(\pi_1, t_1) \rightarrow C^*(\pi_2, t_2)$ with $\pi_2 = \rho \circ \pi_1$ and $t_2 = \rho \circ t_1$. The set of equivalence classes of representations is an ordered set by the order $\preceq$. The universal representation $(\bar{\pi}_X, \bar{t}_X)$ is the largest element in this set.

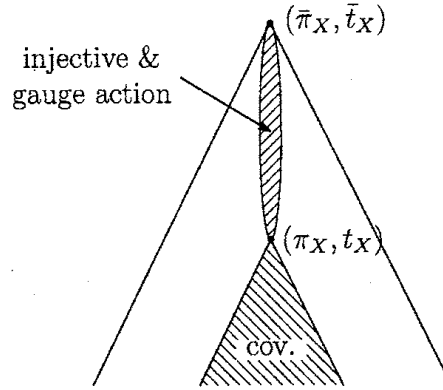
Definition 2.10 A representation (π, t) of X is said to be *injective* if a $*$ -homomorphism π is injective, and said to *admit a gauge action* if for each $z \in \mathbb{T}$, there exists a $*$ -homomorphism $\beta_z: C^*(\pi, t) \rightarrow C^*(\pi, t)$ such that $\beta_z(\pi(a)) = \pi(a)$ and $\beta_z(t(\xi)) = zt(\xi)$ for all $a \in A$ and $\xi \in X$.

By the universality, the representation (π_X, t_X) on $\mathcal{O}_X$ admits a gauge action. We denote this action by $\gamma: \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_X)$ and call it the *gauge action* on $\mathcal{O}_X$. We can also see that (π_X, t_X) is injective by using Fock representation [Ka6].

Theorem 2.11 ([Ka6, Theorem 6.4], [Ka7, Proposition 7.14]) *Each of the following three conditions characterizes the representation (π_X, t_X) on the Pimsner algebra $\mathcal{O}_X$:*

- (i) (π_X, t_X) is the largest in the set of all covariant representations.
- (ii) (π_X, t_X) is the smallest in the set of all injective representations admitting gauge actions.
- (iii) (π_X, t_X) is the only injective covariant representation admitting a gauge action.

(i) is nothing but the definition of (π_X, t_X) . The uniqueness part of (iii) is called the *gauge-invariant uniqueness theorem*. (ii) gives characterizations of (π_X, t_X) and $\mathcal{O}_X$ without using the covariance nor the ideal J_X .



The most important part of the proof of Theorem 2.11 is an analysis of the fixed point algebra $\mathcal{O}_X^\gamma$ of the gauge action (see the proof of the next theorem).

Theorem 2.12 (see [DS, Theorem 3.1], [Ka6, Theorems 7.1, 7.2])

$$A: \text{nuclear} \Rightarrow \mathcal{O}_X^\gamma: \text{nuclear} \iff \mathcal{O}_X: \text{nuclear}.$$

$$A: \text{exact} \iff \mathcal{O}_X^\gamma: \text{exact} \iff \mathcal{O}_X: \text{exact}.$$

Sketch of Proof. The two equivalences

$$“\mathcal{O}_X^\gamma: \text{nuclear} \iff \mathcal{O}_X: \text{nuclear}”, \quad “\mathcal{O}_X^\gamma: \text{exact} \iff \mathcal{O}_X: \text{exact}”$$

follow from the general fact on fixed point algebras by actions of compact groups (see [DLRZ]). We sketch the proof of “ A : nuclear $\Rightarrow \mathcal{O}_X^\gamma$: nuclear” (the corresponding statement for exactness can be proven similarly).

Suppose that A is nuclear, and we will prove that $\mathcal{O}_X^\gamma$ is nuclear. We set $Y_0 = \pi_X(A) \subset \mathcal{O}_X$ and

$$Y_{n+1} = t_X(X)Y_n := \overline{\text{span}}\{xy \in \mathcal{O}_X \mid x \in t_X(X), y \in Y_n\}$$

for $n \in \mathbb{N}$. Then we have

$$\mathcal{O}_X = \overline{\text{span}}\left(\bigcup_{n,m \in \mathbb{N}} Y_n Y_m^*\right), \quad \mathcal{O}_X^\gamma = \overline{\text{span}}\left(\bigcup_{n \in \mathbb{N}} Y_n Y_n^*\right).$$

We set $B_n = Y_n Y_n^*$ and $B_{[0,n]} = B_0 + B_1 + \cdots + B_n$. Then we have $\mathcal{O}_X^\gamma = \varinjlim B_{[0,n]}$. It suffices to show that the C^* -algebra $B_{[0,n]}$ is nuclear for all $n \in \mathbb{N}$. We will prove this by induction on n . The C^* -algebra $B_{[0,0]} = B_0 \cong A$ is nuclear by the assumption. Suppose we will prove that $B_{[0,n-1]}$ is nuclear. The C^* -algebra B_n is strongly Morita equivalent to the C^* -algebra $Y_n^* Y_n \subset \mathcal{O}_X$ which is isomorphic to an ideal of A . Hence B_n is nuclear. Since B_n is an ideal of $B_{[0,n]}$ and $B_{[0,n]} = B_{[0,n-1]} + B_n$, we have $B_{[0,n]}/B_n \cong B_{[0,n-1]}/(B_{[0,n-1]} \cap B_n)$ which is nuclear.

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_{[0,n-1]} \cap B_n & \longrightarrow & B_{[0,n-1]} & \longrightarrow & B_{[0,n-1]}/(B_{[0,n-1]} \cap B_n) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & B_n & \longrightarrow & B_{[0,n]} & \longrightarrow & B_{[0,n]}/B_n \longrightarrow 0 \end{array}$$

Therefore $B_{[0,n]}$ is nuclear being an extension of nuclear C^* -algebras. This completes the proof. $\blacksquare$

Remark 2.13 $\mathcal{T}_X$ is nuclear (resp. exact) if and only if A is nuclear (resp. exact). There is an example of a C^* -correspondence X over a non-nuclear C^* -algebra A such that $\mathcal{O}_X$ is nuclear (see [Ka6, Example 7.7]).

There have been some results on the ideal structures of Pimsner algebras ([Ka7], [MT1]), and a criterion for their simplicity in a special case ([Sc]). However we do not know when they are simple in general. On the K -theory of Pimsner algebras, we have the following (see [Pi, Theorem 4.9] and [Ka6, Theorem 8.6, Proposition 8.8]).

Theorem 2.14 *The Pimsner algebra $\mathcal{O}_X$ satisfies the Universal Coefficient Theorem of [RS], if both A and J_X satisfy it. We have the following exact sequence;*

$$\begin{array}{ccccc} K_0(J_X) & \xrightarrow{\iota_* - [X]} & K_0(A) & \xrightarrow{(\pi_X)_*} & K_0(\mathcal{O}_X) \\ \uparrow & & & & \downarrow \\ K_1(\mathcal{O}_X) & \xleftarrow{(\pi_X)^*} & K_1(A) & \xleftarrow{\iota_* - [X]} & K_1(J_X). \end{array}$$

3 Topological quivers

In this section, we give methods to construct C^* -correspondences over commutative C^* -algebras.

Definition 3.1 ([MT2]) A *topological quiver* $\mathcal{Q} = (E^0, E^1, d, r, \lambda)$ consists of two locally compact spaces E^0 and E^1 , a continuous open map $d: E^1 \rightarrow E^0$, a continuous map $r: E^1 \rightarrow E^0$, and a family of Radon measures $\lambda = \{\lambda_v\}_{v \in E^0}$ on E^1 satisfying the following two conditions:

- (i) $\text{supp } \lambda_v = d^{-1}(v)$ for all $v \in E^0$,
- (ii) $v \mapsto \int_{E^1} \xi(e) d\lambda_v(e)$ is an element of $C_c(E^0)$ for all $\xi \in C_c(E^1)$.

Take a topological quiver $\mathcal{Q} = (E^0, E^1, d, r, \lambda)$. We set $A := C_0(E^0)$. For $\xi, \eta \in C_c(E^1)$,

$$v \mapsto \int_{E^1} \overline{\xi(e)} \eta(e) d\lambda_v(e)$$

is an element of $C_c(E^0)$. We denote this function by $\langle \xi, \eta \rangle \in A$. The linear space $C_c(E^1)$ is an A -bimodule by

$$f\xi g: E^1 \ni e \mapsto f(r(e))\xi(e)g(d(e))$$

for $f, g \in A$ and $\xi \in C_c(E^1)$. Let X be the completion of $C_c(E^0)$ with respect to the norm defined by $\|\xi\| = \|\langle \xi, \xi \rangle\|^{1/2}$. The A -valued inner product and the A -bimodule structure are naturally extended to X . Thus X is a C^* -correspondence over A .

Definition 3.2 The Pimsner algebra $\mathcal{O}_X$ of the C^* -correspondence X over A constructed above is said to be the *C^* -algebra associated to $\mathcal{Q}$* , and denoted by $C^*(\mathcal{Q})$.

A quadruple $E = (E^0, E^1, d, r)$ consisting of two locally compact spaces E^0 and E^1 , a local homeomorphism $d: E^1 \rightarrow E^0$, and a continuous map $r: E^1 \rightarrow E^0$, is called a *topological graph* ([Ka1]). For a topological graph $E = (E^0, E^1, d, r)$, the quintuple $\mathcal{Q}_E = (E^0, E^1, d, r, \lambda)$ is a topological quiver, where λ_v is the counting measures on $d^{-1}(v)$ for $v \in E^0$. The C^* -algebra $C^*(\mathcal{Q}_E)$ is denoted by $\mathcal{O}(E)$ in [Ka1]. When $d: E^1 \rightarrow E^0$ is a branched covering between Riemann surfaces, the counting measures λ_v on $d^{-1}(v)$ for $v \in E^0$ with multiplicities at branched points satisfy two conditions in Definition 3.1. Thus we get a topological quiver, and the C^* -algebras associated to this type of topological quivers are analyzed in [KW].

For C^* -algebras associated to topological quivers, we know the conditions for the simplicity ([MT2, Theorem 10.2], see also [Ka3, Theorem 8.12]).

By Theorems 2.12 and 2.14, the class of the C^* -algebras associated to topological quivers are included in the class of nuclear C^* -algebras satisfying the Universal Coefficient Theorem. There may be possibilities that all separable simple nuclear C^* -algebras satisfying the Universal Coefficient Theorem can be obtained as C^* -algebras associated to topological quivers. In fact, the following C^* -algebras were shown to be obtained as C^* -algebras associated to topological quivers (or actually topological graphs [Ka2, Ka4]):

- (i) all AF-algebras,
- (ii) many ASH-algebras including all simple AT-algebras with real rank zero,
- (iii) all classifiable Kirchberg algebras.

We do not know whether the following examples arise as C^* -algebras associated to topological quivers:

- (i) a simple C^* -algebra with a finite and an infinite projection found in [Ro],
- (ii) all TAF-algebras classified in [L],
- (iii) the Jiang and Su algebra $\mathcal{Z}$ defined in [JS].

A dynamical system $(C_0(\Omega), G, \alpha)$ of a commutative C^* -algebra $C_0(\Omega)$ gives rise to an action of G on the space Ω . Such an action defines a groupoid $\Omega \rtimes G$ which is called a *transformation group*, and the crossed product $C_0(\Omega) \rtimes_\alpha G$ is isomorphic to the C^* -algebra of this groupoid [Re]. From a topological graph E , we can construct a groupoid $\mathcal{G}_E$ using negative orbits so that the C^* -algebra $\mathcal{O}(E)$ is isomorphic to the C^* -algebra of the groupoid $\mathcal{G}_E$. This observation may help when we try to extend the construction in this section to the more general setting involving general groups.

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