

Strongly compatible total orders on free monoids

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In this paper, we deal with a combinatorial problem which the first author raised in relation to term-rewriting systems. To 'complete' a grammar (as a term-rewriting system), one uses a compatible total order on the set X^* of all words (an order $\leq$ on X^* is said compatible if $x < y$ implies $uxv < uyv$ for all u, v) which does not have an infinite descending sequence. An order $\leq$ on X^* is said length-sensitive if $|x| < |y|$ implies $x < y$, where $|x|$ for a word x denotes the length of x . The length-sensitivity assures the non-existence of an infinite descending sequence. Usually, a total order that is used for the above purpose is the length-sensitive lexicographic order (which is clearly compatible). The problem raised by the first author was to describe the all length-sensitive compatible total orders on X^* . He soon noticed that if we replace the compatibility by a somewhat stronger property which we shall call the strong compatibility, then the description of the orders becomes strikingly simple especially when $\text{Card}(X) = 2$ ($\text{Card}(X)$ denotes the cardinality of the set X). In this paper, we take the strong compatibility as a basic property on orders, and present some results on such orders.

Let $X = \{a_1, a_2, \dots, a_n\}$ be a set. We call X the alphabet and its elements letters. The free monoid X^* on X is the set of all words over X (that is, finite sequences of letters including the empty sequence, denoted 1, with concatenation as the operation). We treat X itself as a subset of X^* by identifying each letter a with the word consisting of ' a ' alone. For any word x in X^* , the length of a word x is the number of letters occurring in x , and denoted by $|x|$. For each natural number n , X^n denotes the set $\{w \in X^* : |w| = n\}$, while $X^{[n]}$ denotes the set $\{w \in X^* : |w| \leq n\}$. We basically follow the notations in [B-P].

We consider an order relation $\leq$ on X^* with the following conditions:

- (P1) for letters, $a_1 < a_2 < \dots < a_n$ ($x < y$ means that $x \leq y$ and $x \neq y$), and for any words $x, y \in X^*$,
 (P2) (The length sensitivity) if $|x| < |y|$, then $x < y$.
 (P3) (The compatibility) if $x < y$, then $uxv < uyv$ for all $u, v \in X^*$.
 (P4) (The strong compatibility) if $x = x_1x_2$, $y = y_1y_2$ and $|x_1| = |y_1|$, $|x_2| = |y_2|$, then $x < y$ implies $x_1ux_2 < y_1uy_2$ for all $u \in X^*$, where $x_1, x_2, y_1, y_2 \in X^*$.
 (P5) $\leq$ is a total order on X^* .

By (P1), we mean that we may (and do) fix 'the' ordering on the letters of X without loss of generality. The length-sensitivity (P2) is our another presumption on orders on X^* and will be implicit on any orders which will appear in the paper. It is easy to check that (P4) implies (P3).

By the lexicographic order, we mean the order $\leq$ on X^* determined by the rule: if $x = x_1x_2\dots x_m$, $y = y_1y_2\dots y_m$ and $x_1\dots x_{i-1} = y_1\dots y_{i-1}$ and $x_i < y_i$, then $x < y$, where $x_i, y_i \in X$ ($i = 1, \dots, m$). We denote this order by $\leq_{\text{lex}}$.

If $x = x_1x_2\dots x_m \in X^*$, where $x_i \in X$ ($i = 1, \dots, m$), then we denote $x_mx_{m-1}\dots x_2x_1$ by $\tilde{x}$.

By the anti-lexicographic order, we mean the order $\leq$ on X^* such that for any $x, y \in X^*$, $x <_{\text{lex}} y$ implies $\tilde{x} < \tilde{y}$. We denote this order by $\leq_{\text{al}}$.

It is easy to see that both $<_{\text{lex}}$ and $\leq_{\text{al}}$ are strongly compatible.

Proposition 1. Suppose that $\leq$ is a strongly compatible total order on X^* . If $\leq$ coincides with $\leq_{\text{lex}}$ on $X^{[3]}$, then $\leq$ in fact equals $\leq_{\text{lex}}$ on the entire X^* . In particular, if $\text{Card}(X) \geq 3$, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

Proof. We show the first assertion by induction on the length of words. Suppose that on $X^{[m-1]}$, $\leq$ coincides with $\leq_{\text{lex}}$ ($m \geq 4$). Let $x, y \in X^m$ be $x = x_1x_2\dots x_m$ and $y = y_1y_2\dots y_m$ and let i be the least such that $x_i \neq y_i$. WLOG, we may assume that $x_i < y_i$. Then, since $x <_{\text{lex}} y$, we have to show that $x < y$.

Case 1 : $i \geq 2$. Then, $x_i\dots x_m < y_i\dots y_m$ by the induction hypothesis.

So, using the strong compatibility, we have that $x = x_1 \cdots x_{i-1} x_i \cdots x_m < y_1 \cdots y_{i-1} y_i \cdots y_m = y$.

Case 2 : $i = 1$ and $x_j \leq y_j$ for some $j \neq i$. Then, $x_1 \cdots x_{j-1} x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_{j+1} \cdots y_m$ by the induction hypothesis. So, using the strong compatibility, we have that $x = x_1 \cdots x_{j-1} x_j x_{j+1} \cdots x_m < y_1 \cdots y_{j-1} y_j y_{j+1} \cdots y_m = y$.

Case 3 : $i = 1$ and $x_j > y_j$ for all $j \neq i$. Since by the case 1, we have that $x \leq x_1 a_n^{m-1}$ and $y_1 a_1^{m-1} \leq y$, it suffices to show that $x_1 a_n^{m-1} < y_1 a_1^{m-1}$. Suppose on the contrary, that $y_1 a_1^{m-1} < x_1 a_n^{m-1}$. Then, by the strong compatibility, we have that $y_1 x_1 a_1^{m-1} < x_1^2 a_n^{m-1}$. But, on the other hand, we have:

$$\begin{aligned} x_1^2 a_n^{m-1} &= x_1 (x_1 a_n) a_n^{m-2} \\ &< x_1 (a_n x_1) a_n^{m-2} = x_1 a_n (x_1 a_n^{m-2}) \\ &< x_1 a_n (a_n^{m-3} a_1^2) = (x_1 a_n^{m-2}) a_1^2 \\ &< (y_1 x_1 a_1^{m-3}) a_1^2 = y_1 x_1 a_1^{m-1}, \text{ a contradiction.} \end{aligned}$$

We have proved the first assertion.

To show the second assertion, suppose that $\leq$ coincides with $\leq_{\text{lex}}$ on $X^{[2]}$. It suffices to show that $\leq$ coincides with $\leq_{\text{lex}}$ on $X^{[2]}$. Since in the above argument, m could be 3 until we showed the claim that $x_1 a_n^{m-1} < y_1 a_1^{m-1}$, we need only show the claim for $m = 3$, i.e., that $x_1 a_n^2 < y_1 a_1^2$. Suppose that $y_1 a_1^2 < x_1 a_n^2$. Then, we have that $y_1 a_n a_1^2 < x_1 a_n^3$. But, we have, if $x_1 > a_1$,

$$\begin{aligned} x_1 a_n^3 &= (x_1 a_n) a_n^2 \\ &< (y_1 a_1) a_n^2 = y_1 (a_1 a_n) a_n \\ &< y_1 (x_1 a_1) a_n = y_1 x_1 (a_1 a_n) \\ &< y_1 x_1 (a_n a_1) = y_1 (x_1 a_n) a_1 \\ &< y_1 (a_n x_1) a_1 = y_1 a_n a_1^2, \quad \text{a contradiction,} \end{aligned}$$

and if $x_1 = a_1$,

$$\begin{aligned} x_1 a_n^3 &= (x_1 a_n) a_n^2 \\ &< (a_2 a_1) a_n^2 = a_2 (a_1 a_n) a_n \\ &< a_2 (a_2 a_1) a_n = a_2^2 (a_1 a_n) \\ &< a_2^2 (a_n a_1) = a_2 (a_2 a_n) a_1 \\ &< a_2 (a_n a_1) a_1 \leq y_1 a_n a_1^2, \quad \text{again a contradiction.} \end{aligned}$$

We have proved the proposition.

An entirely similar proof shows the following:

Proposition 2. Suppose that $\leq$ is a strongly compatible total order

on X^* . If $\leq$ coincides with $\leq_{al}$ on $X^{[3]}$, then $\leq$ in fact equals $\leq_{al}$ on the entire X^* .

In particular, if $\text{Card}(X) \geq 3$, then $X^{[3]}$ above can be replaced by $X^{[2]}$.

From now on until the end of the proof of Theorem 7, we restrict our attention to orders on X^* with $\text{Card}(X) = 2$. We define four orders other than $<_{lex}$ or $<_{al}$ and will show that these four orders together with $<_{lex}$ and $<_{al}$ exhaust all the strongly compatible total orders on X^* when $\text{Card}(X) = 2$.

Let $c \in X$. For $x \in X^*$, the number of c occurring in x is denoted by $|x|_c$. We consider the following conditions on an order $\leq$ on $X^* = \{a, b\}^*$ ($a < b$): for $x, y \in X^*$ with $|x| = |y|$,

- (a) if $|x|_b < |y|_b$, then $x < y$,
- (b) if $|x|_b = |y|_b$ and $x <_{lex} y$, then $x < y$,
- (c) if $|x|_b = |y|_b$ and $y <_{al} x$, then $x < y$,
- (b') if $|x|_b = |y|_b$ and $x <_{al} y$, then $x < y$,
- (c') if $|x|_b = |y|_b$ and $y <_{lex} x$, then $x < y$.

It is clear that an order on X^* satisfying (a) and one of (b), (c), (b'), (c') for all $x, y \in X^*$ is a strongly compatible total order on X^* . The next four lemmas show that similar assertions to Proposition 1,2 are true for these four orders.

Lemma 3. Let $X = \{a, b\}$ with $a < b$. Suppose that $\leq$ is a strongly compatible total order on X^* . Then, if $\leq$ coincides with the ordering defined by the conditions (a) and (b) on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 4 Let $X, \leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (c) on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 5. Let $X, \leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (b') on $X^{[4]}$, then they coincide on the entire X^* .

Lemma 6. Let $X, \leq$ be as in the lemma 3. Then, If $\leq$ coincides with the ordering defined by the conditions (a) and (c') on $X^{[4]}$, then they coincide on the entire X^* .

Because the proofs of these lemmas are similar, we exhibit only

the proof of Lemma 3.

Proof of Lemma 3.

We show by induction on the length of a word. Suppose that on $X^{[m-1]}$, $\leq$ coincides with the order defined by the conditions (a) and (b) ($m \geq 5$). Let $x, y \in X^m$ be $x = x_1x_2 \cdots x_m$ and $y = y_1y_2 \cdots y_m$. We have to show that if either $|x|_b = |y|_b$ and $x <_{\text{lex}} y$, or $|x|_b < |y|_b$, then $x < y$.

Case 1 : $|x|_b = |y|_b = k$ and $x <_{\text{lex}} y$. If $x_1 \cdots x_{i-1} = y_1 \cdots y_{i-1}$ and $x_i < y_i$ ($2 \leq i \leq m$), then $|x_i \cdots x_m|_b = |y_i \cdots y_m|_b$ and $x_i \cdots x_m <_{\text{lex}} y_i \cdots y_m$, so that by the induction hypothesis, $x_i \cdots x_m < y_i \cdots y_m$. By the compatibility of $\leq$, $x = x_1 \cdots x_{i-1} x_i \cdots x_m < y_1 \cdots y_{i-1} y_i \cdots y_m = y$. From this fact, we can assume that x, y do not have a common initial segment, and obtain that $x < ab^k a^{m-k-1}$ and $ba^{m-k} b^{k-1} < y$. Therefore, it suffices to show that $ab^k a^{m-k-1} < ba^{m-k} b^{k-1}$. If $2k > m$, then we have :

$$|ab^{m-k} a^{m-k-1}| = |ba^{m-k} b^{m-k-1}| = 2m - 2k < m.$$

Since $|ab^{m-k} a^{m-k-1}|_b = |ba^{m-k} b^{m-k-1}|_b = m - k$ and $ab^{m-k} a^{m-k-1} <_{\text{lex}} ba^{m-k} b^{m-k-1}$, we have that $ab^{m-k} a^{m-k-1} < ba^{m-k} b^{m-k-1}$. Then, from the strong compatibility of $\leq$, we have :

$$ab^k a^{m-k-1} = (ab^{m-k}) b^{2k-m} a^{m-k-1} < (ba^{m-k}) b^{2k-m} b^{m-k-1} = ba^{m-k} b^{k-1}.$$

If $2k < m$, then we similarly obtain that $ab^k a^{k-1} < ba^k b^{k-1}$, so that

$$ab^k a^{m-k-1} = (ab^k) a^{m-2k} a^{k-1} < (ba^k) a^{m-2k} b^{k-1} = ba^{m-k} b^{k-1}.$$

In the case of $2k = m$, suppose that $ba^{m-k} b^{k-1} < ab^k a^{m-k-1}$. Then, $ba^k b^{k-1} < ab^k a^{k-1}$, so that $ba^{k+1} b^{k-1} < a^2 b^k a^{k-1}$. On the other hand, we have :

$$\begin{aligned} a^2 b^k a^{k-1} &= a(ab^{k-2} b^2 a) a^{k-2} \\ &< a(bb^{k-2} a^2 b) a^{k-2} \quad (\text{since } abba < baab) \\ &= (ab^{k-1})(a^2 b) a^{k-2} \\ &< (ba^{k-1})(a^2 b) b^{k-2} \quad (\text{since } ab^{k-1} a^{k-2} < ba^{k-1} b^{k-2}) \\ &= ba^{k+1} b^{k-1}, \quad \text{a contradiction.} \end{aligned}$$

Case 2 : $|x|_b < |y|_b$. Put $|x|_b = s$ and $|y|_b = t$, with $s < t$. If $s+t < m$, then $x_i = y_i = a$ for some i ($1 \leq i \leq m$), and if $s+t > m$, then $x_i = y_i = b$ for some i ($1 \leq i \leq m$). In the both cases, we have :

$$\begin{aligned} |x_1 \cdots x_{i-1} x_{i+1} \cdots x_m|_b &< |y_1 \cdots y_{i-1} y_{i+1} \cdots y_m|_b, \quad \text{so that} \\ x_1 \cdots x_{i-1} x_{i+1} \cdots x_m &< y_1 \cdots y_{i-1} y_{i+1} \cdots y_m. \end{aligned}$$

Thus, $x = x_1 \cdots x_{i-1} x_i x_{i+1} \cdots x_m < y_1 \cdots y_{i-1} y_i y_{i+1} \cdots y_m = y$.

So, suppose that $s+t = m$. Then, from the case 1, we obtain that $x \leq b^s a^t$ and $a^s b^t \leq y$. Thus, it suffices to show that $b^s a^t <$

$a^s b^t$. If $t-s \geq 2$, then since $b^s a^{t-1} < a^s b^{t-1}$ by induction, we have $b^s a^t < a^s b^t$. Hence, let $t = s+1$ (so $m = 2s+1$), and assume on the contrary to our purpose that $a^s b^{s+1} < b^s a^{s+1}$. Then, $a^s a^s b b^{s+1} < a^s b^s b a^{s+1}$. But, on the other hand, we have that

$$\begin{aligned} a^s b^s b a^{s+1} &= (a^s b^{s+1}) a^{s+1} \\ &< (b^s a^{s+1}) a^{s+1} \\ &= (b^{s-1}) b a^{s-1} (a^{s+1}) a^2 \\ &< (a^{s-1}) b a^{s-1} (a b^s) a^2 \quad (\text{since } b^{s-1} a^{s+1} < a^{s-1} a b^s) \\ &< (a^{s-1}) a a^{s-1} (a b^s) b^2 \quad (\text{since } b a^{s-1} a^2 < a a^{s-1} b^2) \text{ ---} (*) \\ &= a^s a^s b b^{s+1}, \quad \text{a contradiction.} \end{aligned}$$

At (*), we can use the induction hypothesis since $|b a^{s-1} a^2| = s+2 < 2s+1 (= m)$ by our hypothesis that $m \geq 5$. We have proved the lemma.

Now, our main theorem:

Theorem 7. Let $X = \{a, b\}$ with $a < b$. Suppose that $\leq$ is a strongly compatible total order on X^* . Then, $\leq$ must be one of the following six orders on X^* :

- (1) the lexicographic order,
- (2) the anti-lexicographic order,
- (3)~(6) one which satisfies the condition (a) and one of the conditions (b), (c), (b'), (c').

Proof. For the sake of Proposition 1-2 and Lemma 3-6, it suffices to check that if $\leq$ is a strongly compatible total order on $\{a, b\}^*$, then it coincides with $<_{\text{lex}}$ or $<_{\text{al}}$ on $X^{[3]}$, or with one of the other four orders on $X^{[4]}$. So, let $\leq$ be a strongly compatible total order on $\{a, b\}^*$. Then, since $a < b$, by the compatibility of $\leq$, there are only two possibilities on X^2 :

- I. $aa < ab < ba < bb$, and
- II. $aa < ba < ab < bb$.

Suppose that the possibility I is the case. Then, the strong compatibility of $\leq$ imposes on the ordering on X^3 that $aaa < aab < aba < (abb \text{ or } baa) < bab < bba < bbb$. The strong compatibility alone can not determine if $abb < baa$ or $baa < abb$. Therefore, we get two possibilities:

- I-(i). $aaa < aab < aba < abb < baa < bab < bba < bbb$, or
- I-(ii). $aaa < aab < aba < baa < abb < bab < bba < bbb$.

If I-(i) is the case, then $\leq$ coincides with $<_{\text{lex}}$ on $X^{[3]}$, and we are done. So, suppose that I-(ii) is the case. Then, the strong compatibility of $\leq$ imposes on the ordering on X^4 that

$$aaaa < aaab < aaba < abaa < baaa < aabb < abab <$$

$< (\text{abba or baab}) <$

$< \text{baba} < \text{bbaa} < \text{abbb} < \text{babb} < \text{bbab} < \text{bbba} < \text{bbbb}.$

Here, again the strong compatibility fails to determine if $\text{abba} < \text{baab}$ or $\text{baab} < \text{abba}$. If $\text{abba} < \text{baab}$, then $\leq$ coincides on $X^{[4]}$ with the order defined by the conditions (a) and (b), and if $\text{baab} < \text{abba}$, then $\leq$ coincides on $X^{[4]}$ with the order defined by the conditions (a) and (c). Hence, we have shown the case I above. The case II can be treated similarly.

When $\text{Card}(X) \geq 3$, a description of the strongly compatible total orders becomes much more complicated even when $\text{Card}(X) = 3$. We are going to show that there are infinitely many strongly compatible total orders on X^* when $\text{Card}(X) \geq 3$.

Let $X = \{a_1, a_2, \dots, a_n\}$ ($n \geq 3$). Let $f : X^* \rightarrow \mathbf{R}$ be a mapping from X^* into the set of real numbers satisfying :

(1) for $a_i \in X$, $f(a_1) \leq f(a_2) \leq \dots \leq f(a_n)$,

(2) for $x = x_1 x_2 \dots x_m \in X^*$, where $x_i \in X$ ($i = 1, \dots, m$), $f(x) =$

$$\sum_{i=1}^m f(x_i).$$

Then, let $\leq_f$ be the order on X^* defined by:

(i) if $|x| < |y|$, then $x \leq_f y$,

(ii) if $|x| = |y|$ and $f(x) < f(y)$, then $x \leq_f y$,

(iii) if $|x| = |y|$, $f(x) = f(y)$ and $x <_{\text{lex}} y$ then $x \leq_f y$.

Then, it is clear that $\leq_f$ is a strongly compatible total order on X^* for every f satisfying (1) and (2) above.

Proposition 8. Let $X = \{a_1, a_2, \dots, a_n\}$ ($n \geq 3$). Then, the cardinality of the set of all strongly compatible total orders on X^* is that of the set of real numbers.

Proof. Let a, b, c be three distinct letters in X with $a < b < c$. For each real number $s \in [0, 1]$, let f_s be a map : $X^* \rightarrow \mathbf{R}$ satisfying (1), (2) above and such that $f_s(a) = 0$, $f_s(b) = s$, and $f_s(c) = 1$. Denote $\leq_{f_s}$ by $\leq_s$. We claim that if $0 < s < t < 1$, then $\leq_s$ and $\leq_t$ are not the same orders, which will show the theorem. To show the claim, let p, q be natural numbers such that $s < \frac{q}{p} < t$ (hence, $ps < q$

$< pt$). Consider the words $x = b^p$ and $y = c^q a^{p-q}$. Since $f_s(x) = ps < q = f_s(y)$, $x <_s y$. On the other hand, since $f_t(x) = pt > q = f_t(y)$, $y <_t x$. Hence, $\leq_s$ and $\leq_t$ are not the same.

Finally, we answer the following question :

Q. Does there exist a compatible total order which is not strongly compatible?

The answer is 'positive'. To show an example, we modify the above $\leq_f$ so that an evaluation of a word also counts a weight on the 'location' of each letter in the word.

Let the alphabet $X = \{ a_1, a_2, \dots, a_n \}$. As before, f puts weight (a real number) on each letter:

$$f(a_1) < f(a_2) < \dots < f(a_n).$$

Let s be a positive real. Then, f_s evaluates a word $x = x_1 x_2 \dots x_m$ (where $x_i \in X$) by :

$$f_s(x) = \sum_{i=1}^m f(x_i) s^{i-1}.$$

Then, let $\leq_f^s$ be the order on X^* defined by:

- (i) if $|x| < |y|$, then $x <_f^s y$,
- (ii) if $|x| = |y|$ and $f_s(x) < f_s(y)$, then $x <_f^s y$.

It is clear that if s is a transcendental real, then $\leq_f^s$ is a total order. To see that $\leq_f^s$ is compatible, it suffices to observe that $f_s(uv) = f_s(u) + f_s(v)s^{|u|}$ for $u, v \in X^*$. But, for some s , $\leq_f^s$ is not strongly compatible. For example, let s be a transcendental number in the interval $(1, \frac{1+\sqrt{5}}{2})$, and let $f(a) = 0$, $f(b) = 1$ for $a, b \in X$.

Then, it is easy to see that $s^2 < s+1$ and $aab <_f^s bba$. If $\leq_f^s$ were strongly compatible, then $aaa^{k-2}b <_f^s bba^{k-2}a$ for every k . But, for a sufficiently large k , $f_s(aaa^{k-2}b) = s^k > s+1 = f_s(bba^{k-2}a)$ and $bba^{k-2}a <_f^s aaa^{k-2}b$. Hence, this $\leq_f^s$ is not strongly compatible.

Appendix.

In this appendix, we present some complimentary results on $\leq_f^s$ -orders in the case that the alphabet $X = \{a, b\}$.

First of all, we get rid of an unnecessary complication on f .

Suppose that we are given an order $\leq_f^s$ on $\{a,b\}^*$ for some f and (a transcendental) s . Then, for $x = x_1 x_2 \cdots x_m$, (where $x_i \in \{a,b\}$), we have:

$$f_s(x) = \sum_{i=1}^m f(x_i) s^{i-1} = (f(b) - f(a)) \cdot \left[\sum_{i=1}^m \delta_i(x) s^{i-1} \right] + \sum_{i=1}^m f(a) s^{i-1},$$

where $\delta_i(x) = 1$ if $x_i = b$, else 0. From this, we conclude that an order $\leq_f^s$ on $\{a,b\}^*$ with $f(a) < f(b)$ is equivalent to the order $\leq_g^s$ with $g(a) = 0$ and $g(b) = 1$. Hence, in the followings, we consider only orders $\leq_f^s$ with $f(a) = 0$ and $f(b) = 1$, and omit the subscript f as $\leq^s$.

Proposition A. For $s \geq 2$ (resp. $0 < s \leq \frac{1}{2}$), $\leq^s$ coincides with $\leq_{al}$ (resp. $\leq_{lex}$).

Proof. We only prove the case for $s \geq 2$. It suffices to show that for every $k \geq 1$, $1 + s + s^2 + \cdots + s^{k-1} < s^k$ because L.H.S. = $f_s(b^k a)$ and R.H.S. = $f_s(a^k b)$. But, since $s \geq 2$, $\frac{s^k - 1}{s - 1} \leq s^{k-1}$, i.e., $\frac{s^k - 1}{s - 1} < s^k$, i.e., $1 + s + s^2 + \cdots + s^{k-1} < s^k$.

Proposition B. For $s \in (\frac{1}{2}, 1) \cup (1, 2)$, $\leq^s$ is not strongly compatible.

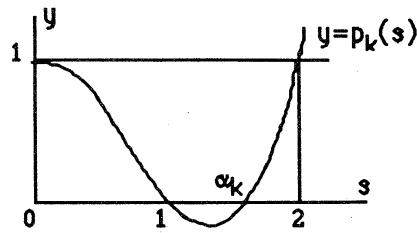
Proof. We prove only the case for $s \in (1, 2)$. We first show that $\leq^s \neq \leq_{al}$, for which it suffices to prove that there is a $k \geq 1$ such that $s^k < 1 + s + s^2 + \cdots + s^{k-1}$ since then $a^k b <^s b^k a$. But, if $s \in (1, 2)$, then for a sufficiently large k , it holds that $1 < (2-s)s^k$, i.e., $1 - s^k < (1-s)s^k$, i.e., $\frac{s^k - 1}{s - 1} > s^k$, i.e., $1 + s + s^2 + \cdots + s^{k-1} > s^k$. On the other hand, there is an m such that $1 + s + s^2 + \cdots + s^{k-1} < s^{k+m}$. For such an m , we have that $a^k a^m b^s > b^k a^m a$. Hence, $\leq^s$ is not strongly compatible.

Put for each $k \geq 2$, $p_k(s) = s^{k+1} - 2s^k + 1$. Then, as in the above proof, for each $k \geq 2$, and for each $s \in (1, 2)$,

$$p_k(s) > 0 \quad \text{iff} \quad a^k b <^s b^k a$$

$$(<) \qquad \qquad \qquad (s>)$$

For each $k \geq 2$, since $p_k(s) = s^k \{(k+1)s - 2\}$, it is easy to see that there is a unique root α_k of $p_k(s) = 0$ in the interval $(1, 2)$.



Then, $\{\alpha_k\}_{k \geq 2}$ is a strictly increasing sequence (converging to 2).

For, if $2 \leq k < m$, then $p_m(\alpha_k) = \alpha_k^{m+1} - 2\alpha_k^m + 1 = \alpha_k^{m-k}(\alpha_k^{k+1} -$

$2\alpha_k^k + 1) + 1 - \alpha_k^{m-k} = 1 - \alpha_k^{m-k} < 0$, which means that $\alpha_k < \alpha_m$.

The convergence is proved by a similar argument to the proof of Proposition B.

Proposition C. There are at least countably many compatible total orders on $\{a, b\}^*$.

Proof. Let $2 \leq k < m$ and $s \in (\alpha_k, \alpha_{k+1})$, $t \in (\alpha_m, \alpha_{m+1})$. Then, it suffices to show that $<^s$ and $<^t$ are distinct orders. Since $s < \alpha_{k+1} \leq \alpha_m$, $p_m(s) < 0$, i.e., $b^m a <^s a^m b$. But, since $\alpha_m < t$, $p_m(t) > 0$, i.e., $a^m b <^t b^m a$. Hence, $<^t \neq <^s$.

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