

A DE LA VALLÉE POUSSIN TYPE THEOREM IN BANACH SPACES

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1. Introduction

Let $C_{2\pi}$ denote the Banach space of all 2π -periodic, continuous functions f on the real line $\mathbb{R}$ with the norm

$$\|f\|_{\infty} = \max\{|f(t)| : |t| \leq \pi\}.$$

Let $\mathbb{N}$ be the set of all positive integers, and put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For each $n \in \mathbb{N}_0$, we denote by $\mathfrak{T}_n$ the set of all trigonometric polynomials of degree at most n . For a given $f \in C_{2\pi}$, we define

$$E_n(C_{2\pi}; f) = \inf\{\|f - g\|_{\infty} : g \in \mathfrak{T}_n\},$$

which is called the best approximation of degree n to f with respect to $\mathfrak{T}_n$.

Let $a \in \mathbb{N}, a \geq 2$ and let $\Omega \neq 0$ be a non-negative, monotone decreasing function on $[a, \infty)$ satisfying the conditions

$$\lim_{x \rightarrow \infty} \Omega(x) = 0 \tag{1}$$

and

$$\int_a^{\infty} \frac{\Omega(x)}{x} dx < \infty. \tag{2}$$

Then the classical theorem of de la Vallée Poussin states that: Let $f \in C_{2\pi}$ and $r \in \mathbb{N}_0$. If

$$E_n(C_{2\pi}; f) = O\left(\frac{\Omega(n)}{n^r}\right) \quad (n \rightarrow \infty),$$

then f is r -times continuously differentiable on $\mathbb{R}$ and

$$\omega(C_{2\pi}; f^{(r)}, \delta) = O\left(\delta \int_a^{a/\delta} \Omega(x) dx + \int_{1/\delta}^\infty \frac{\Omega(x)}{x} dx\right) \quad (\delta \rightarrow +0),$$

where

$$\omega(C_{2\pi}; f^{(r)}, \delta) = \sup\{\|f^{(r)}(\cdot - t) - f^{(r)}(\cdot)\|_\infty : |t| \leq \delta\}$$

denotes the modulus of continuity of $f^{(r)}$ (cf. [5]).

A statement analogous to this result also holds for the Banach space $L_{2\pi}^p$ consisting of all 2π -periodic, p -power Lebesgue integrable functions f on $\mathbb{R}$ with the norm

$$\|f\|_p = \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t)|^p dt\right)^{1/p} \quad (1 \leq p < \infty)$$

using the integral modulus of continuity (cf. [17]). Furthermore, in [3] these results were generalized by means of the higher order moduli of continuity and yielded the inverse theorems of Bernstein-type on the degree of the best approximation with respect to $\mathfrak{T}_n$ (cf. [2], [9], [10], [20]).

The purpose of this paper is to extend the above-mentioned results to arbitrary Banach spaces, and in particular, homogeneous Banach spaces (cf. [8], [11], [18]) which include $C_{2\pi}$ and $L_{2\pi}^p$, $1 \leq p < \infty$, as particular cases. For this purpose, we consider the following setting:

Let X be a complex Banach space with norm $\|\cdot\|_X$, and let $B[X]$ denote the Banach algebra of all bounded linear operators of X into itself with the usual operator norm $\|\cdot\|_{B[X]}$. Let $\mathbb{Z}$ denote the set of all integers, and let $\{P_j : j \in \mathbb{Z}\}$ be a sequence of projection operators in $B[X]$ satisfying the following conditions:

- (P-1) The projections $P_j, j \in \mathbb{Z}$, are mutually orthogonal, i.e., $P_j P_n = \delta_{j,n} P_n$ for all $j, n \in \mathbb{Z}$, where $\delta_{j,n}$ denotes Kronecker's symbol.
- (P-2) $\{P_j : j \in \mathbb{Z}\}$ is fundamental, i.e., the linear span of $\cup_{j \in \mathbb{Z}} P_j(X)$ is dense in X .
- (P-3) $\{P_j : j \in \mathbb{Z}\}$ is total, i.e., if $f \in X$ and $P_j(f) = 0$ for all $j \in \mathbb{Z}$, then $f = 0$.

For each $n \in \mathbb{N}_0$, let M_n be the linear span of $\{P_j(X) : |j| \leq n\}$, which is a closed linear subspace of X . For a given $f \in X$, we define

$$E_n(X; f) = \inf\{\|f - g\|_X : g \in M_n\},$$

which is called the best approximation of degree n to f with respect to M_n . Clearly,

$$E_0(X; f) \geq E_1(X; f) \geq \cdots \geq E_n(X; f) \geq E_{n+1}(X; f) \geq \cdots \geq 0,$$

and Condition (P-2) implies that

$$\lim_{n \rightarrow \infty} E_n(X; f) = 0 \quad \text{for every } f \in X.$$

In this paper, we derive certain smoothness properties of an element $f \in X$ from the hypothesis that $\{E_n(X; f) : n \in \mathbb{N}_0\}$ tends to zero with a given rapidity. We refer to [16] for detailed treatments and [15] (cf. [13], [14]) for the study of the direct theorems of Jackson-type (cf. [7]) which estimates the magnitude of $E_n(X; f)$ in terms of the moduli of continuity of higher orders of f with respect to a strongly continuous group of multiplier operators on X associated with Fourier series expansions corresponding to $\{P_j : j \in \mathbb{Z}\}$.

2. Moduli of continuity and Bernstein-type inequality

For any $f \in X$, we associate its (formal) Fourier series expansion

with respect to $\{P_j : j \in \mathbb{Z}\}$

$$f \sim \sum_{j=-\infty}^{\infty} P_j(f).$$

An operator $T \in B[X]$ is called a multiplier operator on X if there exists a sequence $\{\tau_j : j \in \mathbb{Z}\}$ of complex numbers such that for every $f \in X$,

$$T(f) \sim \sum_{j=-\infty}^{\infty} \tau_j P_j(f),$$

and the following notation is used:

$$T \sim \sum_{j=-\infty}^{\infty} \tau_j P_j \quad (3)$$

(cf. [4], [11], [12], [21]).

Let $M[X]$ denote the set of all multiplier operators on X , which is a commutative closed subalgebra of $B[X]$ containing the identity operator I . Let $\{T_t : t \in \mathbb{R}\}$ be a family of operators in $M[X]$ satisfying

$$\|T_t\|_{B[X]} \leq 1 \quad \text{for all } t \in \mathbb{R} \quad (4)$$

and having the expansions

$$T_t \sim \sum_{j=-\infty}^{\infty} e^{-ijt} P_j \quad (t \in \mathbb{R}).$$

Then $\{T_t : t \in \mathbb{R}\}$ becomes a strongly continuous group of operators in $B[X]$ and we have

$$G^r(P_j(g)) = (-ij)^r P_j(g) \quad (j \in \mathbb{Z}, g \in X, r \in \mathbb{N}) \quad (5)$$

and

$$G^r(f) \sim \sum_{j=-\infty}^{\infty} (-ij)^r P_j(f) \quad (f \in D(G^r), r \in \mathbb{N}),$$

where G is the infinitesimal generator of $\{T_t : t \in \mathbb{R}\}$ with domain $D(G)$ (cf. [11; Proposition 2]). For the basic theory of semigroups of operators on Banach spaces we refer to [1] and [6].

For each $r \in \mathbb{N}_0$ and $t \in \mathbb{R}$, we define

$$\Delta_t^0 = I, \quad \Delta_t^r = (T_t - I)^r = \sum_{m=0}^r (-1)^{r-m} \binom{r}{m} T_{mt} \quad (r \geq 1),$$

which stands for the r -th iteration of $T_t - I$. Then Δ_t^r belongs to $M[X]$ and

$$\|\Delta_t^r\|_{B[X]} \leq 2^r, \quad \Delta_t^r \sim \sum_{j=-\infty}^{\infty} (e^{-ijt} - 1)^r P_j.$$

If $r \in \mathbb{N}_0$, $f \in X$ and $\delta \geq 0$, then we define

$$\omega_r(X; f, \delta) = \sup\{\|\Delta_t^r(f)\|_X : |t| \leq \delta\},$$

which is called the r -th modulus of continuity of f with respect to $\{T_t : t \in \mathbb{R}\}$. This quantity has the following properties ([15; Lemma 1]):

Lemma 1. *Let $r \in \mathbb{N}$ and $f \in X$*

(a)

$$\omega_r(X; f, \delta) \leq 2^r \|f\|_X \quad (\delta \geq 0).$$

(b) $\omega_r(X; f, \cdot)$ is a non-decreasing function defined on $[0, \infty)$ and $\omega_r(X; f, 0) = 0$.

(c)

$$\omega_{r+s}(X; f, \delta) \leq 2^r \omega_s(X; f, \delta) \quad (s \in \mathbb{N}_0, \delta \geq 0).$$

In particular, we have

$$\lim_{\delta \rightarrow +0} \omega_r(X; f, \delta) = 0.$$

(d)

$$\omega_r(X; f, \xi\delta) \leq (1 + \xi)^r \omega_r(X; f, \delta) \quad (\xi, \delta \geq 0).$$

(e) *If $0 < \delta \leq \xi$, then*

$$\omega_r(X; f, \xi)/\xi^r \leq 2^r \omega_r(X; f, \delta)/\delta^r.$$

(f) *If $f \in D(G^r)$, then*

$$\omega_{r+s}(X; f, \delta) \leq \delta^r \omega_s(X; G^r(f), \delta) \quad (s \in \mathbb{N}_0, \delta \geq 0).$$

(g) $\omega_r(X; \cdot, \delta)$ is a seminorm on X .

If k is a function in $L^1_{2\pi}$ having the Fourier series expansion

$$k(t) \sim \sum_{j=-\infty}^{\infty} \hat{k}(j) e^{ijt}$$

with its Fourier coefficients

$$\hat{k}(j) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(t) e^{-ijt} dt \quad (j \in \mathbb{Z})$$

and if $T \in B[X]$, then we define the convolution operator $k * T$ by

$$(k * T)(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} k(t) T_t(T(f)) dt \quad (f \in X),$$

which exists as a Bochner integral (cf. [11]). Obviously, $k * T$ belongs to $B[X]$ and

$$\|k * T\|_{B[X]} \leq \|k\|_1 \|T\|_{B[X]} \quad (6)$$

because of (4). In particular, if T is an operator in $M[X]$ having the expansion (3), then $k * T \in M[X]$ and there holds

$$k * T \sim \sum_{j=-\infty}^{\infty} \hat{k}(j) \tau_j P_j, \quad (7)$$

which is an immediate consequence of [15; Lemma 2].

Now we need the following Bernstein-type inequality in order to prove the main theorem.

Lemma 2. *Let $n \in \mathbb{N}_0$ and $r \in \mathbb{N}$. Then*

$$\|G^r(f)\|_X \leq (2n)^r \|f\|_X \quad (8)$$

for all $f \in M_n$.

Proof. By (5), (8) is trivial for $n = 0$. Let $n \in \mathbb{N}$, and let $k_n(t) = 2nF_n(t) \sin nt$, where

$$F_n(t) = 1 + 2 \sum_{j=1}^n \left(1 - \frac{j}{n}\right) \cos jt = \frac{1}{n} \left(\frac{\sin(nt/2)}{\sin(t/2)} \right)^2$$

is the Fejér kernel. Since

$$f = \sum_{j=-n}^n P_j(f), \quad (9)$$

(5) implies $f \in D(G^r)$ and

$$G^r(f) = \sum_{j=-n}^n G^r(P_j(f)) = \sum_{j=-n}^n (-ij)^r P_j(f). \quad (10)$$

Since

$$\frac{i}{n} k_n(t) = F_n(t) e^{int} - F_n(t) e^{-int},$$

we obtain

$$\frac{i}{n} \hat{k}_n(j) = \frac{j}{n} \quad (|j| \leq n).$$

Therefore, in view of (7), (9) and (10), we have

$$\begin{aligned} (k_n * I)(f) &= \sum_{j=-n}^n P_j((k_n * I)(f)) = \sum_{j=-n}^n \hat{k}_n(j) P_j(f) \\ &= \sum_{j=-n}^n (-ij) P_j(f) = G(f), \end{aligned}$$

and so (6) yields

$$\|G(f)\|_X = \|(k_n * I)(f)\|_X \leq \|k_n\|_1 \|f\|_X \leq 2n \|f\|_X.$$

Thus (8) follows from induction on r . $\square$

3. The main theorem

Recall that M_n is the closed linear subspace of X spanned by $\{P_j(X) : |j| \leq n\}$. Here we suppose that for each given $f \in X$ and each $n \in \mathbb{N}_0$, there exists an element $f_n \in M_n$ of the best approximation of f with respect to M_n , i.e.,

$$E_n(X; f) = \|f - f_n\|_X. \quad (11)$$

Remark 1. *If the dimension of M_n is finite, then for any $f \in X$ there exists an element of the best approximation of f with respect to M_n . In particular, if $\{\varphi_j, \varphi_j^*\}_{j \in \mathbb{Z}}$ is a fundamental, total, biorthogonal system (cf. [11; Remark 8]) and if M_n is the linear span of $\{\varphi_j : |j| \leq n\}$, then for every $f \in X$ there exists an element of the best approximation of f with*

respect to M_n . Also, if X is a Hilbert space, then for each $f \in X$ there exists a unique element of the best approximation of f with respect to M_n . For the general theory of the best approximation in normed linear spaces, we refer to [19].

Theorem 1. Let Ω be as in Section 1. Let $f \in X$ and $r \in \mathbb{N}_0$. If

$$E_n(X; f) = O\left(\frac{\Omega(n)}{n}\right) \quad (n \rightarrow \infty), \quad (12)$$

then $f \in D(G^r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(X; G^r(f), \delta) = O\left(\delta^k \int_a^{a/\delta} x^{k-1} \Omega(x) dx + \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x} dx\right) \quad (\delta \rightarrow +0). \quad (13)$$

Proof. Let f_n be an element of the best approximation of f with respect to M_n . Then by (11) and (12), we have

$$\|f - f_{a^n}\|_X \leq C_1 a^{-nr} \Omega(a^n) \quad (n \geq 1), \quad (14)$$

where C_1 is a positive constant independent of n . Put

$$g_2 = f_{a^2}, \quad g_n = f_{a^n} - f_{a^{n-1}} \quad (n \geq 3). \quad (15)$$

Then it follows from (14) that

$$\|g_n\|_X \leq \|f_{a^n} - f\|_X + \|f - f_{a^{n-1}}\|_X \leq C_1 \frac{(1 + a^r) \Omega(a^{n-1})}{a^{nr}} \quad (n \geq 3),$$

and so Lemma 2 yields

$$\|G^r(g_n)\|_X \leq 2^r C_1 (1 + a^r) \Omega(a^{n-1}) \quad (n \geq 3). \quad (16)$$

By (2), we have

$$\sum_{n=3}^{\infty} \Omega(a^{n-1}) \leq \frac{a}{a-1} \int_a^{\infty} \frac{\Omega(x)}{x} dx < \infty,$$

which together with (16) implies that there exists an element $g \in X$ such that

$$g = \sum_{n=2}^{\infty} G^r(g_n). \quad (17)$$

Also, (1), (14) and (15) imply that

$$f = \sum_{n=2}^{\infty} g_n. \quad (18)$$

Since G^r is a closed linear operator, it follows from (17) and (18) that $f \in D(G^r)$ and

$$G^r(f) = \sum_{n=2}^{\infty} G^r(g_n). \quad (19)$$

Therefore, by Lemma 1 (a) and (g), we have that for each $m \geq 2$,

$$\begin{aligned} \omega_k(X; G^r(f), \delta) &\leq \omega_k\left(X; \sum_{n=2}^m G^r(g_n), \delta\right) + \omega_k\left(X; \sum_{n=m+1}^{\infty} G^r(g_n), \delta\right) \\ &\leq \sum_{n=2}^m \omega_k(X; G^r(g_n), \delta) + 2^k \sum_{n=m+1}^{\infty} \|G^r(g_n)\|_X = I_1 + I_2, \end{aligned}$$

say. By (10), Lemma 1 (f), Lemma 2 and (16), we have

$$\begin{aligned} \omega_k(X; G^r(g_n), \delta) &\leq \delta^k \|G^k(G^r(g_n))\|_X \\ &\leq \delta^k (2a^n)^k \|G^r(g_n)\|_X \quad (n \geq 2) \\ &\leq \delta^k (2a^n)^k 2^r C_1 (1 + a^r) \Omega(a^{n-1}) \quad (n \geq 3). \end{aligned}$$

Thus we obtain that for each $m \geq 2$,

$$\begin{aligned} I_1 &\leq \delta^k (2a^2)^k \|G^r(g_2)\|_X + 2^{k+r} C_1 (1 + a^r) \delta^k \sum_{n=3}^m a^{kn} \Omega(a^{n-1}) \\ &\leq C_2 \delta^k \sum_{n=2}^m \left(a^{k(n-1)} - a^{k(n-1)-1} \right) \Omega(a^{n-1}), \end{aligned}$$

where C_2 is a positive constant independent of δ and m .

Now let $0 < \delta \leq 1/a$, and we choose $m \in \mathbb{N}, m \geq 3$ such that

$$a^{m-2} \leq \delta^{-1} < a^{m-1}.$$

First, we consider the case of $\Omega(a^2) = 0$. Then (14) implies $f = g_2 \in M_{a^2}$, and so (10), Lemma 1 (f) and Lemma 2 yield that

$$\omega_k(X; G^r(f), \delta) \leq (2a^2)^k \|G^r(f)\|_X \delta^k. \quad (20)$$

Also, we have

$$\int_a^{a/\delta} x^{k-1} \Omega(x) dx \geq \int_a^{a^{m-1}} x^{k-1} \Omega(x) dx \geq \int_a^{a^2} x^{k-1} \Omega(x) dx > 0,$$

which together with (20) clearly implies (13).

Next, let $\Omega(a^2) > 0$. Then we have

$$\begin{aligned} \sum_{n=2}^m \left(a^{k(n-1)} - a^{k(n-1)-1} \right) \Omega(a^{n-1}) &\leq (a^{2k} - a^{2k-1}) \Omega(a) \\ &+ \sum_{n=3}^m \int_{a^{k(n-1)-1}}^{a^{k(n-1)}} \Omega(x^{1/k}) dx \leq \frac{\Omega(a)}{\Omega(a^2)} \int_{a^{2k-1}}^{a^{2k}} \Omega(x^{1/k}) dx \\ &+ \int_{a^{2k-1}}^{a^{k(m-1)}} \Omega(x^{1/k}) dx \leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{a^{k(m-1)}} \Omega(x^{1/k}) dx \\ &\leq \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) \int_{a^k}^{(a/\delta)^k} \Omega(x^{1/k}) dx. \end{aligned}$$

Therefore putting $u = x^{1/k}$, we get

$$I_1 \leq C_2 \left(\frac{\Omega(a)}{\Omega(a^2)} + 1 \right) k \delta^k \int_a^{a/\delta} u^{k-1} \Omega(u) du.$$

Also, by (16) we have

$$\begin{aligned} I_2 &\leq 2^{k+r} C_1 (1 + a^r) \sum_{n=m+1}^{\infty} \Omega(a^{n-1}) \\ &\leq 2^{k+r} C_1 (1 + a^r) \frac{a}{a-1} \int_{a^{m-1}}^{\infty} \frac{\Omega(x)}{x} dx \\ &\leq 2^{k+r} C_1 (1 + a^r) \frac{a}{a-1} \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x} dx. \end{aligned}$$

Hence, we arrive at

$$\omega_k(X; G^r(f), \delta) \leq C_3 \left(\delta^k \int_a^{a/\delta} x^{k-1} \Omega(x) dx + \int_{1/\delta}^{\infty} \frac{\Omega(x)}{x} dx \right),$$

where C_3 is a positive constant independent of δ . This implies (13) and the proof of the theorem is now complete. $\square$

Applying Theorem 1 to the case where

$$\Omega(x) = \frac{1}{x^\alpha}, \quad \alpha > 0,$$

we have the following corollary.

Corollary 1. *Let $\alpha > 0$, $f \in X$ and $r \in \mathbb{N}_0$. If*

$$E_n(X; f) = O\left(\frac{1}{n^{r+\alpha}}\right) \quad (n \rightarrow \infty),$$

then $f \in D(G^r)$ and for every $k \in \mathbb{N}$,

$$\omega_k(X; G^r(f), \delta) = \begin{cases} O(\delta^\alpha) & (\alpha < k, \delta \rightarrow +0) \\ O(\delta^k |\log \delta|) & (\alpha = k, \delta \rightarrow +0) \\ O(\delta^k) & (\alpha > k, \delta \rightarrow +0). \end{cases}$$

In the remaining part of this section, we restrict ourselves to the case where X is a homogeneous Banach space, i.e., X satisfies the following properties:

- (H-1) X is a linear subspace of $L_{2\pi}^1$ with a norm $\|\cdot\|_X$ under which it is a Banach space.
- (H-2) X is continuously embedded in $L_{2\pi}^1$, i.e., there exists a constant $C > 0$ such that $\|f\|_1 \leq C\|f\|_X$ for all $f \in X$.
- (H-3) The translation operator T_t defined by

$$T_t(f)(\cdot) = f(\cdot - t) \quad (f \in X),$$

is isometric on X for each $t \in \mathbb{R}$.

- (H-4) For each $f \in X$, the mapping $t \mapsto T_t(f)$ is strongly continuous on $\mathbb{R}$.

Typical examples of homogeneous Banach spaces are $C_{2\pi}$ and $L_{2\pi}^p$, $1 \leq p < \infty$. For other examples see [11] (cf. [8], [18]).

Now we define the sequence $\{P_j : j \in \mathbb{Z}\}$ of projection operators in $B[X]$ by

$$P_j(f)(\cdot) = \hat{f}(j)e^{ij\cdot} \quad (f \in X),$$

which satisfies Conditions (P-1), (P-2) and (P-3) just as Section 1 (cf. [8], [11]). Notice that $M_n = \mathfrak{T}_n$ and for all $f \in X$ we have

$$\Delta_t^0(f) = f, \quad \Delta_t^r(f)(\cdot) = \sum_{m=0}^r (-1)^{r-m} \binom{r}{m} f(\cdot - mt) \quad (t \in \mathbb{R}, r \in \mathbb{N}).$$

Consequently, in the above setting all the results obtained in this paper hold, and so by Theorem 1, we have the inverse theorem of the generalized de la Vallée Poussin type (cf. [3], [5], [17]) in arbitrary homogeneous Banach spaces. Furthermore, for $r = 0$ and $0 < \alpha < 1$, Corollary 1 gives a generalization of [18; Theorem 9.4.5.1] in the context of the use of the k -th modulus of continuity $\omega_k(X; f, \delta)$. Also, for $k = 2$, Corollary 1 establishes the theorem of Zygmund type (cf. [22]) in arbitrary homogeneous Banach spaces.

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