

**Cubic hyper-resolutions of analytic varieties
with hypersurface ordinary singularities
of dimension ≤ 5**

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§1 Hypersurface ordinary singularities of dimension ≤ 5

Let $(Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ be a pure-dimensinal hypersurface germ and let $\nu : (X^n, \nu^{-1}(o)) \rightarrow (Y^n, o)$ be the normalization map. We define $f := \iota \circ \nu : (X^n, f^{-1}(o)) \rightarrow (\mathbb{C}^{n+1}, o)$, where $\iota : (Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ is the inclusion map.

1.1 Definition. we say (Y^n, o) is an *ordinary singularity* if

- (i) $(X^n, f^{-1}(o))$ is non-singular, and
- (ii) $f := \iota \circ \nu : (X^n, f^{-1}(o)) \rightarrow (\mathbb{C}^{n+1}, o)$ is *simultaneously stable*, i.e., small deformation of the multi-germ f of a holomorphic map is *trivial*.

For an ordinary singularity $(Y^n, o) \subset (\mathbb{C}^{n+1}, o)$ with $f := \iota \circ \nu : (X^n, f^{-1}(o)) \rightarrow (\mathbb{C}^{n+1}, o)$ being the same as above, we put

$$f^{-1}(o) := \{p_1, p_2, \dots, p_k\},$$

$$R(f)_{p_i} := \mathcal{O}_{X, p_i} / f^* \mathfrak{m}_o \cdot \mathcal{O}_{X, p_i}$$

($1 \leq i \leq k$, $\mathfrak{m}_o$ is the maximal ideal of $\mathcal{O}_{\mathbb{C}^{n+1}, o}$), and

$$C_i := \{q \in X^n | R(f)_q \simeq R(f)_{p_i}\} \text{ (contact class of } f \text{ at } p_i).$$

1.2 Proposition ([8, Proposition 7.1], [5]). $f := \iota \circ \nu : (X^n, f^{-1}(o)) \rightarrow (\mathbb{C}^{n+1}, o)$ is *simultaneously stable* iff both of the following conditions are satisfied:

- (i) $f_i := f|_{p_i} : (X^n, p_i) \rightarrow (\mathbb{C}^{n+1}, o)$ is *stable* for any i ($1 \leq i \leq k$),
- (ii) $(df)_{p_1}(T_{C_1, p_1}), \dots, (df)_{p_k}(T_{C_k, p_k})$ are in *genaral position* in $T_{\mathbb{C}^{n+1}, o}$, where T_{C_i, p_i} denotes the tangent space of C_i at p_i and so on.

1.3 Proposition ([7]). Let $f : (\mathbb{C}^n, o) \rightarrow (\mathbb{C}^m, o)$ be a holomorphic map germ. Assume that (i) $(n, m) \in \{\text{nice range}\}$ (cf. [6]), (ii) $n < m$, and (iii) $n \leq \frac{2}{3}m + 1$. Then f is *stable* iff $R(f)_o$ is isomorphic to one of the following $\mathbb{C}$ -algebras:

$$A_0 := \mathbb{C}[[x]]/(x), \quad A_1 := \mathbb{C}[[x]]/(x^2), \quad A_2 := \mathbb{C}[[x]]/(x^3).$$

When $n < m$, the normal forms of holomorphic maps f with $R(f)_o \simeq A_\ell$ ($0 \leq \ell \leq 2$) are given as follows (cf. [4]):

(i) In the case of $R(f)_o \simeq A_0$:

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n) \\ y_i \circ f = 0 & (n+1 \leq i \leq m), \end{cases}$$

(ii) In the case of $R(f)_o \simeq A_1$:

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n-1) \\ y_n \circ f = x_n^2 \\ y_{n+i} \circ f = x_i x_n & (1 \leq i \leq m-n \leq n-1), \end{cases}$$

(iii) In the case of $R(f)_o \simeq A_2$:

$$\begin{cases} y_i \circ f = x_i & (1 \leq i \leq n-1) \\ y_n \circ f = x_n^3 + x_1 x_n \\ y_{n+i} \circ f = x_{2i} x_n + x_{2i+1} x_n^2 & (1 \leq i \leq m-n, \ 2(m-n)+1 \leq n-1). \end{cases}$$

1.4 Remark. When $m = n+1$, (n, m) satisfies the conditions (i), (ii) and (iii) in Theorem 1.3 iff $1 \leq n \leq 5$, and the case (iii) above occurs only when $n = 4, 5$.

Using these facts, we can calculate the defining equations of ordinary singularities of dimension ≤ 5 .

1.5 Proposition([8]). The defining equations of hypersurface ordinary singularities of dimension $n \leq 5$ in $\mathbb{C}^{n+1}$ are given as follows:

i) $n = 1$:	ii) $n = 2$:	iii) $n = 3$:
a) ₁ $y_1 = 0$	a) _k $y_1 \cdots y_k = 0 \ (1 \leq k \leq 3)$	a) _k $y_1 \cdots y_k = 0 \ (1 \leq k \leq 4)$
a) ₂ $y_1 y_2 = 0$	b) $y_1^2 - y_2^2 y_3 = 0$	b) $y_1^2 - y_2^2 y_3 = 0$
		a) ₁ + b) $y_4(y_1^2 - y_2^2 y_3) = 0$

iii) $n = 4$:

a) _k $y_1 \cdots y_k = 0 \ (1 \leq k \leq 5)$
b) $y_1^2 - y_2^2 y_3 = 0$
a) ₁ + b) $y_4(y_1^2 - y_2^2 y_3) = 0$
a) ₂ + b) $y_4 y_5(y_1^2 - y_2^2 y_3) = 0$
c) $y_5^3 + 2y_1 y_3 y_5^2 + (y_1^2 y_3^2 - 3y_2 y_3 y_4 + y_1 y_2^2) y_5 - \{y_3^3 y_4 + y_2(y_2^2 + y_1 y_3^2)\} y_4 = 0$

iv) $n = 5$:

a) _k $y_1 \cdots y_k = 0 \ (1 \leq k \leq 6)$
b) $y_1^2 - y_2^2 y_3 = 0$
a) ₁ + b) $y_4(y_1^2 - y_2^2 y_3) = 0$
a) ₂ + b) $y_4 y_5(y_1^2 - y_2^2 y_3) = 0$
a) ₃ + b) $y_4 y_5 y_6(y_1^2 - y_2^2 y_3) = 0$
b) + b) $(y_1^2 - y_2^2 y_3)(y_4^2 - y_5^2 y_6) = 0$

$$\begin{aligned} c) \quad & y_5^3 + 2y_1y_3y_5^2 + (y_1^2y_3^2 - 3y_2y_3y_4 + y_1y_2^2)y_5 - \{y_3^3y_4 + y_2(y_2^2 + y_1y_3^2)\}y_4 = 0 \\ a)_1 + c) \quad & y_6[y_5^3 + 2y_1y_3y_5^2 + (y_1^2y_3^2 - 3y_2y_3y_4 + y_1y_2^2)y_5 - \{y_3^3y_4 + y_2(y_2^2 + y_1y_3^2)\}y_4] = 0 \end{aligned}$$

PROOF: For example, we shall show how the equation iii) c) follows. In the case of $n=4$, the normal form of a holomorphic map f with $R(f)_o \simeq A_2$ is given by

$$(1.1) \quad \begin{cases} y_i \circ f = x_i \quad (1 \leq i \leq 3) \\ y_4 \circ f = x_4^3 + x_1x_4 \\ y_5 \circ f = x_2x_4 + x_3x_4^2. \end{cases}$$

Substituting $x_1 = y_1, x_2 = y_2, x_3 = y_3$ into the last two equations, we have

$$\begin{cases} x_4^3 + y_1x_4 - y_4 = 0 \\ y_3x_4^2 + y_2x_4 - y_5 = 0. \end{cases}$$

We regard this as a simultaneous equation for x_4 with coefficients in the polynomial ring $\mathbb{C}[y_1, \dots, y_5]$. To eliminate x_4 , we calculate the *resultant*

$$\begin{vmatrix} 1 & 0 & y_1 & -y_4 & 0 \\ 0 & 1 & 0 & y_1 & -y_4 \\ y_3 & y_2 & -y_5 & 0 & 0 \\ 0 & y_3 & y_2 & -y_5 & 0 \\ 0 & 0 & y_3 & y_2 & -y_5 \end{vmatrix}$$

of the equation(cf. [12, Chapter 11]). Then we get the equation iii) c) in the proposition. For more details, see [8].

Q.E.D.

§2 Cubic hyper-resolutions of hypersurface ordinary singularities of dimension ≤ 5

We denote by $\mathbb{Z}$ the integer ring.

2.1 Definition. For $n \in \mathbb{Z}$ with $n \geq 0$ the *augmented n -cubic category*, denoted by $\square_n^+$, is defined to be a category whose objects $\text{Ob}(\square_n^+)$ and the set of homomorphisms $\text{Hom}_{\square_n^+}(\alpha, \beta)$ ($\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n), \beta = (\beta_0, \beta_1, \dots, \beta_n) \in \text{Ob}(\square_n^+)$) are given as follows:

$$\text{Ob}(\square_n^+) := \{\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{Z}^{n+1} \mid 0 \leq \alpha_i \leq 1 \text{ for } 0 \leq i \leq n\},$$

$$\text{Hom}_{\square_n^+}(\alpha, \beta) := \begin{cases} \alpha \rightarrow \beta \text{ (an arrow from } \alpha \text{ to } \beta) & \text{if } \alpha_i \leq \beta_i \text{ for } 0 \leq i \leq n \\ \emptyset & \text{otherwise.} \end{cases}$$

For $n = -1$ we understand $\square_{-1}^+$ to be the punctual category $\{*\}$, i.e., the category consisting of a single point.

Notice that $\text{Ob}(\square_n^+)$ can be considered as a finite ordered set whose order is defined by $\alpha \leq \beta \iff \alpha \rightarrow \beta$ for $\alpha, \beta \in \text{Ob}(\square_n^+)$.

2.2 Definition. A $\square_n^+$ -analytic variety is defined to be a contravariant functor X from $\square_n^+$ to the category of complex analytic varieties ($\text{An}/\mathbb{C}$). It is also called an *augmented n -cubic analytic variety*.

2.3 Definition. Let X, Y be $\square_n^+$ -analytic varieties. We define a morphism $\Phi : X \rightarrow Y$ to be a natural transformation from the functor X to the one Y over the identity functor $\text{id} : \square_n^+ \rightarrow \square_n^+$.

2.4 Definition. For a $\square_n^+$ -analytic variety X , a contravariant functor Y from $\square_1^+$ to the category of $\square_n^+$ -analytic varieties is called a *2-resolution of X* if Y is defined by a cartesian square of morphisms of $\square_n^+$ -analytic varieties

$$(2.1) \quad \begin{array}{ccc} Y_{11} & \longrightarrow & Y_{01} \\ \downarrow & & \downarrow f \\ Y_{10} & \longrightarrow & Y_{00} \end{array}$$

which satisfies the following conditions:

- (i) $Y_{00} = X$,
- (ii) Y_{01} is a smooth $\square_n^+$ -analytic variety, i.e., a contravariant functor from $\square_n^+$ to the category of smooth analytic varieties,
- (iii) the horizontal arrows are closed immersion of $\square_n^+$ -analytic varieties
- (iv) f is a proper morphism between $\square_n^+$ -analytic varieties, and
- (v) f induces an isomorphism from $Y_{01\beta} - Y_{11\beta}$ to $Y_{00\beta} - Y_{10\beta}$ for any $\beta \in \text{Ob}(\square_n^+)$.

We think of the cartesian square in (2.1) as a morphism from the $\square_{n+1}^+$ -complex analytic variety $Y_{1..}$ to the one $Y_{0..}$ and write it as $Y_{1..} \rightarrow Y_{0..}$. For a 2-resolution Z of $Y_{1..}$, we define the $\square_{n+3}^+$ -analytic variety $rd(Y, Z)$ by

$$rd(Y, Z) := \begin{array}{ccc} Z_{11} & \longrightarrow & Z_{01} \\ \downarrow & & \downarrow \\ Z_{10} & \longrightarrow & Y_{0..} \end{array}$$

and call it the *reduction* of $\{Y, Z\}$.

2.5 Definition. Let X be an analytic variety and let $\{X^1, X^2, \dots, X^n\}$ be a sequence of $\square_r^+$ -analytic varieties X^r ($1 \leq r \leq n$) such that

- (i) X^1 is a 2-resolution of X ,
- (ii) X^{r+1} is a 2-resolution of X_1^r for $1 \leq r \leq n-1$.

Then, by induction on n , we define

$$Z. := rd(X.^1, X.^2, \dots, X.^n) := rd(rd(X.^1, X.^2, \dots, X.^{n-1}), X.^n).$$

With this notation, if Z_α are smooth for all $\alpha \in \square_n$, we call $Z.$ an *augmented n -cubic hyper-resolution* of X .

2.6 Definition. We call the cartesian square in (2.1) the *2-resolution of X by normalization* if it satisfies that (i) $f : Y_{01} \rightarrow Y_{00}$ is the normalization, (ii) Y_{10} is the discriminant of f (i.e., the smallest, closed $\square_n^+$ -analytic variety of Y_{00} with $Y_{01} - f^{-1}(Y_{10}) \simeq Y_{00} - Y_{10}$), and (iii) $Y_{11} = f^{-1}(Y_{10})$.

2.7 Definition. Let X be an analytic variety. If there exists an augmented n -cubic hyper-resolution $Z. := rd(X.^1, X.^2, \dots, X.^n)$ of X such that X^{r+1} is the 2-resolution of X_1^r by normalization for every r with $0 \leq r \leq n-1$ (we understand $X_1^0 = X$), then we say that an augmented cubic hyper-resolution of X is obtained by *successive normalizations*.

2.8 Example. Let $(Y, o) \subset (\mathbb{C}^5, o)$ be the hypersurface ordinary singularity defined by the equation iii) c) in Proposition 1.5. We shall show that an augmented cubic hyper-resolution of Y is obtained by *successive normalizations*. The map $f := \iota \circ \nu : (\mathbb{C}^4, o) \rightarrow (\mathbb{C}^5, o)$, the composite of the normalization $\nu : (\mathbb{C}^4, o) \rightarrow (Y, o)$ and the inclusion $\iota : (Y, o) \subset (\mathbb{C}^5, o)$, is given by (1.1). Let

$$D_i(f) := \{x \in \mathbb{C}^4 \mid \#f^{-1}(f(x)) \geq i\}, \quad (i = 2, 3)$$

denote the i -ple point locus of f and let

$$D_i(Y) := \{y \in Y \mid \mu_y(Y) \geq i\}, \quad (i = 2, 3)$$

denote that of Y , where $\mu_y(Y)$ is the *multiplicity* of Y at $y \in Y$. By calculation, we can see that each of these loci is defined by the following equation:

$$(2.2) \quad D_2(f) : x_2^2 + (x_3x_4)x_2 + x_3^2(x_4^2 + x_1) = 0$$

(this equation follows from that a point $p \in \mathbb{C}^4$ belongs to $D_2(f)$ iff the equation $f(p) = f(x)$ has other roots than p),

$$D_3(f) : x_2 = x_3 = 0,$$

$$(2.3) \quad D_2(Y) : y_3y_5 + y_2^2 + y_1y_3^2 = y_2y_5 + y_3^2y_4 = 0$$

(since $D_2(Y) = f(D_2(f))$, we obtain this by eliminating $x_1, \dots, x_4$ from the equation of $D_2(f)$ in (2.2) and the equation in (1.1)),

$$D_3(Y) : y_2 = y_3 = y_5 = 0.$$

Note that $Sing(D_2(f)) = D_3(f)$ and $Sing(D_2(Y)) = D_3(Y)$, where $Sing(Z)$ denotes the singular locus of $Z = D_2(f), D_2(Y)$. We define a $\square_1^+$ -analytic variety X^1 to be

$$\begin{array}{ccc}
X_{11}^1 := D_2(f) & \xrightarrow{j_1} & \mathbb{C}^4 =: X_{01}^1 \\
\mu_1 := \nu_1|_{D_2(f)} \downarrow & & \downarrow \nu_1 \\
X_{10}^1 := D_2(Y) & \xrightarrow{i_1} & Y =: X_{00}^1,
\end{array}
\quad (2.4)$$

where $\nu_1 := \nu$, the normalization of Y , and the horizontal arrows are inclusions. This diagram is nothing but the 2-resolution of Y by *normalization*. We regard the map $\mu_1 : D_2(f) \rightarrow D_2(Y)$ as a $\square_0^+$ -analytic variety. We are now going to show that a 2-resolution of the $\square_0^+$ -analytic variety $\mu_1 : D_2(f) \rightarrow D_2(Y)$ is also obtained by *normalization*.

Step(1) First, we shall show that the strict transform $D_2(Y)^*$ of $D_2(Y)$ by the blowing-up $\sigma : \hat{\mathbb{C}}^5 \rightarrow \mathbb{C}^5$ of $\mathbb{C}^5$ with non-singular center $D_3(Y)$ becomes non-singular, and that the restriction map $\nu_{20} := \sigma|_{D_2(Y)^*} : D_2(Y)^* \rightarrow D_2(Y)$ is the normalization. We put

$$\begin{aligned}
g_1 &:= y_3 y_5 + y_2^2 + y_1 y_3^2, \\
g_2 &:= y_2 y_5 + y_3^2 y_4
\end{aligned}$$

(cf. (2.3)) and let $\mathcal{I}_{D_2(Y)}$ denote the ideal sheaf of $D_2(Y)$ in $\mathcal{O}_{\mathbb{C}^5}$, which is generated by g_1 and g_2 as a $\mathcal{O}_{\mathbb{C}^5}$ -module. Here we should note that Buchberger's algorithm to compute the (*reduced*) *Groebner basis* of an ideal of the polynomial ring (cf. [2, Chapter 2, §7]) works as well for computing the *standard basis* (cf. [1, Corollary 4.2.1]) of an ideal of the convergent power series ring. Hence, applying this algorithm to $\{g_1, g_2\}_o := \mathcal{I}_{D_2(Y),o}$, the stalk of $\mathcal{I}_{D_2(Y)}$ at the origin $o \in \mathbb{C}^5$, we can find that g_1, g_2 and

$$g_3 := y_2^3 - y_3^3 y_4 + y_1 y_2 y_3^2$$

constitute the standard basis of $\mathcal{I}_{D_2(Y),o}$. Since $\mu_{D_3(Y),o}(g_i) = \mu_o(g_i)$, $i = 1, 2, 3$, where $\mu_{D_3(Y),o}(g_i)$ denotes the multiplicity of g_i along $D_3(Y)$ at the origin $o \in \mathbb{C}^5$, which is defined to be the largest μ such that the germ of g_i at o belongs to $(\mathcal{I}_{D_3(Y),o})^\mu$, the stalk $\mathcal{I}_{D_2(Y)^*,x}$ of the ideal sheaf $\mathcal{I}_{D_2(Y)^*}$ at $x \in \sigma^{-1}(o)$ is generated by the strict transforms g_i^* of g_i , $i=1, 2, 3$, by σ as a $\mathcal{O}_{\hat{\mathbb{C}}^5,o}$ -module ([1, Lemma 7.1]). In fact, calculating in terms of local coordinates, we can see that $\mathcal{I}_{D_2(Y)^*,x}$, $x \in \sigma^{-1}(o)$, is generated by g_1^*, g_2^* since $g_3 = y_2 g_1 - y_3 g_2$. The blowing-up $\sigma : \hat{\mathbb{C}}^5 \rightarrow \mathbb{C}^5$ of $\mathbb{C}^5$ with non-singular center $D_3(Y) : y_2 = y_3 = y_5 = 0$ is explicitly described as follows:

$$\begin{aligned}
\hat{\mathbb{C}}^5 &:= \{(y_1, \dots, y_5) \times (\xi_2 : \xi_3 : \xi_5) \in \mathbb{C}^5 \times \mathbb{P}^2 \mid y_i \xi_j - y_j \xi_i = 0, i, j = 2, 3, 5\}, \\
\sigma &:= \text{Pr}_{\mathbb{C}^5|_{\hat{\mathbb{C}}^5}} : \hat{\mathbb{C}}^5 \rightarrow \mathbb{C}^5, \text{ the restriction of the projection } \text{Pr}_{\mathbb{C}^5} : \mathbb{C}^5 \times \mathbb{P}^2 \rightarrow \mathbb{C}^5 \text{ to } \hat{\mathbb{C}}^5.
\end{aligned}$$

Let $U_i, i = 2, 3, 5$, denote the open subset of $\hat{\mathbb{C}}^5$ defined by $\xi_i \neq 0$. On U_3 , we can take $(y_1, y_3, y_4, u_2, u_5)$ as a local coordinate system, where $u_i = \frac{\xi_i}{\xi_3}, i = 2, 5$, and σ is written as

$$\sigma : (y_1, y_3, y_4, u_2, u_5) \rightarrow (y_1, y_3 u_2, y_3, y_4, y_3 u_5) = (y_1, y_2, y_3, y_4, y_5)$$

in term of this local coordinate system. Hence the strict transforms g_1^*, g_2^* of g_1, g_2 by σ is given by

$$(2.5) \quad g_1^* = y_3^{-2}(\sigma^{-1}(g_1)) = u_5 + u_2^2 + y_1,$$

$$g_2^* = y_3^{-2}(\sigma^{-1}(g_2)) = y_4 + u_2 u_5.$$

Since the rank of the Jacobian matrix $\partial(g_1^*, g_2^*)/\partial(y_1, y_3, y_4, u_2, u_5)$ is maximal throughout $D_2(Y)^* \cap U_3$, we conclude that $D_2(Y)^*$ is non-singular in U_3 . By (2.5), the map $\nu_{20|D_2(Y)^* \cap U_3} : D_2(Y)^* \cap U_3 \rightarrow D_2(Y) \cap \sigma(U_3)$ is obviously a finite map. Therefore, $\nu_{20|D_2(Y)^* \cap U_3} : D_2(Y)^* \cap U_3 \rightarrow D_2(Y) \cap \sigma(U_3)$ is nothing but the normalization, since ν_{20} gives rise to an isomorphism between $D_2(Y)^* \cap U_3 - \sigma^{-1}(D_3(Y))$ and $D_2(Y) \cap \sigma(U_3) - D_3(Y)$. On other $U_i, i = 2, 5$, we can also check that $D_2(Y)^* \cap U_i$ is non-singular and the map $\nu_{20|D_2(Y)^* \cap U_i} : D_2(Y)^* \cap U_i \rightarrow D_2(Y) \cap \sigma(U_i)$ is the normalization. Hence the map $\nu_{20} : D_2(Y)^* \rightarrow D_2(Y)$ is the non-singular normalization of $D_2(Y)$.

Step(2) Secondly, we shall show that the normalization of $D_2(f)$ is non-singular. The defining equation of $D_2(f)$ in (2.2) is transformed as follows:

$$\begin{aligned} & x_2^2 + (x_3 x_4) x_2 + x_3^2 (x_4^2 + x_1) \\ &= \{x_2 + \frac{1}{2} x_3 x_4 + \sqrt{-x_1 - \frac{3}{4} x_4^2} \cdot x_3\} \{x_2 + \frac{1}{2} x_3 x_4 - \sqrt{-x_1 - \frac{3}{4} x_4^2} \cdot x_3\} \\ &= (z + \sqrt{xy})(z - \sqrt{xy}) = z^2 - xy^2, \end{aligned}$$

where $x := -x_1 - \frac{3}{4} x_4^2$, $y := x_3$, $z := x_2 + \frac{1}{2} x_3 x_4$. Note that $D_3(f)$ is given by $y = z = 0$. The map $\nu_{21} : D_2(f)^* := \mathbb{C}^3 \rightarrow D_2(f) \subset \mathbb{C}^4$ defined by $(u, v, x_4) \rightarrow (u^2, v, uv, x_4) = (x, y, z, x_4)$ is the normalization of $D_2(f)$, since ν_{21} gives rise to an isomorphism between $D_2(f)^* - \{v = 0\}$ and $D_2(f) - D_3(f)$. Therefore the normalization of $D_2(f)$ is non-singular

Step(3) We consider the following diagram:

$$\begin{array}{ccc} D_2(Y)^* & \xleftarrow{\tilde{\mu}_1} & D_2(f)^* \\ \nu_{20} \downarrow & & \downarrow \nu_{21} \\ D_2(Y) & \xleftarrow{\mu_1} & D_2(f), \end{array}$$

where $\tilde{\mu}_1$ is the lifting of μ_1 . This gives the normalization of $\square_0^+$ -analytic variety $\mu_1 : D_2(f) \rightarrow D_2(Y)$ and, further, gives rise to the following 2-resolution of it:

$$\begin{array}{ccccc}
X_{111}^2 := \nu_{21}^{-1}(D_3(f)) & \xrightarrow{j_{21}} & D_2(f)^* & =: & X_{011}^2 \\
\tilde{\lambda}_1 \swarrow & \mu_{21} \downarrow & \tilde{\mu}_1 \swarrow & \downarrow & \\
X_{110}^2 := \nu_{20}^{-1}(D_3(Y)) & \xrightarrow{j_{20}} & D_2(Y)^* & =: & X_{010}^2 \\
\mu_{20} \downarrow & \downarrow & \downarrow & \downarrow & \\
X_{101}^2 & \xrightarrow{i_{21}} & D_2(f) & =: & X_{001}^2 = X_{11}^1 \\
\downarrow & \lambda_1 \swarrow & \downarrow & \mu_1 \swarrow & \\
X_{100}^2 := D_3(Y) & \xrightarrow{i_{20}} & D_2(Y) & =: & X_{000}^2 = X_{10}^1
\end{array}$$

(2.6)

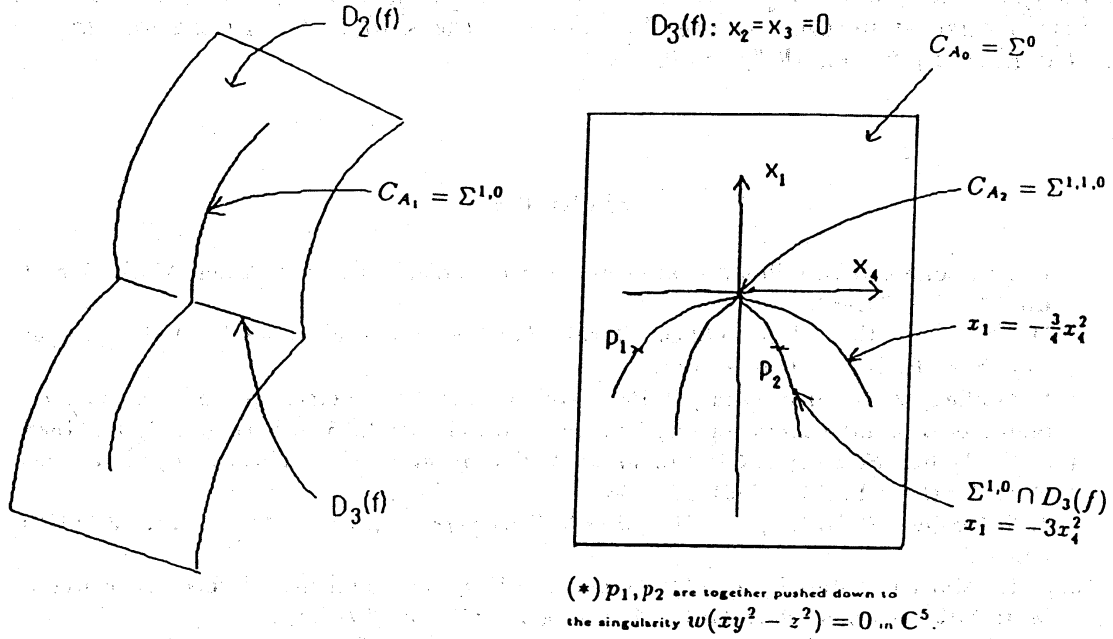
where $i_{2\alpha}, j_{2\alpha}, \alpha = 0, 1$, are inclusion maps, and $\lambda_1 := \mu_1|_{D_3(f)}$, $\tilde{\lambda}_1 := \tilde{\mu}_1|_{\nu_{21}^{-1}(D_3(f))}$. Since all of $D_2(Y)^*$, $D_2(f)^*$, $D_3(Y)$, $D_3(f)$, $\nu_{20}^{-1}(D_3(Y))$ and $\nu_{21}^{-1}(D_3(f))$ are non-singular, replacing $\mu_1 : D_2(f) \rightarrow D_2(Y)$ by $\nu_1 : \mathbb{C}^4 \rightarrow Y$ (this means to form the *reduction* of (2.4) and (2.6)), we obtain the following cubic hyper-resolution of Y :

$$\begin{array}{ccc}
\nu_{21}^{-1}(D_3(f)) & \xrightarrow{j_{21}} & D_2(f)^* \\
\tilde{\lambda}_1 \swarrow & \mu_{21} \downarrow & \tilde{\mu}_1 \swarrow \\
\nu_{20}^{-1}(D_3(Y)) & \xrightarrow{j_{20}} & D_2(Y)^* \\
\mu_{20} \downarrow & \downarrow & \downarrow \\
D_3(f) & \xrightarrow{i_1 \circ \nu_{20}} & \mathbb{C}^4 \\
\downarrow & \lambda_1 \swarrow & \downarrow \\
D_3(Y) & \xrightarrow{i_1 \circ i_{20}} & Y
\end{array}$$

$j_1 \circ \nu_{21}$
 $j_1 \circ i_{21}$
 ν_1

Therefore we conclude that an augmented hyper-resolution of Y is obtained by *successive normalizations*.

2.9 Remark. The various singular point loci of the map f in Example 2.8 are described as follows:



$$\begin{aligned}
 C_{A_i} &:= \{ x \in \mathbb{C}^4 \mid R(f)_x \simeq A_i \} \quad (i = 0, 1, 2), \\
 \Sigma^i &:= \{ x \in \mathbb{C}^4 \mid \dim \text{Ker } df_x = i \} \quad (i = 0, 1), \\
 \Sigma^{1,i} &:= \{ x \in \Sigma^1 \mid \dim \text{Ker } d(f|_{\Sigma^1})_x = i \} \quad (i = 0, 1), \\
 \Sigma^{1,1,0} &:= \{ x \in \Sigma^{1,1} \mid \dim \text{Ker } d(f|_{\Sigma^{1,1}})_x = 0 \}.
 \end{aligned}$$

2.10 Theorem. Let X be an analytic variety with hypersurface ordinary singularities of dimension ≤ 5 , then a cubic hyper-resolution of X is obtained by successive normalizations.

PROOF: In the similar manner to prove Proposition 2.15 in [10, I], we can prove this, using the calculation in Example 2.8 above.

Q.E.D.

As a by-product, we obtain the following.

2.11 Corollary([10],[11]). Let $\pi : \mathfrak{X} \rightarrow M$ be a locally trivial family of compact complex projective varieties with hypersurface ordinary singularities of dimension ≤ 5 , parametrized by a complex manifold M . We define $R_{\mathbb{Z}}^{\ell}(\pi) := R^{\ell}\pi_{*}\mathbb{Z}_{\mathfrak{X}}$ (modulo torsion) ($0 \leq \ell \leq 2(\dim \mathfrak{X} - \dim M)$), $R_{\mathbb{Q}}^{\ell}(\pi) := R_{\mathbb{Z}}^{\ell}(\pi) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $R_{\mathcal{O}}^{\ell}(\pi) := R^{\ell}\pi_{*}(\pi^{*}\mathcal{O}_M) \simeq R^{\ell}\pi_{*}(DR_{\mathfrak{X}/M})$, where $\pi^{*}\mathcal{O}_M$ is the topological inverse of the structure sheaf of M by the map $\pi : \mathfrak{X} \rightarrow M$ and $DR_{\mathfrak{X}/M}$ the

cohomological relative de Rham complex of the family $\pi : \mathfrak{X} \rightarrow M$. Then there exist a family of increasing sub-local systems $\mathbf{W}$ (weight filtration) on $R_{\mathbb{Q}}^{\ell}(\pi)$ and a family of decreasing holomorphic subbundles $\mathbf{F}$ (Hodge filtration) on $R_{\mathbb{C}}^{\ell}(\pi)$ such that $\{R_{\mathbb{Z}}^{\ell}(\pi), (R_{\mathbb{Q}}^{\ell}(\pi), \mathbf{W}[\ell]), (R_{\mathbb{C}}^{\ell}(\pi), \mathbf{W}[\ell], \mathbf{F})\}$ is a variation of mixed Hodge structure, where $\mathbf{W}[\ell]$ denotes the shift of the filtration degree to the right by ℓ , i.e., $\mathbf{W}[\ell]_q := \mathbf{W}_{q-\ell}$.

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