

ON THE MAPPING DEGREE SETS FOR 3-MANIFOLDS

SHICHENG WANG

ABSTRACT. This note records the recent results on the following questions: Let M and N be a closed orientable 3-manifolds, $D(M, N)$ be the set of degrees of maps from M to N , denote $D(M, M)$ by $D(M)$.

- (1) For which N , is the set $D(M, N)$ finite for any M ?
- (2) If $D(M)$ is unbounded, what is $D(M)$?
- (3) When is a self-map of degree ± 1 on M homotopic to a homeomorphisms?

Some of those results were presented at the RIMS Seminar at Akita Shirakami during September 13-17, 2010. For the proofs of those results, see [DeW2], [DeSW], [Wa1], [Du], [SWW], [SWWZ], [Sun].

1991 Mathematics Subject Classification. 57M99, 55M25.

Key words and phrases. 3-manifolds, mapping degrees, finiteness.

1. INTRODUCTION

Let M and N be two closed oriented 3-dimensional manifolds. Let $D(M, N)$ be the set of degrees of maps from M to N , that is

$$D(M, N) = \{d \in \mathbb{Z} \mid f: M \rightarrow N, \deg(f) = d\}.$$

We will simply use $D(N)$ to denote $D(N, N)$, the set of self-mapping degrees of N .

The calculation of $D(M, N)$ is a classical topic which often appeared in the literatures. According to [CT], Gromov thought it is a fundamental problem in topology to determine the set $D(M, N)$ for any dimension n .

Specially the calculation of $D(M)$, the integer set naturally associated to each closed orientable manifold M which presents an interesting connections between topology and number theory.

The result is simple and well-known for dimension $n = 1, 2$. For dimension $n > 3$, there are some interesting special results (See [DW] for recent ones and references therein), but it is difficult to get general results, since there are no classification results for manifolds of dimension $n > 3$.

The case of dimension 3 becomes the most attractive in this topic. Since Thurston's geometrization conjecture, which has been confirmed, implies that closed orientable 3-manifolds can be classified in reasonable sense.

A basic property of $D(M, N)$ is reflected in the following:

Question 1.1. (see [Wa2, Question 1.3] and [Re, Problem A]): *For which closed orientable 3-manifolds N , is the set $D(M, N)$ finite for any given closed oriented 3-manifold M ?*

It is clear if $D(N)$ is unbounded, then $D(M, N)$ is unbounded for some M . For each M , it is clear $\{0, 1\} \subset D(M)$, and if $D(M)$ is bounded then $D(M) \subset \{0, 1, -1\}$.

Question 1.2. *Let M be a closed orientable 3-manifold.*

- (1) *When is $D(M)$ bounded?*
- (2) *If $D(M)$ is unbounded, what is $D(M)$?*

Remark 1.3. The still unknown part for $D(M)$ is that if $D(M)$ is bounded, when does $-1 \in D(M)$?

The following related question is also natural and interesting.

Question 1.4. *For which closed orientable 3-manifolds M , whether there is a selfmap of degree ± 1 on M which is not homotopic to a homeomorphism on M ?*

Under Thurston's picture of 3-manifold, which is confirmed now, Question 1.2 (1) is answered 20 years ago; Question 1.1 and Question 1.2 (2) were answered very recently; the answer of Question 3 is known for Haken manifold and hyperbolic manifolds long times ago, and the answer is complete now for prime 3-manifolds. In Sections 2, 3 and 4, we will present those answers as well as how those answers are developed.

To end this section, we present the picture of 3-manifold which will be used to present the answers. All terminologies not defined are standard, see [He], [Sc] and [IR].

The picture of 3-manifolds: Each closed orientable 3-manifold N has a unique prime decomposition $N_1 \# \dots \# N_k$, the prime factors are unique up to the order and up to homeomorphisms. Each closed orientable prime 3-manifold N has a unique geometric decomposition such that each geometric piece supports one of the following eight geometries: H^3 , $\overline{PSL}(2, R)$, $H^2 \times E^1$, Sol, Nil, E^3 , S^3 and $S^2 \times E^1$ (where H^n , E^n and S^n are n -dimensional hyperbolic space, Euclidean space and sphere respectively), for details see [Th] and [Sc]. Moreover each geometric piece of N with non-trivial geometric decomposition supports either H^3 -geometry or $H^2 \times E^1$ -geometry, hence each 3-manifold supporting one of the remaining six geometry is closed. Furthermore each 3-manifold supporting geometries of either $H^2 \times E^1$, or E^3 , or $S^2 \times E^1$ is covered by a trivial circle bundle, and each 3-manifold supporting geometries of either Sol, or Nil, or E^3 is covered by a torus bundle. Call prime closed orientable 3-manifold N a *non-trivial graph manifold* if N has non-trivial geometric decomposition but contains no hyperbolic piece.

Acknowledgement. The author thanks the invitation and the support of the organizing committee of the RIMS seminar at Akita Shirakami during September 13-17, 2010.

The author is partially supported by grant No.11071006 of the National Natural Science Foundation of China and Ph.D. grant No. 5171042-055 of the Ministry of Education of China.

2. ABOUT $D(M, N)$

This section is based on [DeW2] and [DeSW].

The answer of Question 1.1 is the following

Theorem 2.1. *Let N be a closed orientable 3-manifold. Then there is a closed orientable 3-manifold M such that $|D(M, N)| = \infty$ if and only if $|D(R)| = \infty$ for each prime factor R of N .*

In the following we will make a brief recall of the development of Theorem 2.1.

The development of Theorem 2.1: It is a common sense for many people that $|D(N)| = \infty$ for 3-manifold N which is either a product of a surface and the circle, or N is covered by the 3-sphere. The first significant result in this direction is due to Milnor and Thurston in the later 1970's. By using the minimum integer number of 3-simplices to build N [MT, Theorem 2], they proved

Theorem 2.2. *For each given hyperbolic 3-manifold N , $|D(M, N)| < \infty$ for any M .*

Gromov [G] introduced the simplicial volume $\|N\|$ for a manifold N , which is approximately the minimum real number of 3-simplices to build N . Gromov and Thurston proved that $\|N\|$ is proportional to the hyperbolic volume of N in the case of N is a hyperbolic 3-manifold, and then Soma proved $\|N\|$ is proportional to the sum of the hyperbolic volume of the hyperbolic pieces in the geometric decomposition of N (see [G], [Th], [So]). $\|*\|$ respects the mapping degrees, i.e. for any map $f: M \rightarrow N$ then $\|M\| \geq |\deg(f)| \cdot \|N\|$. Then it is deduced that

Theorem 2.3. *Suppose N is a closed orientable 3-manifold. If a prime factor of N has a hyperbolic piece in its geometric decomposition, then $|D(M, N)| < \infty$ for any M .*

Brooks and Goldman [BG1] [BG2] introduced the Seifert volume $SV(*)$ for closed orientable 3-manifolds which also respects the mapping degrees and is non-zero for each 3-manifold supporting the $\widetilde{PSL}(2, R)$ geometry. Then it is deduced that

Theorem 2.4. *Suppose N is a closed orientable 3-manifold. If a prime factor of N supports $\widetilde{PSL}(2, R)$ geometry. Then $|D(M, N)| < \infty$ for any M .*

Both Theorems 2.3 and 2.4 were already known in the early 1980's. The following result is known no later than early 1990's (see [Wa1] for example).

Theorem 2.5. *Suppose N is a closed orientable 3-manifold. Then $|D(N)| = \infty$ if and only if either N is covered by a torus bundle or a trivial circle bundle, or each prime factor of N is covered by S^3 or $S^2 \times E^1$.*

After Theorems 2.3, 2.4 and 2.5, the remaining unknown cases for Question 1.1 are: either a prime factor of N is a non-trivial graph manifold; or N is a non-prime 3-manifold, and $|D(R)| = \infty$ for each prime factor R of N , but some R is not covered by either S^3 or $S^2 \times E^1$.

In 2009 it is proved in [DeW2] that each closed orientable non-trivial graph manifold N has a finite covering $\tilde{N}$ with positive Seifert volume (it is still unknown whether $SV(\tilde{N}) > 0$ implies $SV(N) > 0$ for a finite cover $\tilde{N} \rightarrow N$), and therefore it is deduced that

Theorem 2.6. *Let N be closed orientable non-trivial graph manifold. Then $|D(M, N)| < \infty$ for any closed orientable 3-manifold M .*

Remark 2.7. Two years before [DeW2], Theorem 2.6 is proved under the restriction that M are also graph manifolds [DeW1], by using a standard form of maps between graph manifolds [De1], and the estimation of the $PSL(2, R)$ -volume for a certain special class of graph manifolds.

In 2010 it is proved in [DeSW]

Theorem 2.8. *Let N be a given closed oriented 3-manifold N . If $|\mathcal{D}(R)| = \infty$ for each prime factor R of N , then there is a closed orientable 3-manifold M such that $|D(M, N)| = \infty$.*

Theorems 2.3 2.4, 2.6 and 2.8 (and Theorem 2.5) imply Theorem 2.1.

Remark 2.9. Theorem 2.8 follows from an explicit result [DeSW, Theorem 2.5], which provides the concrete M and the infinite set in $D(M, N)$ for the given N . The proof of Theorem 2.8 is essentially elementary, which does not appear until now mainly due to three reasons:

(1) $|\mathcal{D}(N)|$ may be finite even if $|\mathcal{D}(R)| = \infty$ for each prime factor R of N ; for example $|\mathcal{D}(T^3)| = \infty$ but $|\mathcal{D}(T^3 \# T^3)| < \infty$ for 3-dimensional torus T^3 [Wa1]. Such phenomena puzzled us to wonder if Theorem 2.8 was always true [Wa2, page 460].

(2) The target concerned in Theorem 2.8 became the only unknown case for Question 1.1 after the work [DeW2].

(3) The proof of Theorem 2.8 uses the result of $\mathcal{D}(N)$ which was just completely determined for each N recently ([Du], [SWW], [SWWZ]).

3. ABOUT $D(M)$

This section is based on [Wa1], [SWW], [SWWZ] and [Du].

3.1. Finer classes for calculate $D(N)$ when $D(N)$ is unbounded. To make this section to be complete, we allow it to have some light repeat with Section 2. The following result, which is a re-statement of Theorem 2.5, is known in early 1990's and answered Question 1.2 (1).

Theorem 3.1. *Suppose M is a geometrizable 3-manifold. Then M admits a self-map of degree larger than 1 if and only if M is either*

- (a) *covered by a torus bundle over the circle, or*
- (b) *covered by $F \times S^1$ for some compact surface F with $\chi(F) < 0$, or*
- (c) *each prime factor of M is covered by S^3 or $S^2 \times E^1$.*

Hence for any 3-manifold M not listed in (a)-(c) of Theorem 3.1, $D(M)$ is either $\{0, 1, -1\}$ or $\{0, 1\}$, which depends on whether M admits a self map of degree -1 . To determine $D(M)$ for geometrizable 3-manifolds listed in (a)-(c) of Theorem 1.0, let's have a close look of them.

For short, we often call a 3-manifold supporting Nil geometry a *Nil 3-manifold*, and so on. Among Thurston's eight geometries, six of them belong to the list (a)-(c) in Theorem 1.0. 3-manifolds in (a) are exactly those supporting either E^3 , or Sol or Nil geometries. E^3 3-manifolds, Sol 3-manifolds, and some Nil 3-manifolds are torus bundle or semi-bundles; Nil 3-manifolds which are not torus bundles or semi-bundles are Seifert fibered spaces having Euclidean orbifolds with three singular points. 3-manifolds in (b) are exactly those supporting $H^2 \times E^1$ geometry; 3-manifolds supporting S^3 or $S^2 \times E^1$ geometries form a proper subset of (3). Now we divide all 3-manifolds in the list (a)-(c) in Theorem 3.1 into the following five classes:

Class 1. M supporting either S^3 or $S^2 \times E^1$ geometries;

Class 2. each prime factor of M supporting either S^3 or $S^2 \times E^1$ geometries, but M is not in Class 1;

Class 3. torus bundles and torus semi-bundles;

Class 4. Nil 3-manifolds not in Class 3;

Class 5. M supporting $H^2 \times E^1$ geometry. We will present $D(M)$ for M in all those five classes. To do this, we need first to coordinate 3-manifolds in each class, then state the results of $D(M)$ in term of those coordinates. This is carried in the next subsection.

3.2. Main Results. Class 1. According to [Or] or [Sc], the fundamental group of a 3-manifold supporting S^3 -geometry is among the following eight types: $\mathbb{Z}_p$, D_{4n}^* , T_{24}^* , O_{48}^* , I_{120}^* , $T'_{8 \cdot 3^q}$, $D'_{n' \cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a 3-manifold supporting S^3 -geometry, $\pi_1(N)$ belongs to the previous seven ones, and $|\pi_1(N)|$ is coprime to m . The cyclic group $\mathbb{Z}_p$ is realized by lens space $L(p, q)$, each group in the remaining types is realized by a unique 3-manifold supporting S^3 -geometry. Note also the sub-indices of those seven types groups are exactly their orders, and the order of the groups in the last type is $m|\pi_1(N)|$. There are only two closed orientable 3-manifolds supporting $S^2 \times \mathbb{E}^1$ geometry: $S^2 \times S^1$ and $RP^3 \# RP^3$.

Theorem 3.2. (1) $D(M)$ for M supporting S^3 -geometry are listed below:

$\pi_1(M)$	$D(M)$
$\mathbb{Z}_p$	$\{k^2 \mid k \in \mathbb{Z}\} + p\mathbb{Z}$
D_{4n}^*	$\{h^2 \mid h \in \mathbb{Z}; 2 \nmid h \text{ or } h = n \text{ or } h = 0\} + 4n\mathbb{Z}$
T_{24}^*	$\{0, 1, 16\} + 24\mathbb{Z}$
O_{48}^*	$\{0, 1, 25\} + 48\mathbb{Z}$
I_{120}^*	$\{0, 1, 49\} + 120\mathbb{Z}$
$T'_{8 \cdot 3^q}$	$\begin{cases} \{k^2 \cdot (3^{2q-2p} - 3^q) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \mid q) \\ \{k^2 \cdot (3^{2q-2p} - 3^{q+1}) \mid 3 \nmid k, q \geq p > 0\} + 8 \cdot 3^q\mathbb{Z} & (2 \nmid q) \end{cases}$
$D'_{n' \cdot 2^q}$	$\{k^2 \cdot [1 - (n')^{2^q-1}]^i \cdot [1 - 2^{(2p-q)(n'-1)}]^j \mid i, j, k, p \in \mathbb{Z}, q \geq p > 0\} + n' 2^q \mathbb{Z}$
$\mathbb{Z}_m \times \pi_1(N)$	$\left\{ d \in \mathbb{Z} \mid \begin{array}{l} d = h + \pi_1(N) \mathbb{Z}, h \in D(N) \\ d = k^2 + m\mathbb{Z}, k \in \mathbb{Z} \end{array} \right\}$

(2) $D(S^2 \times S^1) = D(RP^3 \# RP^3) = \mathbb{Z}$.

Class 2. We assume that each 3-manifold P supporting S^3 -geometry has the canonical orientation induced from the canonical orientation on S^3 . When we change the orientation of P , the new oriented 3-manifold is denoted by $\bar{P}$. Moreover, lens space $L(p, q)$ is orientation reversed homeomorphic to $L(p, p - q)$, so we can write all the lens spaces connected summands as $L(p, q)$. Now we can decompose each 3-manifold in Class 2 as

$$M = (mS^2 \times S^1) \# (m_1 P_1 \# n_1 \bar{P}_1) \# \cdots \# (m_s P_s \# n_s \bar{P}_s) \\ \# (L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \# \cdots \# (L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t})),$$

where all the P_i are 3-manifolds with finite fundamental group different from lens spaces, all the P_i are different from each other, and all the positive integer p_i are different from each other. Define

$$D_{iso}(M) = \{deg(f) \mid f : M \rightarrow M, f \text{ induces an isomorphism on } \pi_1(M)\}.$$

Theorem 3.3. (1) $D(M) = D_{iso}(m_1 P_1 \# n_1 \bar{P}_1) \cap \cdots \cap D_{iso}(m_s P_s \# n_s \bar{P}_s) \cap D_{iso}(L(p_1, q_{1,1}) \# \cdots \# L(p_1, q_{1,r_1})) \cap \cdots \cap D_{iso}(L(p_t, q_{t,1}) \# \cdots \# L(p_t, q_{t,r_t}));$
 (2) $D_{iso}(mP \# n\bar{P}) = \begin{cases} D_{iso}(P) & \text{if } m \neq n, \\ D_{iso}(P) \cup (-D_{iso}(P)) & \text{if } m = n; \end{cases}$
 (3) $D_{iso}(L(p, q_1) \# \cdots \# L(p, q_n)) = H^{-1}(C).$

The notions H and C in Theorem 3.3 (3) is defined as below:

Let $U_p = \{\text{all units in ring } \mathbb{Z}_p\}$, $U_p^2 = \{a^2 \mid a \in U_p\}$, which is a subgroup of U_p . We consider the quotient $U_p/U_p^2 = \{a_1, \dots, a_m\}$, every a_i corresponds with a coset A_i of U_p^2 . For the structure of U_p , see [IR] page 44. Define H to be the natural projection from $\{n \in \mathbb{Z} \mid \gcd(n, p) = 1\}$ to U_p/U_p^2 .

Define $\bar{A}_s = \{L(p, q_i) \mid q_i \in A_s\}$ (with repetition allowed). In U_p/U_p^2 , define $B_l = \{a_s \mid \#\bar{A}_s = l\}$ for $l = 1, 2, \dots$, there are only finitely many l such that $B_l \neq \emptyset$. Let $C_l = \{a \in U_p/U_p^2 \mid a_i a \in B_l, \forall a_i \in B_l\}$ if $B_l \neq \emptyset$ and $C_l = U_p/U_p^2$ otherwise. Define $C = \bigcap_{l=1}^{\infty} C_l$.

Class 3. To simplify notions, for a diffeomorphism ϕ on torus T , we also use ϕ to present its isotopy class and its induced 2 by 2 matrix on $\pi_1(T)$ for a given basis.

A *torus bundle* is $M_\phi = T \times I / (x, 1) \sim (\phi(x), 0)$ where ϕ is a diffeomorphism of the torus T and I is the interval $[0, 1]$. Then the coordinates of M_ϕ is given as below:

(1) M_ϕ admits E^3 geometry, ϕ conjugates to a matrix of finite order n , where $n \in \{1, 2, 3, 4, 6\}$;

(2) M_ϕ admits Nil geometry, ϕ conjugates to $\pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$, where $n \neq 0$;

(3) M_ϕ admits Sol geometry, ϕ conjugates to $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $|a + d| > 2, ad - bc = 1$.

A *torus semi-bundle* $N_\phi = N \cup_\phi N$ is obtained by gluing two copies of N along their torus boundary ∂N via a diffeomorphism ϕ , where N is the twisted I -bundle over the Klein bottle. We have the double covering $p : S^1 \times S^1 \times I \rightarrow N = S^1 \times S^1 \times I / \tau$, where τ is an involution such that $\tau(x, y, z) = (x + \pi, -y, 1 - z)$.

Denote by l_0 and l_∞ on ∂N be the images of the second S^1 factor and first S^1 factor on $S^1 \times S^1 \times \{1\}$. A *canonical coordinate* is an orientation of l_0 and l_∞ , hence there are four choices of canonical coordinate on ∂N . Once canonical coordinates on each ∂N are chosen, ϕ is identified with an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $GL_2(\mathbb{Z})$ given by $\phi(l_0, l_\infty) = (l_0, l_\infty)$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

With suitable choice of canonical coordinates of ∂N , N_ϕ has coordinates as below:

(1) N_ϕ admits E^3 geometry, $\phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$;

(2) N_ϕ admits Nil geometry, $\phi = \begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$, where $z \neq 0$;

(3) N_ϕ admits Sol geometry, $\phi = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, where $abcd \neq 0, ad - bc = 1$.

Theorem 3.4. $D(M_\phi)$ is in the table below for torus bundle M_ϕ , where $\delta(3) = \delta(6) = 1, \delta(4) = 0$.

M_ϕ	ϕ	$D(M_\phi)$
E^3	finite order $k = 1, 2$	$\mathbb{Z}$
E^3	finite order $k = 3, 4, 6$	$\{(kt + 1)(p^2 - \delta(k)pq + q^2) \mid t, p, q \in \mathbb{Z}\}$
Nil	$\pm \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}, n \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, a + d > 2$	$\{p^2 + \frac{(d-a)pr}{c} - \frac{br^2}{c} \mid p, r \in \mathbb{Z},$ either $\frac{br}{c}, \frac{(d-a)r}{c} \in \mathbb{Z}$ or $\frac{p(d-a)-br}{c} \in \mathbb{Z}\}$

(2) $D(N_\phi)$ is listed in the table below for torus semi-bundle N_ϕ , where $\delta(a, d) = \frac{ad}{gcd(a, d)^2}$.

N_ϕ	ϕ	$D(N_\phi)$
E^3	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$\mathbb{Z}$
E^3	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\{2l + 1 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 1 & 0 \\ z & 1 \end{pmatrix}, z \neq 0$	$\{l^2 \mid l \in \mathbb{Z}\}$
Nil	$\begin{pmatrix} 0 & 1 \\ 1 & z \end{pmatrix}$ or $\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, z \neq 0$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\}$
Sol	$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, abcd \neq 0, ad - bc = 1$	$\{(2l + 1)^2 \mid l \in \mathbb{Z}\},$ if $\delta(a, d)$ is even or $\{(2l + 1)^2 \mid l \in \mathbb{Z}\} \cup \{(2l + 1)^2 \cdot \delta(a, d) \mid l \in \mathbb{Z}\},$ if $\delta(a, d)$ is odd

To coordinate 3-manifolds in Class 4 and Class 5, we first recall the well known coordinates of Seifert fibered spaces.

Suppose an oriented 3-manifold M' is a circle bundle with a given section F , where F is a compact surface with boundary components $c_1, \dots, c_n$ with $n > 0$. On each boundary component of M' , orient c_i and the circle fiber h_i so that the product of their orientations match with the induced orientation of M' (call such pairs $\{(c_i, h_i)\}$ a section-fiber coordinate system). Now attach n solid tori S_i to the n boundary tori of M' such that the meridian of S_i is identified with slope $r_i = c_i^{\alpha_i} h_i^{\beta_i}$ where $\alpha_i > 0, (\alpha_i, \beta_i) = 1$. Denote the resulting manifold by $M(\pm g; \frac{\beta_1}{\alpha_1}, \dots, \frac{\beta_s}{\alpha_s})$ which has the Seifert fiber structure extended from the circle bundle structure of M' , where g is the genus of the section F of M , with the sign $+$ if F is orientable and $-$ if F is nonorientable, here 'genus' of nonorientable surfaces means the number of RP^2 connected summands. Call $e(M) = \sum_{i=1}^s \frac{\beta_i}{\alpha_i} \in \mathbb{Q}$ the Euler number of the Seifert fibration.

Class 4. If a Nil manifold M is not a torus bundle or torus semi-bundle, then M has one of the following Seifert fibreing structures: $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$, $M(0; \frac{\beta_1}{3}, \frac{\beta_2}{3}, \frac{\beta_3}{3})$, or $M(0; \frac{\beta_1}{2}, \frac{\beta_2}{4}, \frac{\beta_3}{4})$, where $e(M) \in \mathbb{Q} - \{0\}$.

Theorem 3.5. For 3-manifold M in Class 4, we have

- (1) $D(M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \pmod{6}, m, n \in \mathbb{Z}\};$
- (2) $D(M(0; \frac{\beta_1}{3}, \frac{\beta_2}{3}, \frac{\beta_3}{3})) = \{l^2 | l = m^2 + mn + n^2, l \equiv 1 \pmod{3}, m, n \in \mathbb{Z}\};$
- (3) $D(M(0; \frac{\beta_1}{2}, \frac{\beta_2}{4}, \frac{\beta_3}{4})) = \{l^2 | l = m^2 + n^2, l \equiv 1 \pmod{4}, m, n \in \mathbb{Z}\}.$

Class 5. All manifolds supporting $H^2 \times E^1$ geometry are Seifert fibered spaces M such that $e(M) = 0$ and the Euler characteristic of the orbifold $\chi(O_M) < 0$.

Suppose $M = (g; \frac{\beta_{1,1}}{\alpha_1}, \dots, \frac{\beta_{1,m_1}}{\alpha_1}, \dots, \frac{\beta_{n,1}}{\alpha_n}, \dots, \frac{\beta_{n,m_n}}{\alpha_n})$, where all the integers $\alpha_i > 1$ are different from each other, and $\sum_{i=1}^n \sum_{j=1}^{m_i} \frac{\beta_{i,j}}{\alpha_i} = 0$.

For each α_i and each $a \in U_{\alpha_i}$, define $\theta_a(\alpha_i) = \#\{\beta_{i,j} \mid p_i(\beta_{i,j}) = a\}$ (with repetition allowed), p_i is the natural projection from $\{n \mid \gcd(n, \alpha_i) = 1\}$ to U_{α_i} . Define $B_l(\alpha_i) = \{a \mid \theta_a(\alpha_i) = l\}$ for $l = 1, 2, \dots$, there are only finitely many l such that $B_l(\alpha_i) \neq \emptyset$. Let $C_l(\alpha_i) = \{b \in U_{\alpha_i} \mid ab \in B_l(\alpha_i), \forall a \in B_l(\alpha_i)\}$ if $B_l(\alpha_i) \neq \emptyset$ and $C_l(\alpha_i) = U_{\alpha_i}$ otherwise. Finally define $C(\alpha_i) = \bigcap_{l=1}^{\infty} C_l(\alpha_i)$, and $\bar{C}(\alpha_i) = p_i^{-1}(C(\alpha_i))$.

Theorem 3.6. $D(M(g; \frac{\beta_{1,1}}{\alpha_1}, \dots, \frac{\beta_{1,m_1}}{\alpha_1}, \dots, \frac{\beta_{n,1}}{\alpha_n}, \dots, \frac{\beta_{n,m_n}}{\alpha_n})) = \bigcap_{i=1}^n \bar{C}(\alpha_i).$

3.3. A brief comment of the topic and organization of the paper. Theorem 3.1 was appeared in [Wa1]. The proof of the "only if" part in Theorem 3.1 is based on the results on simplicial volume developed by Gromov, Thurston and Soma (see [So]), and various classical results by others on 3-manifold topology and group theory ([He], [SW], [R]). The proof of "if" part in Theorem 3.1 is a sequence elementary constructions, which were essentially known before, for example see [HL] and [KM] for (3). That graph manifolds admit no self-maps of degrees > 1 also follows from a recent work [De2].

The table in Theorem 3.2 is quoted from [Du], which generalizes the earlier work [HKWZ], which is presented as below.

Proposition 3.7. For 3-manifold M supporting S^3 geometry,

$$D_{iso}(M) = \{k^2 + l|\pi_1(M)|, \text{ where } k \text{ and } |\pi_1(M)| \text{ are co-prime}\}.$$

The topic of mapping degrees between (and to) 3-manifolds covered by S^3 has been discussed for long time and has much relation with other topics (see [Wa2] for details). We just mention several papers: in very old papers [Rh] and [Ol], the degrees of maps between any given pairs of lens spaces are obtained by using equivalent maps between spheres; in [HWZ], $D(M, L(p, q))$ can be computed for any 3-manifold M ; and in a recent one [MP], an algorithm (or formula) is given for the degrees of maps between given pairs of 3-manifolds covered by S^3 in term of their Seifert invariants.

Theorem 3.4 is proved in [SWW].

Theorem 3.3, Theorem 3.5 and Theorem 3.6 are proved in [SWWZ].

3.4. Some examples of computation.

Example 3.8. Let $M_1 = (P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# 2L(7, 3))$ and $M_2 = (2P \# \bar{P}) \# (L(7, 1) \# L(7, 2) \# L(7, 3))$, where P is the Poincare homology three sphere. Apply Theorem 3.3 we have

$$D(M_1) = \{840n + i \mid n \in \mathbb{Z}, i = 1, 71, 121, 169, 191, 239, 241, 289, 311, 359, 361,$$

409, 431, 479, 481, 529, 551, 599, 601, 649, 671, 719, 769, 839.}

$$D(M_2) = \{840n + i \mid n \in \mathbb{Z}, i = 1, 121, 169, 289, 361, 529.\}$$

Example 3.9. By Theorem 3.4, for the torus bundle M_ϕ , $\phi = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, among the first 20 integers > 0 , exactly $1, 4, 5, 9, 11, 16, 19, 20 \in D(M_\phi)$.

Example 3.10. For Nil 3-manifold $M = M(0; \frac{\beta_1}{2}, \frac{\beta_2}{3}, \frac{\beta_3}{6})$,

$$D(M) = \{l^2 \mid l = m^2 + mn + n^2, l \equiv 1 \pmod{6}, m, n \in \mathbb{Z}\}.$$

The numbers in $D(M)$ smaller than 10000 are exactly 1, 49, 169, 361, 625, 961, 1369, 1849, 2401, 3721, 4489, 5329, 6241, 8291, 9409.

Example 3.11. For $H^2 \times E^1$ manifold $M = M(2; \frac{1}{5}, \frac{1}{5}, -\frac{2}{5}, \frac{1}{7}, \frac{2}{7}, -\frac{3}{7})$, apply Theorem 3.6 we have $D(M) = \{5n + 1 \mid n \in \mathbb{Z}\} \cap \{7n + i \mid n \in \mathbb{Z}, i = 1, 2, 4\} = \{35n + i \mid n \in \mathbb{Z}, i = 1, 11, 16\}$.

4. REALIZATION OF SELF-MAP OF DEGREE ± 1 BY A HOMEOMORPHISMS

This section is based on [Sun].

Given a closed orientable n -manifold M , it is natural to ask, whether all the degree ± 1 self-maps on M can be homotopic to homeomorphisms. Without specific description, all the manifolds below are closed and orientable.

If the property stated above holds for M , we say M has property H. In particular, if all the degree 1 (-1) self-maps on M can be homotopic to homeomorphisms, we say M has property 1H ($-1H$). M has property H if and only if M has both property 1H and property $-1H$. We can observe that, if M admits an orientation-reversing self-homeomorphism, then M has property 1H if and only if M has property $-1H$. So we mostly only concern property 1H.

Below we would like to determine which prime 3-manifolds, which are the basic part of 3-manifolds, has property H.

It is known that each degree ± 1 self-map map f on M induces an isomorphism $f_* : \pi_1(M) \rightarrow \pi_1(M)$.

Hyperbolic 3-manifolds and Haken manifolds have property H by the celebrated Mostow rigidity theorem [M] and Waldhausen's theorem on Haken manifolds (see 13.6 of [He]).

This two theorems cover most cases of irreducible 3-manifolds, including: the manifolds with nontrivial JSJ decomposition, hyperbolic manifolds, Seifert manifolds M with incompressible surface. So the remaining cases are:

Class 1. manifolds supporting S^3 -geometry;

Class 2. Seifert manifolds supporting Nil or $\widetilde{PSL(2, R)}$ geometries with orbifold $S^2(p, q, r)$;

4.1. Main Results. Class 1. According to [Or] or [Sc], the fundamental group of a 3-manifold supporting S^3 -geometry is among the following eight types: $\mathbb{Z}_p$, D_{4n}^* , T_{24}^* , O_{48}^* , I_{120}^* , $T'_{8 \cdot 3^q}$, $D'_{n' \cdot 2^q}$ and $\mathbb{Z}_m \times \pi_1(N)$, where N is a S^3 3-manifold, $\pi_1(N)$ belongs to the

previous seven ones, and $|\pi_1(N)|$ is coprime to m . The cyclic group Z_p is realized by lens space $L(p, q)$, each group in the remaining types is realized by a unique S^3 -manifold.

Theorem 4.1. *For M supporting S^3 -geometry, M has property 1H if and only if M belongs to one of the following classes:*

- i) S^3 ;
- ii) $L(p, q)$ satisfies one of the following:
 - a) $p = 2, 4, p_1^{e_1}, 2p_1^{e_1}$;
 - b) $p = 2^s (s > 2), 4p_1^{e_1}, p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}, q^2 \equiv 1 \pmod{p}$ and $q \neq \pm 1$;
- iii) $\pi_1(M) = \mathbb{Z}_m \times D_{4k}^*$, $(m, k) = (1, 2^k), (p_1^{e_1}, 2), (1, p_2^{e_2})$ or $(p_1^{e_1}, p_2^{e_2})$;
- iv) $\pi_1(M) = D'_{2^{k+2}p_1^{e_1}}$;
- v) $\pi_1(M) = T_{24}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times T_{24}^*$;
- vi) $\pi_1(M) = T'_{8 \cdot 3^{k+1}}$;
- vii) $\pi_1(M) = O_{48}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times O_{48}^*$;
- viii) $\pi_1(M) = I_{120}^*$ or $\mathbb{Z}_{p_1^{e_1}} \times I_{120}^*$.

Where all the p_1, p_2 are odd prime numbers, e_1, e_2, k, m are positive integers.

By [HKWZ] and elementary number theory, among all the S^3 -manifolds, only S^3 and lens spaces admit degree -1 self-maps. When considering about property $-1H$, it is reasonable to restrict the manifold to be $L(p, q)$.

Proposition 4.2. *$L(p, q)$ has property $-1H$ if and only if $L(p, q)$ belongs to one of the following classes:*

- i) $4|p$ or some odd prime factor of p is in $4k + 3$ type;
- ii) $q^2 \equiv -1 \pmod{p}$ and $p = 2, p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k + 1$ type prime number.

Essentially, it is known that the manifolds in Class 2 have property H. However, the author can't find a proper reference and he can just copy the proof of Theorem 3.9 of [Sc] to prove this result.

Theorem 4.3. *For Seifert manifolds M supporting Nil or $\widetilde{PSL(2, R)}$ geometries with orbifold $S^2(p, q, r)$, M has property H.*

Synthesize from Mostow and Waldhausen's theorem and Theorem 4.1, 4.3, Proposition 4.2, we get the following consequence:

Theorem 4.4. *Suppose M is a prime geometrizable 3-manifold.*

- 1) M has property 1H if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes stated in Theorem 4.1
- 2) M has property $-1H$ if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes stated in Proposition 4.2.
- 3) M has property H if and only if M belongs to one of the following classes:
 - i) M does not support S^3 -geometry;
 - ii) M is in one of the classes except ii) stated in Theorem 4.1;
 - iii) $L(p, q)$ satisfies one of the following:
 - a) $p = 2, 4$;
 - b) $p = p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k + 3$ type prime number;

- c) $p = p_1^{e_1}, 2p_1^{e_1}$, where p_1 is $4k+1$ type prime number and $q^2 \equiv -1 \pmod{p}$;
 d) $p = 2^s (s > 2), 4p_1^{e_1}$, $q^2 \equiv 1 \pmod{p}$, $q \not\equiv \pm 1$;
 e) $p = p_1^{e_1} p_2^{e_2}, 2p_1^{e_1} p_2^{e_2}$, where one of p_1, p_2 is $4k+3$ type prime number, $q^2 \equiv 1 \pmod{p}$, $q \not\equiv \pm 1$.

Indeed the proof of above theorems in [Sun] give much stronger results. For simplicity, we only explain the situation for 1H.

Let $K(M) = \{\phi \in \text{Out}(\pi_1(M)) \mid \exists f : M \rightarrow M, f_* \in \phi, \deg(f) = 1\}$. It is known $K(M)$ is 1 – 1 corresponds with $\{\text{degree 1 self-maps } f \text{ on } M\}/\text{homotopy}$.

Let $K'(M) = \{\phi \in \text{Out}(\pi_1(M)) \mid \phi \text{ is realized by orientation preserving homeomorphism}\}$, which is a subgroup of $K(M)$. $K'(M)$ is 1 – 1 corresponds with $\mathcal{MCG}^+(M)$, the orientation preserving subgroup of mapping class group of M .

To determine whether M has property 1H, we need only determine whether $K(M) = K'(M)$, or whether $|K(M)| = |\mathcal{MCG}^+(M)|$. Define the realization coefficient of M to be

$$RC(M) = \frac{|K(M)|}{|K'(M)|}.$$

So M has property 1H if and only if $RC(M) = 1$. The $RC(M)$ is completely determined for each 3-manifold support S^3 -geometry in [Sun].

REFERENCES

- [BG1] R. Brooks, W. Goldman, *The Godbillon-Vey invariant of a transversely homogeneous foliation*, Trans. Amer. Math. Soc. **286** (1984), no. 2, 651–664.
 [BG2] R. Brooks, W. Goldman, *Volumes in Seifert space*, Duke Math. J. **51** (1984), 529–545.
 [CT] J. Carlson, D. Toledo, *Harmonic mappings of Kähler manifolds to locally symmetric spaces*, Inst. Hautes Études Sci. Publ. Math. **69** (1989), 173–201.
 [De1] P. Derbez, *Nonzero degree maps between closed orientable three-manifolds*, Trans. Amer. Math. Soc. **359** (2007), no. 8, 3887–3911.
 [De2] P. Derbez, *Topological rigidity and Gromov simplicial volume*, Comm. Math. Helv. **85**, Issue 1, (2010) 1–37.
 [DeSW] P. Derbez, H.B. Sun, S.C. Wang, *Finiteness of mapping degree sets for 3-manifolds*, to appear in Acta Math. Sin. (Engl. Ser.)
 [DeW1] P. Derbez, S.C. Wang, *Finiteness of mapping degrees and $PSL(2, R)$ -volume on graph manifolds*, Algebr. Geom. Topol. **9** (2009), no. 3, 1727–1749.
 [DeW2] P. Derbez, S.C. Wang, *Graph manifolds have virtually positive Seifert volume*, math.GT (math.AT) arXiv:0909.3489.
 [Du] X.M. Du, *On Self-mapping Degrees of S^3 -geometry 3-manifolds*, Acta Math. Sin. (Engl. Ser.) **25**, no. 8 (2009), 1243–1252.
 [DW] H.B. Duan, S.C. Wang, *Non-zero degree maps between 2n-manifolds*, Acta Math. Sin. (Engl. Ser.) **20**, no. 1 (2004), 1–14.
 [G] M. Gromov, *Volume and bounded cohomology*, Inst. Hautes Études Sci. Publ. Math. **56** (1982), 5–99.
 [HKWZ] C. Hayat-Legend, E. Kudryavtseva, S.C. Wang, H. Zieschang, *Degrees of self-mappings of Seifert manifolds with finite fundamental groups*, Dedicated to the memory of Marco Reni. Rend. Istit. Mat. Univ. Trieste **32**, suppl. 1 (2001), 131–147.
 [HWZ] C. Hayat-Legend, S.C. Wang, H. Zieschang, *Degree-one maps onto lens spaces*, Pacific J. Math. **176** (1996), 19–32.
 [He] J. Hempel, *3-manifolds*, Princeton University Press And University of Tokyo Press, 1976.
 [HL] H. Hendriks, F. Laudenbach, *Scindement d’une équivalence d’homotopie en dimension 3*, (French) Ann. Sci. École Norm. Sup. (4) **7** (1974), 203–217.

- [IR] K. Ireland and M. Rosen, *A classical introduction to modern number theory*, second edition, GTM 84, Springer-Verlag, 2003.
- [KM] J. Kalliongis, D. McCullough, π_1 -injective mappings of compact 3-manifolds, Proc. London Math. Soc. (3) **52**, no. 1 (1986), 173–192.
- [MP] S.V. Matveev, A.A. Perfiliev, *Periodicity of degrees of mappings between Seifert manifolds*, (Russian) Dokl. Akad. Nauk **395**, no. 4 (2004), 449–451.
- [MT] J. Milnor, W. Thurston, *Characteristic numbers of 3-manifolds*, Enseignement Math. (2), **23** (1977), no. 3-4, 249–254.
- [M] G. D. Mostow, *Strong rigidity of locally symmetric spaces*, Ann. of Math. Studies **78**, Princeton, 1973.
- [Ol] P. Olum, *Mappings of manifolds and the notion of degree*, Ann. of Math. **58** (1953), 458–480
- [Or] P. Orlik, *Seifert manifolds*, Lect. Notes Math., **291**, Springer, Berlin-Heidelberg-New York, 1972.
- [Re] A. Reznikov, *Volumes of discrete groups and topological complexity of homology spheres*, Math. Ann. **306** (1996), no. 3, 547–554.
- [Rh] G. de Rham *Sur l'analyse situs des varietes n dimensions*, J. Math. **10** (1931), 115–200.
- [R] Y.W. Rong, *Degree one maps between geometric 3-manifolds*, Trans. Amer. Math. Soc. **332**, No. 1 (1992), 441–436.
- [RW] Y.W. Rong and S.C. Wang, *The preimage of submanifolds*, Math. Proc. Camb. Phil. Soc. **112** (1992), 271–279.
- [Sc] P. Scott, *The geometries of 3-manifolds*, Bull. Lond. Math. Soc. **15** (1983), 401–487.
- [SW] P. Scott, C.T.C. Wall, *Topological methods in group theory. Homological group theory*, (Proc. Sympos., Durham (1979), 137–203), London Math. Soc. Lecture Note Ser. 36, Cambridge Univ. Press, Cambridge-New York, 1979.
- [So] T. Soma, *The Gromov invariant of links*, Invent. Math. **64** (1981), 445–454.
- [Sun] H.B. Sun, *Degree ± 1 self-maps and self-homeomorphisms on prime 3-manifolds*, Algebr. Geom. Topol. **10**, no. 2 (2010), 867–890.
- [SWW] H.B. Sun, S.C. Wang, J.C. Wu, *Self-mapping degrees of torus bundles and semi-bundles*, Osaka J. Math. **47**, no. 1 (2010), 131–155.
- [SWWZ] H. B. Sun, S.C. Wang, J.C. Wu, H. Zheng, *Self-mapping Degrees of 3-Manifolds.*, math.GT (math.AT) arXiv:0810.1801, to appear in Osaka J. Math..
- [Th] W. Thurston, *Three-dimensional manifolds, Kleinian groups and hyperbolic geometry*, Bull. Am. Math. Soc. **6** (1982), 357–381.
- [Wa1] S.C. Wang, *The π_1 -injectivity of self-maps of nonzero degree on 3-manifolds*, Math. Ann. **297** (1993), 171–189.
- [Wa2] S.C. Wang, *Non-zero degree maps between 3-manifolds*, Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 457–468, 2002.

DEPARTMENT OF MATHEMATICS, PEKING UNIVERSITY, BEIJING, CHINA
 E-mail address: wangsc@math.pku.edu.cn