

A Reverse of Generalized Ando-Hiai Inequality

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§1 Introduction

The Löwner-Heinz inequality says that the function t^α is operator monotone for $\alpha \in [0, 1]$.

$$(LH) \quad A \geq B \geq 0 \implies A^\alpha \geq B^\alpha \quad \text{for } \alpha \in [0, 1].$$

It induces the α -geometric operator mean defined for $\alpha \in [0, 1]$ as

$$A \#_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}.$$

if $A > 0$, i.e., A is invertible, by the Kubo-Ando theory [1] see also [7].

Ando and Hiai [2] proposed a log-majorization inequality, whose essential part is the following operator inequality. We say it the Ando-Hiai inequality, simply (AH).

$$(AH) \quad A \#_\alpha B \leq I \implies A^r \#_\alpha B^r \leq I \quad (r \geq 1).$$

In some sense, (AH) might be motivated by the following operator inequality, which is a super extension of Löwner-Heinz inequality, the case of $r = 0$ in below:

Furuta inequality (FI). If $A \geq B \geq 0$, then for each $r \geq 0$,

$$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geq (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geq (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geq 0$ and $q \geq 1$ with $(1+r)q \geq p+r$.

By using the mean theoretic notation, the Furuta inequality has the following expression by virtue of (LH):

(FI) If $A \geq B > 0$, then

$$A^{-r} \#_{\frac{1+r}{p+r}} B^p \leq A \quad \text{for } p \geq 1 \text{ and } r \geq 0.$$

Moreover, to make a simultaneous extension of both (FI) and (AH), Furuta added variables as in the extension of (LH) to (FI). Actually he established so-called grand Furuta inequality, simply (GFI). It is sometimes said to be generalized Furuta inequality, [3], [6] and [9].

Grand Furuta inequality (GFI). If $A \geq B > 0$ and $t \in [0, 1]$, then

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ and $p, s \geq 1$.

It is easily seen that

$$(\text{GFI}) \text{ for } t = 1, r = s \iff (\text{AH})$$

$$(\text{GFI}) \text{ for } t = 0, (s = 1) \iff (\text{FI}).$$

Based on an idea of Furuta inequality, we proposed two variables version of Ando-Hiai inequality, [4] and [5].

Generalized Ando-Hiai inequality (GAH). For $A, B > 0$ and $\alpha \in [0, 1]$, if $A \#_{\alpha} B \leq I$, then

$$A^r \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s \leq I \quad \text{for } r, s \geq 1.$$

It is obvious that the case $r = s$ in (GAH) is just Ando-Hiai inequality.

Now we consider two one-sided versions of (GAH):

(1) For $A, B > 0$ and $\alpha \in [0, 1]$, if $A \#_{\alpha} B \leq I$, then

$$A^r \#_{\frac{\alpha r}{\alpha r + 1 - \alpha}} B \leq I \quad r \geq 1.$$

(2) For $A, B > 0$ and $\alpha \in [0, 1]$, if $A \#_{\alpha} B \leq I$, then

$$A \#_{\frac{\alpha}{\alpha + (1-\alpha)s}} B^s \leq I \quad s \geq 1.$$

It is known in [5] that (1) and (2) are equivalent and they are equivalent to (FI). Furthermore generalized Ando-Hiai inequality (GAH) is understood as the case $t = 1$ in (GFI):

§2 Generalisation of Seo's result

Recently Seo showed the following reverse inequality of (AH) in [8].

Theorem S. Let A, B be positive invertible operators. Then

$$A \#_{\alpha} B \leq I \implies A^r \#_{\alpha} B^r \leq \|(A \#_{\alpha} B)^{-1}\|^{1-r} I \quad (0 \leq r \leq 1).$$

Thus we give a generalization of Theorem S. As a matter of fact, we show the following theorem.

Theorem 1. Let A, B be positive invertible operators and $\alpha \in [0, 1]$. Then

$$A \#_{\alpha} B \leq I \implies A^r \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s \leq \|(A \#_{\alpha} B)^{-1}\|^{\frac{\alpha(r-s) + (1-r)s}{\alpha r + (1-\alpha)s}} I \quad (0 \leq r, s \leq 1)$$

To prove this, we need some lemmas.

Lemma 2. (Araki-Cordes inequality)

$$\|A^p B^p A^p\| \leq \|ABA\|^p \quad \text{for } A, B \geq 0 \text{ and } 0 \leq p \leq 1.$$

It is well-known that Lemma 2 is equivalent to (LH), and so the following lemma is regarded as a reverse of both (LH) and Lemma 2.

Lemma 3. If $A \geq B > 0$, then

$$\left\| A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}} \right\| B^p \geq A^p \quad (0 \leq p \leq 1).$$

proof. $A \geq B > 0$ implies $A^p \geq B^p$ for all $0 \leq p \leq 1$ and then

$$I \geq A^{-\frac{p}{2}} B^p A^{-\frac{p}{2}} \geq \left\| A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}} \right\|^{-1}.$$

$$\text{Hence we have } \left\| A^{\frac{p}{2}} B^{-p} A^{\frac{p}{2}} \right\| B^p \geq A^p.$$

Finally, we cite a convenient formula on exponent of operators.

Lemma 4. Let A, B be an invertible operators. Then

$$(BAB^*)^r = BA^{\frac{1}{2}}(A^{\frac{1}{2}}B^*BA^{\frac{1}{2}})^{r-1}A^{\frac{1}{2}}B^* \quad (r \in \mathbf{R})$$

We prove Theorem 1 in the below.

Proof of Theorem 1. If we put

$$C = A^{-\frac{1}{2}}BA^{-\frac{1}{2}} \text{ then the assumption } A\#_{\alpha}B \leq I \text{ implies } C^{\alpha} \leq A^{-1}.$$

By Lemma 3, we have

$$A^r = A^{\frac{1}{2}}A^{r-1}A^{\frac{1}{2}} \leq \left\| A^{-\frac{1-r}{2}}C^{-\alpha(1-r)}A^{-\frac{1-r}{2}} \right\| A^{\frac{1}{2}}C^{\alpha(1-r)}A^{\frac{1}{2}}.$$

In addition, by Lemma 2, we have

$$A^r \leq \left\| A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}} \right\|^{1-r} A^{\frac{1}{2}}C^{\alpha(1-r)}A^{\frac{1}{2}}.$$

On the other hand, multiplying $C^{-\frac{1}{2}}$ on both sides of $C^{\alpha} \leq A^{-1}$, we have

$$C^{\alpha-1} \leq (C^{\frac{1}{2}}AC^{\frac{1}{2}})^{-1}.$$

We apply this to Lemma 3. Namely we have

$$\begin{aligned}
B^s &= \left(A^{\frac{1}{2}}CA^{\frac{1}{2}}\right)^s \\
&= A^{\frac{1}{2}}C^{\frac{1}{2}}\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{s-1}C^{\frac{1}{2}}A^{\frac{1}{2}} \text{ by Lemma 4} \\
&\leq \left\|\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1-s}{2}}C^{-(\alpha-1)(1-s)}\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1-s}{2}}\right\|A^{\frac{1}{2}}C^{\frac{1}{2}}C^{(\alpha-1)(1-s)}C^{\frac{1}{2}}A^{\frac{1}{2}} \\
&\leq \left\|\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1}{2}}C^{-(\alpha-1)}\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right\|^{1-s}A^{\frac{1}{2}}C^{\frac{1}{2}}C^{(\alpha-1)(1-s)}C^{\frac{1}{2}}A^{\frac{1}{2}}.
\end{aligned}$$

Let $r(A)$ be the spectral radius of A . Then we have

$$\begin{aligned}
\left\|\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1}{2}}C^{-(\alpha-1)}\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-\frac{1}{2}}\right\| &= r\left(\left(C^{\frac{1}{2}}AC^{\frac{1}{2}}\right)^{-1}C^{-(\alpha-1)}\right) \\
&= r\left(A^{-1}C^{-\alpha}\right) \\
&= r\left(A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right) \\
&= \left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|,
\end{aligned}$$

so that $B^s \leq \left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|^{1-s}A^{\frac{1}{2}}C^{(\alpha-1)(1-s)+1}A^{\frac{1}{2}}$.

Therefore, it follows that

$$\begin{aligned}
A^r \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} B^s &\leq \left\{\left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|^{1-r}A^{\frac{1}{2}}C^{(1-r)\alpha}A^{\frac{1}{2}}\right\} \\
&\quad \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} \left\{\left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|^{1-s}A^{\frac{1}{2}}C^{(\alpha-1)(1-s)+1}A^{\frac{1}{2}}\right\} \\
&= \left\|A^{-\frac{1}{2}}C^{-\alpha}A^{-\frac{1}{2}}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r + (1-\alpha)s}}A^{\frac{1}{2}}\{C^{(1-r)\alpha} \#_{\frac{\alpha r}{\alpha r + (1-\alpha)s}} C^{(\alpha-1)(1-s)+1}\}A^{\frac{1}{2}} \\
&\quad \text{by } aX \#_{\gamma} bY = a^{1-\gamma}b^{\gamma}X \# Y \\
&= \left\|A^{-\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{-\alpha}A^{-\frac{1}{2}}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r + (1-\alpha)s}}A^{\frac{1}{2}}C^{\alpha}A^{\frac{1}{2}} \\
&= \left\|\left\{A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\alpha}A^{\frac{1}{2}}\right\}^{-1}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r + (1-\alpha)s}}(A \#_{\alpha} B) \\
&\leq \left\|(A \#_{\alpha} B)^{-1}\right\|^{\frac{\alpha(r-s)+(1-r)s}{\alpha r + (1-\alpha)s}} I
\end{aligned}$$

Hence the proof is complete.

Remark . Theorem S is obtained by putting $r = s$ in Theorem 1.

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