

On the existence of toric crepant resolution of toric hyperquotient singularities in dimension three

Kohei Sato

sato-kohei@ed.tmu.ac.jp
Tokyo Metropolitan University
Supported by JSPS

Abstract: We show an equivalent condition of the existence of a toric crepant resolution of a hyperquotient singularity which is given by quotients of a three-dimensional affine toric terminal singularity by diagonal group actions.

1 Motivation

X : a normal algebraic variety with isolated Gorenstein singularities

G : a finite group acting on X

Our primary question

What conditions are sufficient for existence of crepant resolutions of quotient singularities $(X/G, x)$?

Assumption

X : an affine toric terminal 3-fold, G : a finite group which acts on X diagonally, $(X/G, x)$: an isolated Gorenstein singularity

Theorem 1.1 (G. K. White, D. Morrison, G. Stevens, V. Danilov and M. Frumkin) *Let X be an affine toric $\mathbb{Q}$ -factorial threefold. Then X is terminal if and only if X is of the type $\frac{1}{r}(a, -a, 1)$ where the integer a is coprime to r . In particular, if X is Gorenstein, then X is smooth.*

Theorem 1.2 *Let X be an affine toric non- $\mathbb{Q}$ -factorial threefold. Then X has a terminal singularity if and only if $X \cong \text{Spec}(C[x, y, z, w]/(xz - yw))$.*

2 Quotient case

Let X be of the type $\frac{1}{r}(a, -a, 1)$. We define $G' \subset GL(3, C)$ as follows where ε_r is a primitive r -th root of 1.

$$G' := \left\langle \begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^{-a} & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix} \right\rangle$$

Definition 2.1 Let $G \subset GL(3, C)$ be a finite diagonal group action on C^3 as $(x_1, x_2, x_3) \mapsto (\varepsilon_k^a x_1, \varepsilon_k^b x_2, \varepsilon_k^c x_3)$ where $\text{GCD}(a, b, c, k) = 1$ and $a, b, c \in \mathbb{Z} \cap [0, r)$. If G contains G' as a normal subgroup, then G is called a diagonal group action on X .

Theorem 2.1 (Y. Ito, D. G. Markushevich, S. S. Roan) *All three-dimensional Gorenstein quotient singularities possess crepant resolutions.*

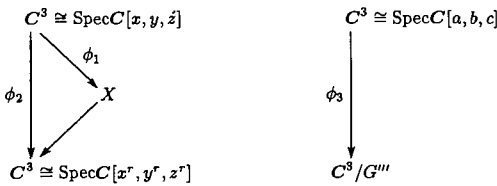
Let G be a diagonal group action on X .

Q. Do there exist diagonal group actions on X such that $(X/G, x)$ are isolated Gorenstein quotient singularities?

Theorem 2.2 (K. Kurano and S. Nishi) *Let n be an odd prime number. Let G be a finite subgroup of $GL(n, C)$ which does not contain any pseudo-reflections. Assume that the C^n/G is Gorenstein with isolated singularity. Then C^n/G has a cyclic quotient singularity.*

Example 2.1 $(X, [0])$: a quotient singularity of type $\frac{1}{r}(a, -a, 1)$

$$G'' := \left\langle \begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^a & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix} \right\rangle$$

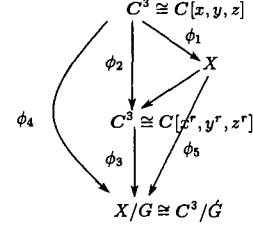


ϕ_2 : the quotient map by G''

$G''' \subset GL(3, C)$: small finite, $(C^3/G''', 0)$: an isolated Gorenstein singularity $\Rightarrow \phi_3$ is a cyclic quotient.

$$G''' := \left\langle \begin{pmatrix} \varepsilon_u^a & 0 & 0 \\ 0 & \varepsilon_v^b & 0 \\ 0 & 0 & \varepsilon_w^k \end{pmatrix} \right\rangle \text{ where } \text{GCD}(u, v, w, k) = 1.$$

We may identify the variables as $x^r = a$, $y^r = b$, $z^r = c$.



ϕ_4 : the composition of quotient maps ϕ_2 and $\phi_3 \Rightarrow \exists$ the quotient map ϕ_5 .

$$\phi_4 \leadsto G = \left\langle \begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^a & 0 \\ 0 & 0 & \varepsilon_r \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} \varepsilon_r^a & 0 & 0 \\ 0 & \varepsilon_r^a & 0 \\ 0 & 0 & \varepsilon_r^a \end{pmatrix} \right\rangle$$

Note 2.1 X : a quotient singularity of the type $\frac{1}{r}(a, -a, 1) \Rightarrow \exists G$: a diagonal group action on X s.t. $(X/G, 0)$ is an isolated Gorenstein singularity $\Rightarrow$ For the G , $\exists$ a crepant resolution.

3 Hypersurface case

Let X be $\text{Spec}(C[x, y, z, w]/(xz - yw))$.

Definition 3.1 Let G be a finite diagonal subgroup (i.e., generated by diagonal matrices) of $GL(4, C)$ acting on C^4 as follows where ε_k is a primitive k -th root of 1 and $a, b, c, d \in [0, r) \cap \mathbb{Z}$.

$$(x, y, z, w) \mapsto (\varepsilon_k^a x, \varepsilon_k^b y, \varepsilon_k^c z, \varepsilon_k^d w)$$

If the group G acts on X , then we call G a diagonal group action on X .

Proposition 3.1 Let G be a diagonal group action on X and g be a generator of G . If the quotient X/G is Gorenstein, then g can be written as

$$\begin{pmatrix} \varepsilon_k^a & 0 & 0 & 0 \\ 0 & \varepsilon_k^b & 0 & 0 \\ 0 & 0 & \varepsilon_k^{-a} & 0 \\ 0 & 0 & 0 & \varepsilon_k^{-b} \end{pmatrix}.$$

4 Main result

Theorem 4.1 Let G be a diagonal group action on $\text{Spec}(C[x, y, z, w]/(xz - yw))$ and $(\text{Spec}(C[x, y, z, w]/(xz - yw))/G, 0)$ be an isolated Gorenstein singularity. There exists a toric crepant resolution of $\text{Spec}(C[x, y, z, w]/(xz - yw))/G$ if and only if G is one of the following;

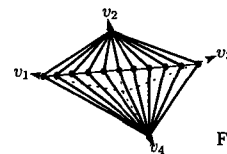
$$\left\langle \begin{pmatrix} \varepsilon_k^a & 0 & 0 & 0 \\ 0 & \varepsilon_k^b & 0 & 0 \\ 0 & 0 & \varepsilon_k^{-a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle \text{ or } \left\langle \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_k^b & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_k^{-b} \end{pmatrix} \right\rangle.$$

Moreover

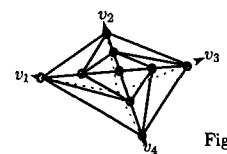
$$\chi(\widetilde{X/G}) = 2|G|$$

where $\widetilde{X/G}$ is a crepant resolution of $\text{Spec}(C[x, y, z, w]/(xz - yw))/G$.

Example 4.1



Fig[1]



Fig[2]

$$\text{Fig [1]: } G = \left\langle \begin{pmatrix} \varepsilon_{10}^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{10}^{-a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\rangle$$

$$\text{Fig [2]: } G = \left\langle \begin{pmatrix} \varepsilon_{12}^a & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \varepsilon_{12}^{-a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \varepsilon_{12}^a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \varepsilon_{12}^{-a} \end{pmatrix} \right\rangle$$

Fig[1] is a toric crepant resolution.

Fig[2] is NOT a crepant resolution.