

The geometry of finite Blaschke products: some duality results

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Abstract

We study geometrical properties of finite Blaschke products. For a Blaschke product B of degree d , the interior curve and the exterior curve are defined. Daep *et al.* proved that the interior curve of B of degree 3 forms an ellipse. However, in the case of $d > 3$, the properties of the interior curve including of the degree are still unknown. On the other hand, it is known that the exterior curve is an algebraic curve of degree at most $d - 1$. In this paper, we give a relation between the interior curve and the exterior curve.

1 Blaschke Products

A Blaschke product of degree d is a rational function defined by

$$B(z) = e^{i\theta} \prod_{k=1}^d \frac{z - a_k}{1 - \overline{a_k}z} \quad (a_k \in \mathbb{D}, \theta \in \mathbb{R}).$$

In the case that $\theta = 0$ and $B(0) = 0$, B is called *canonical*.

For a Blaschke product B of degree d , set $f_1(z) = e^{-\frac{\theta}{d}}z$, and $f_2(z) = \frac{z - (-1)^d a_1 \cdots a_d e^{i\theta}}{1 - (-1)^d a_1 \cdots a_d e^{i\theta} z}$. Then, the composition $f_2 \circ B \circ f_1$ is a canonical one, and geometrical properties with respect to preimages of these two Blaschke products B and $f_2 \circ B \circ f_1$ are same. Hence, we only need to consider a canonical Blaschke product for the following discussions. Here, we remark that there are d distinct preimages $z_1, \dots, z_d$ of $\lambda \in \partial\mathbb{D}$ by B because the derivative of B has no zeros on $\partial\mathbb{D}$ (for instance, see [Mas13]).

In this paper, we will study geometrical properties of curves defined by the preimages of points on the unit circle under B .

2 The interior curves and exterior curves

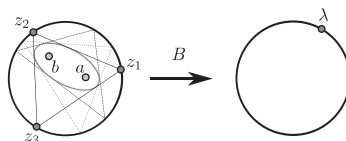
For Blaschke product B of degree d and $\lambda \in \partial\mathbb{D}$, let ℓ_λ be the set of lines joining each distinct two preimages in $B^{-1}(\lambda)$. Then, the envelope of the family of lines $\{\ell_\lambda\}_\lambda$ called the *interior curve* associated with B .

The following is a beautiful property of the interior curve associated with Blaschke product B of degree 3.

Theorem 1 (U. Daep, P. Gorkin, and R. Mortini [DGM02])

Let B be a canonical Blaschke product of degree 3 with zeros 0, a , and b . For $\lambda \in \partial\mathbb{D}$, let z_1, z_2 , and z_3 denote the points mapped to λ under B . Then the lines joining z_j and z_k for $j \neq k$ are tangent to the ellipse E with equation

$$|z - a| + |z - b| = |1 - \overline{a}b|.$$



The ellipse E corresponds to “the inner ellipse” in Poncelet’s Theorem.

For a canonical Blaschke product B of degree d and a point λ on the unit circle, let L_λ be the set of d lines tangent to $\partial\mathbb{D}$ at the d preimages of λ . Then, the trace of the intersection points of each two elements in L_λ as λ ranges over the unit circle, called the *exterior curve* associated with B .

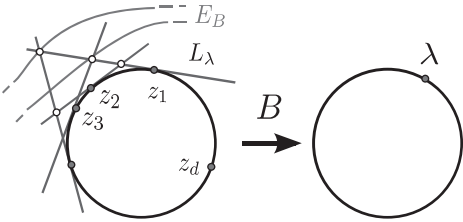


Figure 1: The exterior curve is defined by the trace of the intersection points of tangent lines.

When the degree is low, the interior curve and the exterior curve are described concretely, as follows.

1. For a canonical Blaschke product $B(z) = z \frac{z-a}{1-\bar{a}z}$ of degree 2 with zeros 0 and $a(\neq 0)$,
 - The interior curve is the point a (see [DGM02]).
 - The exterior curve is the line $\bar{a}z + a\bar{z} - 2 = 0$ (see [Fuj17]).
2. For a canonical Blaschke product $B(z) = z \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z}$ of degree 3,
 - The interior curve is the ellipse $|z-a| + |z-b| = |1-\bar{a}b|$ (see [DGM02]).
 - The exterior curve is the conic

$$\bar{a}\bar{b}z^2 + (-|ab|^2 + |a+b|^2 - 1)z\bar{z} + ab\bar{z}^2 - 2(\bar{a}+\bar{b})z - 2(a+b)\bar{z} + 4 = 0. \tag{1}$$

Moreover, the above curve is a non-degenerate conic, i.e. either an ellipse, a circle, a parabola, or a hyperbola (see [Fuj17]).

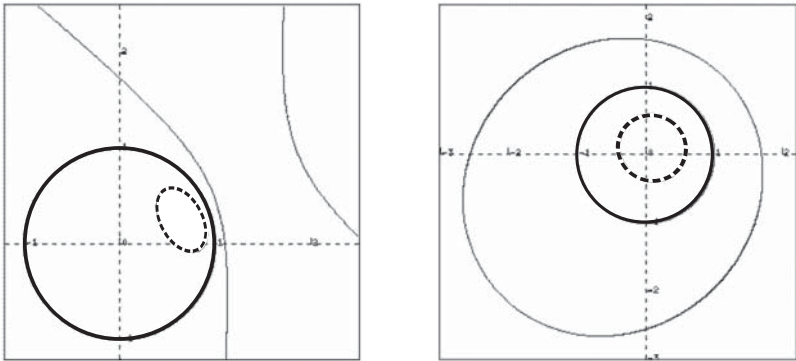


Figure 2: In the case of degree 3: $(a, b) = (\frac{1}{2} + \frac{1}{2}i, \frac{4}{5})$ (left), $(a, b) = (\frac{1}{5}i, \frac{1}{5})$ (right). The thick circle is the unit circle, the dotted curves and the thin curves indicate the interior and the exterior curves, respectively.

3. For a canonical Blaschke product $B(z) = z \frac{z-a}{1-\bar{a}z} \frac{z-b}{1-\bar{b}z} \frac{z-c}{1-\bar{c}z}$ of degree 4,

- The interior curve is defined by the equation of degree 6. To compute a defining equation of this curve we use Risa/Asir, a symbolic computation system. The output of the result is about 200Kb as a text file (see [Fuj13]). So we cannot describe it here.
- The exterior curve is written as follows (see [Fuj17])

$$\begin{aligned} & \overline{\sigma_3} z^3 + (\sigma_1 \overline{\sigma_2} - \sigma_2 \overline{\sigma_3} - \overline{\sigma_1}) z^2 \bar{z} - (\sigma_1 - \sigma_2 \overline{\sigma_1} + \sigma_3 \overline{\sigma_2}) z \bar{z}^2 + \sigma_3 \bar{z}^3 \\ & - 2 \overline{\sigma_2} z^2 - (2 \sigma_1 \overline{\sigma_1} - 2 \sigma_3 \overline{\sigma_3} - 4) z \bar{z} - 2 \sigma_2 \bar{z}^2 + 4 \overline{\sigma_1} z + 4 \sigma_1 \bar{z} - 8 = 0, \end{aligned}$$

where σ_k are the elementary symmetric polynomials on three variables a, b, c of degree k ($k = 1, 2, 3$), i.e.

$$\sigma_1 = a + b + c, \quad \sigma_2 = ab + bc + ca \quad \text{and} \quad \sigma_3 = abc.$$

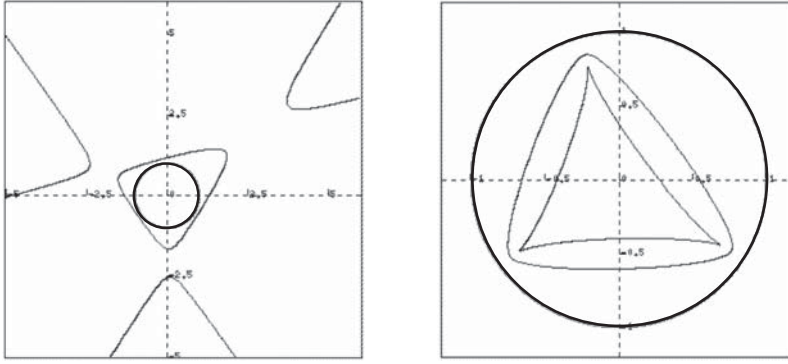


Figure 3: In the case of degree 4: $(a, b, c) = (-0.72 - 0.51i, 0.75 - 0.47i, -0.22 + 0.83i)$. The left figure indicates the exterior curve, and the right figure indicates the interior curve. The thick circles in both figures are the unit circles.

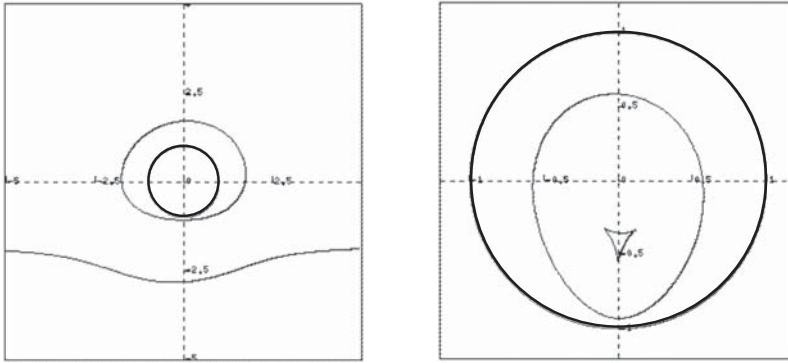


Figure 4: In the case of degree 4: $(a, b, c) = (-0.8i, 0.1 - 0.1i, -0.1 + 0.2i)$. The left figure indicates the exterior curve, and the right figure indicates the interior curve. The thick circles in both figures are the unit circles.

4. For a canonical Blaschke product of degree 5 with zeros $0, a, b, c$ and d ,

- The defining equation of the interior curve is difficult to compute, even though we use symbolic computation systems.
- The exterior curve is written as follows

$$\begin{aligned} & \overline{\sigma_4} z^4 + (\sigma_1 \overline{\sigma_3} - \overline{\sigma_2} - \sigma_2 \overline{\sigma_4}) z^3 \overline{z} - (\sigma_1 \overline{\sigma_1} - \sigma_2 \overline{\sigma_2} + \sigma_3 \overline{\sigma_3} - \sigma_4 \overline{\sigma_4} - 1) z^2 \overline{z}^2 \\ & + (\sigma_3 \overline{\sigma_1} - \sigma_4 \overline{\sigma_2} - \sigma_2) z \overline{z}^3 + \sigma_4 \overline{z}^4 - 2 \overline{\sigma_3} z^3 + 2(2 \overline{\sigma_1} - \sigma_1 \overline{\sigma_2} + \sigma_3 \overline{\sigma_4}) z^2 \overline{z} \\ & - 2(\sigma_2 \overline{\sigma_1} - 2 \sigma_1 - \sigma_4 \overline{\sigma_3}) z \overline{z}^2 - 2 \sigma_3 \overline{z}^3 + 4 \overline{\sigma_2} z^2 + 4(\sigma_1 \overline{\sigma_1} - \sigma_4 \overline{\sigma_4} - 3) z \overline{z} \\ & + 4 \sigma_2 \overline{z}^2 - 8 \overline{\sigma_1} z - 8 \sigma_1 \overline{z} + 16 = 0, \end{aligned} \quad (2)$$

where σ_k are the elementary symmetric polynomials on four variables a, b, c, d of degree k ($k = 1, \dots, 4$) (see [Fuj18]).

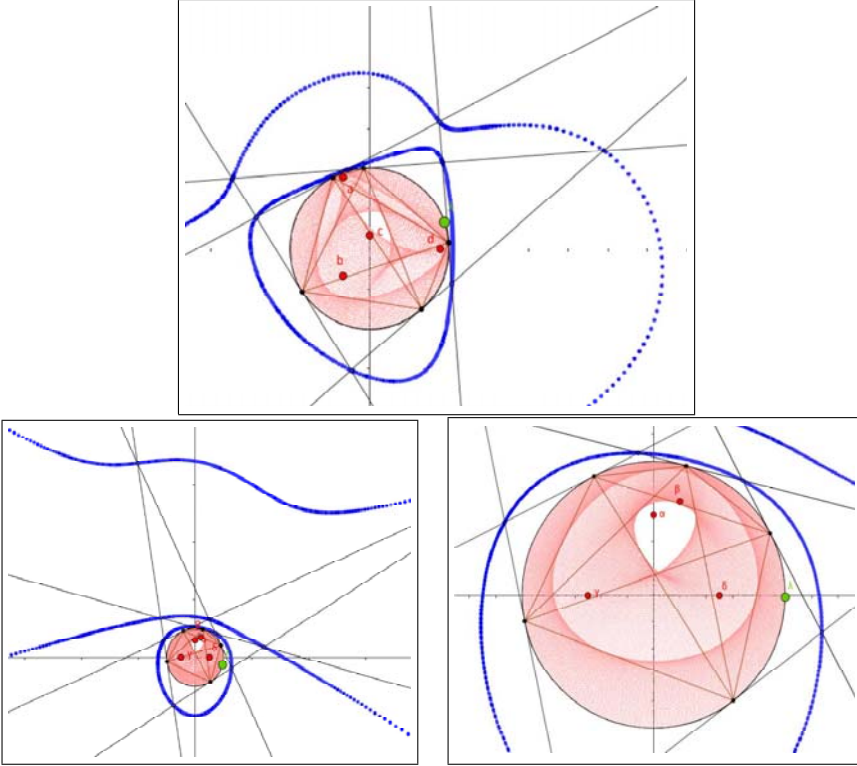


Figure 5: In the case of degree 5: $(a, b, c, d) = (-\frac{1}{3} + \frac{8}{9}i, -\frac{1}{3} - \frac{1}{3}i, \frac{1}{6}i, \frac{8}{9})$ (upper), $(0.6i, 0.2 + 0.7i, -0.5, 0.5)$ (lower), The envelope on the lower right figure is the interior curve corresponding to the lower left. We will give the defining equation of the envelope in the upper figure in Example 2 (cf. Figure 7).

For the degree of the interior curve, we obtained the following.

Theorem 2 ([Fuj17])

Let B be a canonical Blaschke product of degree d . Then, the exterior curve E_B is an algebraic curve of degree at most $d - 1$.

3 Some duality

Is there any relevance between the interior curve and the exterior curve? The following theorem gives a solution to this question.

Theorem 3 ([Fuj18])

Let B be a canonical Blaschke product of degree d , and E_B^* the dual curve of the homogenized exterior curve E_B . Then, the interior curve is given by

$$I_B : u_B^*(-z) = 0,$$

where $u_B^*(z) = 0$ is a defining equation of the affine part of E_B^* .

Equivalently, the converse also holds.

Corollary 4

Let B be a canonical Blaschke product of degree d , and I_B^* be the dual curve of the homogenized interior curve I_B . Then, the exterior curve is given by

$$E_B : v_B^*(-z) = 0,$$

where $v_B^*(z) = 0$ is a defining equation of the affine part of I_B^* .

Remark 1

For every two preimages w_1, w_2 of $\lambda \in \partial\mathbb{D}$ under B , we can consider the intersection point p of two lines tangent to the unit circle at these two points. Then, the line joining w_1 and w_2 is the polar of the unit circle with respect to the pole p .

Example 1

Consider the Blaschke product

$$B(z) = z \cdot \frac{z - \frac{2}{3}}{1 - \frac{2}{3}z} \cdot \frac{z - (\frac{1}{3} + \frac{2}{3}i)}{1 - (\frac{1}{3} - \frac{2}{3}i)z} \quad (3)$$

of degree 3.

Substituting the zero points $a = \frac{2}{3}$, $b = \frac{1}{3} + \frac{2}{3}i$ to the equation (1), we have a defining equation of the exterior curve

$$(18 - 36i)z^2 + 16z\bar{z} - (162 - 108i)z + (18 + 36i)\bar{z}^2 - (162 + 108i)\bar{z} + 324 = 0.$$

Then, the homogenization of the exterior curve is written as

$$u_B(z : t) = (18 - 36i)z^2 + 16z\bar{z} - (162 - 108i)tz + (18 + 36i)\bar{z}^2 - (162 + 108i)t\bar{z} + 324t^2 = 0.$$

Here, we consider the following system of equations,

$$\bar{z} = 2 \frac{\partial}{\partial \zeta} u_B(\zeta : \tau), \quad z = 2 \frac{\partial}{\partial \bar{\zeta}} u_B(\zeta : \tau), \quad t = \frac{\partial}{\partial \tau} u_B(\zeta : \tau), \quad u_B(\zeta : \tau) = 0. \quad (4)$$

Eliminating ζ, τ from (4) we have the defining equation of the dual curve

$$u_B^*(z : t) = (-2187 + 2916i)z^2 - 13770z\bar{z} - (11016 - 6048i)tz - (2187 + 2916i)\bar{z}^2 - (11016 + 6048i)t\bar{z} - 6224t^2 = 0.$$

Taking the reflection of the affine part of the above through the origin, we have

$$(-2187 + 2916i)z^2 - 13770z\bar{z} + (11016 - 6048i)z - (2187 + 2916i)\bar{z}^2 + (11016 + 6048i)\bar{z} - 6224 = 0. \quad (5)$$

This equation gives a defining equation of the interior curve from Theorem 3.

On the other hand, from Theorem 1, the interior curve I_B is the ellipse

$$\left| z - \frac{2}{3} \right| + \left| z - \left(\frac{1}{3} + \frac{2}{3}i \right) \right| = \left| 1 - \frac{2}{3} \left(\frac{1}{3} + \frac{2}{3}i \right) \right|.$$

We can check that the above equation can be expressed as (5).

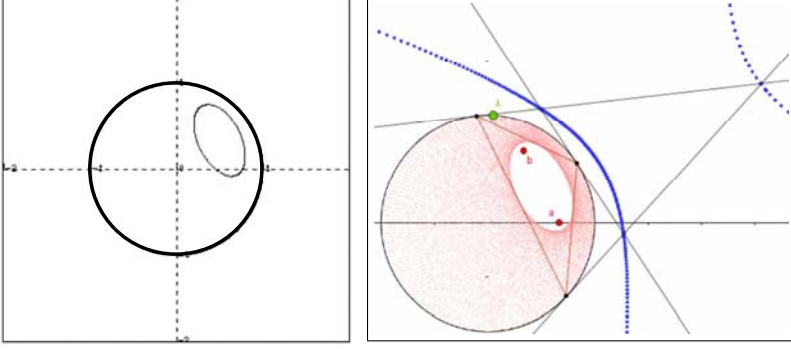


Figure 6: The interior curve I_B of Blaschke product (3) of degree 3: The thick circle is the unit circle. The interior curve I_B in the left figure coincides with the envelope in the right figure.

Example 2

Consider the Blaschke product

$$B(z) = z \cdot \frac{z - (\frac{1}{3} + \frac{8}{9}i)}{1 - (\frac{1}{3} - \frac{8}{9}i)z} \cdot \frac{z - (-\frac{1}{3} - \frac{1}{3}i)}{1 - (-\frac{1}{3} + \frac{1}{3}i)z} \cdot \frac{z - \frac{1}{6}i}{1 + \frac{1}{6}iz} \cdot \frac{z - \frac{8}{9}}{1 - \frac{8}{9}z} \quad (6)$$

of degree 5.

It is difficult to calculate the defining equation of the interior curve directly, but we can obtain the equation by using the result of Theorem 3 as follows.

Substituting the zero points $a = -\frac{1}{3} + \frac{8}{9}i$, $b = -\frac{1}{3} - \frac{1}{3}i$, $c = \frac{1}{6}i$, $d = \frac{8}{9}$ to the equation (2), we have a defining equation of the exterior curve

$$\begin{aligned} & (58320 - 128304i)z^4 + ((409005 + 1248381i)\bar{z} + (-1320948 - 831060i))z^3 \\ & + (1052930\bar{z}^2 + (1076664 - 5064102i)\bar{z} + (-2361960 - 2939328i))z^2 + ((409005 - 1248381i)\bar{z}^3 \\ & + (1076664 + 5064102i)\bar{z}^2 - 20691404\bar{z} + (-3779136 + 12282192i))z + (58320 + 128304i)\bar{z}^4 \\ & - (1320948 - 831060i)\bar{z}^3 - (2361960 - 2939328i)\bar{z}^2 - (3779136 + 12282192i)\bar{z} + 34012224 = 0. \end{aligned} \quad (7)$$

In this case, it is hard to calculate the dual curve using the same procedure as Example 1. Using the condition of a line in the projective plane tangent to the homogenization of (7), we can calculate the defining equation of the dual curve $u_B^*(z, t) = 0$. Taking the reflection of the dual curve through the origin, we have the following defining equation of the interior curve

$$\begin{aligned} I_B : u_B^*(-z, 1) = & -(15399794871190561053331383012853171200 + 93160594377307169577309097056788841600i)z^{12} \\ & + ((-627338421372704312628743046091308312768 - 107925126261883388203551039516469276224i)\bar{z} + \\ & (-421499772859600852951670495185234247424 + 149551173500954916714838575401655831168i))z^{11} \\ & + ((1823650769857883684260878749599785854064 - 2042321228256466521270154024517450929248i)\bar{z}^2 + \\ & (914727195587267285125696970332823606304 + 2263001433392434524791863322001381759072i)\bar{z} + \\ & (666584668669945833556382418834060324864 + 787884861656745276853950006314125400352i))z^{10} \\ & + ((244725126147788719979626610846120719344 + 2559078355112824525530986133948473216064i)\bar{z}^3 + \\ & (-2051873427082483632575157883655929248672 - 3163556186307890315767440645791963539968i)\bar{z}^2 + \\ & (3274182849789405446374298344007339182368 - 3425289026048863734482564201707710449952i)\bar{z} + \\ & (409779582109122705700495209797036379456 - 1606913002279717963932654310794701387328i))z^9 \\ & + ((269714783698348859595549200818004309604 - 2239890217369921286355363892507795438368i)\bar{z}^4 + \\ & (431384656390404126683340488133847643280 + 24278937377697631446690473594940792181608i)\bar{z}^3 + \\ & (-9515426420825710356606520242146484626136 - 1127908459483987789652249078261439511968i)\bar{z}^2 + \\ & (-8786862982962932779213958425651977935088 - 2189422870723397775011507416557912700080i)\bar{z} + \\ & (-2375131273647724514455838299678026432384 + 112496020479043777266882374861772397256i))z^8 \end{aligned}$$

$$\begin{aligned}
& +((-1947598686318801570273994841617761245232-7531942676699568588046820819137788848064i)\bar{z}^5 + \\
& (-30239787667496795082752729390841943935360-19029772621633232686236866704155541605732i)\bar{z}^4 + \\
& (50812484348847288620622628525461987342708-16898143360653319219388432216711716530528i)\bar{z}^3 + \\
& (5005101875654457945642036207038450784984-1381286175504319498890581778800105374680i)\bar{z}^2 + \\
& (5348711907367394432639391022669936055604+12461996129139233591879831636191114330668i)\bar{z} + \\
& (3491483141002171428210091554794478901680+1904777487610891917999689336184998412840i))z^7 \\
& + (18950048315829709992083074551514920754776\bar{z}^6 + (-7013865454572586840637247840786893322768 + \\
& 22735094922870223205021720711157020939220i)\bar{z}^5 + (-35079078778149636378608838308536996801881 + \\
& 14082913005334124986863340699194309294072i)\bar{z}^4 + (3341667917987193061309849064283521257812 - \\
& 33979393309691191200256264258332285687414i)\bar{z}^3 + (-19269811203534257769444879030925253626569 + \\
& 3957698559263561501836858704341055858582i)\bar{z}^2 + (6521838892366202671885062016228944485850 - \\
& 9677658405405658271359826691169142923530i)\bar{z} + (-599453519307130739713498421943088088064 - \\
& 4741925438645558078073192742296840735834i))z^6 \\
& +((-1947598686318801570273994841617761245232+7531942676699568588046820819137788848064i)\bar{z}^7 + \\
& (-7013865454572586840637247840786893322768-22735094922870223205021720711157020939220i)\bar{z}^6 + \\
& 85062633951984362874673659997332552735638\bar{z}^5 + (-8836668873476677766074139484988861021848 + \\
& 70690555049785974451853302925525022587682i)\bar{z}^4 + (-15006583878230023796946358507634048741346 + \\
& 12103853933815344724317722294974467032304i)\bar{z}^3 + (-197628291781550236608449823178969635864 + \\
& 17670103990174632326917979353451719900302i)\bar{z}^2 + (-6947816879763396712491978002840319033864 - \\
& 1252596000340792840677044068629221642892i)\bar{z} + (-3161192041331701773380546212189414967220 + \\
& 2904376175710197377287691186476134380424i))z^5 \\
& + ((269714783698348859595549200818004309604+2239890217369921286355363892507795438368i)\bar{z}^8 + \\
& (-30239787667496795082752729390841943935360+19029772621633232686236866704155541605732i)\bar{z}^7 + \\
& (-35079078778149636378608838308536996801881-14082913005334124986863340699194309294072i)\bar{z}^6 + \\
& (-8836668873476677766074139484988861021848-70690555049785974451853302925525022587682i)\bar{z}^5 - \\
& 24619672195026711578875443645046801543310\bar{z}^4 + (25839420806078751339883853356250230630014 - \\
& 48182200499991600524484928307815862407992i)\bar{z}^3 + (16565814240436534955205981323575555732575 - \\
& 7366232959966176285449733595895897753850i)\bar{z}^2 + (3120867994772439269914301236374195834918 + \\
& 3001155051479635543405883654345014427130i)\bar{z} + (3066925768691058290442360773341067114988 + \\
& 642794874118525636443546803274462800454i))z^4 \\
& + ((244725126147788719979626610846120719344-2559078355112824525530986133948473216064i)\bar{z}^9 + \\
& (431384656390404126683340488133847643280-2427893737769763144669047359490792181608i)\bar{z}^8 + \\
& (50812484348847288620622628525461987342708+16898143360653319219388432216711716530528i)\bar{z}^7 + \\
& (3341667917987193061309849064283521257812+33979393309691191200256264258332285687414i)\bar{z}^6 + \\
& (-15006583878230023796946358507634048741346-12103853933815344724317722294974467032304i)\bar{z}^5 + \\
& (25839420806078751339883853356250230630014+48182200499991600524484928307815862407992i)\bar{z}^4 - \\
& 14648202064105699533212246646936691895418\bar{z}^3 + (-13231775770477161839310708912045264987582 + \\
& 9356713378348101232683143990522557242018i)\bar{z}^2 + (-2000428128405417278320958680496643523176 - \\
& 144460597294005864861187964919822629164i)\bar{z} + (-1007836384993440979534490537203566474900 - \\
& 1705534009383936842007186243637606082148i))z^3 \\
& + ((1823650769857883684260878749599785854064+2042321228256466521270154024517450929248i)\bar{z}^{10} + \\
& (-2051873427082483632575157883655929248672+3163556186307890315767440645791963539968i)\bar{z}^9 + \\
& (-9515426420825710356606520242146484626136+1127908459483987789652249078261439511968i)\bar{z}^8 + \\
& (5005101875654457945642036207038450784984+1381286175504319498890581778800105374680i)\bar{z}^7 + \\
& (-19269811203534257769444879030925253626569-3957698559263561501836858704341055858582i)\bar{z}^6 + \\
& (-197628291781550236608449823178969635864-17670103990174632326917979353451719900302i)\bar{z}^5 + \\
& (16565814240436534955205981323575555732575+7366232959966176285449733595895897753850i)\bar{z}^4 + \\
& (-13231775770477161839310708912045264987582-9356713378348101232683143990522557242018i)\bar{z}^3 + \\
& 7587791904014055406215789564437793454788\bar{z}^2 + (1669498981482568421290894827287110243020 - \\
& 576017700317798274703663588591390763964i)\bar{z} + (-131354331767473962028179002734345324932 + \\
& 95059661459613888839883621584999225032i))z^2
\end{aligned}$$

$$\begin{aligned} &+((-627338421372704312628743046091308312768 + 107925126261883388203551039516469276224i)\bar{z}^{11} + \\ &(914727195587267285125696970332823606304 - 2263001433392434524791863322001381759072i)\bar{z}^{10} + \\ &(3274182849789405446374298344007339182368 + 3425289026048863734482564201707710449952i)\bar{z}^9 + \\ &(-8786862982962932779213958425651977935088 + 2189422870723397775011507416557912700080i)\bar{z}^8 + \\ &(5348711907367394432639391022669936055604 - 12461996129139233591879831636191114330668i)\bar{z}^7 + \\ &(6521838892366202671885062016228944485850 + 9677658405405658271359826691169142923530i)\bar{z}^6 + \\ &(-6947816879763396712491978002840319033864 + 1252596000340792840677044068629221642892i)\bar{z}^5 + \\ &(3120867994772439269914301236374195834918 - 3001155051479635543405883654345014427130i)\bar{z}^4 + \\ &(-2000428128405417278320958680496643523176 + 1444605972940058648611879649198222629164i)\bar{z}^3 + \\ &(1669498981482568421290894827287110243020 + 576017700317798274703663588591390763964i)\bar{z}^2 - \\ &1418342883157293423385575529720968194376\bar{z} + (207143735763635021359351189511345737728 - \\ &196462802394323405821927082437541760816i)z \\ &+ (-15399794871190561053331383012853171200 + 93160594377307169577309097056788841600i)\bar{z}^{12} + \\ &(-421499772859600852951670495185234247424 - 149551173500954916714838575401655831168i)\bar{z}^{11} + \\ &(666584668669945833556382418834060324864 - 787884861656745276853950006314125400352i)\bar{z}^{10} + \\ &(409779582109122705700495209797036379456 + 1606913002279717963932654310794701387328i)\bar{z}^9 + \\ &(-2375131273647724514455838299678026432384 - 112496020479043777266882374861772397256i)\bar{z}^8 + \\ &(3491483141002171428210091554794478901680 - 1904777487610891917999689336184998412840i)\bar{z}^7 + \\ &(-599453519307130739713498421943088088064 + 4741925438645558078073192742296840735834i)\bar{z}^6 + \\ &(-3161192041331701773380546212189414967220 - 2904376175710197377287691186476134380424i)\bar{z}^5 + \\ &(3066925768691058290442360773341067114988 - 642794874118525636443546803274462800454i)\bar{z}^4 + \\ &(-1007836384993440979534490537203566474900 + 1705534009383936842007186243637606082148i)\bar{z}^3 + \\ &(-131354331767473962028179002734345324932 - 950596614596138888839883621584999225032i)\bar{z}^2 + \\ &(207143735763635021359351189511345737728 + 196462802394323405821927082437541760816i)\bar{z} - \\ &35287189479373462425402570572661272592 = 0. \end{aligned}$$

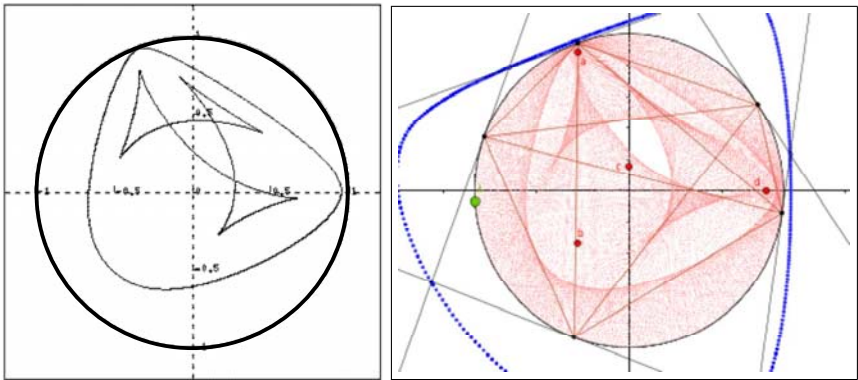


Figure 7: The interior curve of Blaschke product (6) of degree 5: The thick circle is the unit circle. We remark that the interior curve in the left figure coincides with the envelope in the right figure (cf. the upper figure in Figure 5).

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