

Rational curves on Fano threefolds with terminal factorial singularities

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This poster summarizes the results of my project in progress, in which we generalize the results of the paper [BLRT] regarding the Geometric Manin's Conjecture (GMC) for smooth Fano 3-folds to the case of terminal factorial Fano 3-folds. Our goal is to analyze the asymptotic behavior of irreducible components of the Hom scheme $\text{Hom}(\mathbb{P}^1, X, \alpha)$, as $\alpha \in N_1(X)$ grows positively.

0 Settings

- X : terminal factorial Fano 3-fold/ $\mathbb{C}$
- $\alpha \in N_1(X)$: effective curve class on X
- $M \subset \overline{M}_{0,0}(X, \alpha)$: irreducible component generically parametrizing stable maps $f: \mathbb{P}^1 \rightarrow X$ s.t. $f_*\mathbb{P}^1 = \alpha$
- $\mathcal{U} \xrightarrow{\text{ev}} X$: evaluation morphism
↓ universal family
 M
- For a projective variety Y with a nef, big divisor L ,

$$a := a(Y, L) = \inf\{t \in \mathbb{R} \mid K_{\tilde{Y}} + t\phi^*L \in \overline{\text{Eff}}^1(\tilde{Y})\},$$

which is indep. of the choice of a resolution $\phi: \tilde{Y} \rightarrow Y$

- $\kappa := \kappa(\tilde{Y}, K_{\tilde{Y}} + a\phi^*L)$: Iitaka dimension

1 Surfaces with higher a -invariants

a -invariants should detect “pathological” components.

Theorem 1 (cf. [BLRT], §.4). $Y := \text{ev}(\mathcal{U})$

- $Y = X \implies$ general $f \in M$ is free, $\dim M = -K_X \cdot \alpha$
- $Y \neq X \implies a(Y, -K_X) > 1$

Here is the list of surfaces with $a > 1$:

No.	$(Y, -K_X _Y)$	κ	a	contractibility
1	swept out by $-K_X$ -lines	1	2	F
2	$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$	0	2	T (E1 type)
3	$(\mathbb{P}^2, \mathcal{O}(2))$	0	3/2	T (E2 type)
4	$(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, 1))$	0	2	T (E3 type)
5	(quadric cone, $\mathcal{O}(1)$)	0	2	T (E4 type)
6	$(\mathbb{P}^2, \mathcal{O}(1))$	0	3	T (E5 type)
7	its normalization $\cong (\mathbb{P}^2, \mathcal{O}(2))$	0	3/2	F

When X is smooth, the case 7 does not occur by [BLRT].

2 a -covers

Theorem 2 (work in progress, cf. [BLRT], §.5).

Assume M is dominant and general fiber of ev is not irreducible

$\tilde{\mathcal{U}} \rightarrow \mathcal{U}$: smooth resolution

$\tilde{\mathcal{U}} \rightarrow Y \xrightarrow{f} X$: Stein fact. of $\tilde{\mathcal{U}} \rightarrow \mathcal{U} \rightarrow X$

$\implies f: Y \rightarrow X$: a -cover (i.e. $a(Y, -f^*K_X) = 1$)

κ	Iitaka fib. $\pi: Y \dashrightarrow B$	X : smooth case ([BLRT])
2	general fiber maps birationally to a $-K_X$ -conic	
1	(work in progress) $\pi: dP$ fib. if $-K_X$ very ample	
0	$\nexists$ when $X \not\subseteq E5$ divisor	$\nexists$ in any case

3 Movable Bend-and-Break

MBB reduces the irreducibility problem to low deg cases.

Theorem 3 (cf. [BLRT], §.6).

M : dominant component s.t. $-K_X \cdot \alpha \geq 5$

$\implies \exists(f: C \rightarrow X) \in M$ s.t. either

(1) $C = C_1 + C_2$, $f|_{C_i}$: free, or

(2) $C = C_1 + \ell + C_2$, $f|_{C_i}$: free, $f(\ell)$: $-K_X$ -line in $E5$ div.

Moreover, $-K_X \cdot \alpha \geq 10 \implies \exists f \in M$ of the type (1)

4 Geometric Manin's Conjecture

M : Manin component

$\iff \nexists f: Y \rightarrow X$: breaking map, $\nexists N \subset \overline{M}_{0,0}(Y)$: irred. comp.

s.t. f induces a dominant, gen. finite map $f_*: N \dashrightarrow M$

Conjecture 4 ([Tan]).

$\exists c \in \mathbb{Z}_{>0}$, $\exists \alpha \in \text{Nef}_1(X)_{\mathbb{Z}}$ s.t. $\forall \beta \in \alpha + \text{Nef}_1(X)_{\mathbb{Z}}$,

$\exists$ exactly c Manin components in $\overline{M}_{0,0}(X, \beta)$

5 Example: Cubic threefolds

$X \subset \mathbb{P}^4$: factorial terminal cubic 3-fold, $H := -K_X/2$

Theorem 5.

$\ell \in N_1(X)$: H -line class

$\implies \overline{M}_{0,0}(X, \ell)$ is dominant, irreducible of dim 2

(Strategy of the proof).

$q \in \text{Sing}(X)$, C : 1-para. family of lines through q

Prove C is irreducible, $\overline{M}_{0,0}(X, \ell)$ is birational to $\text{Sym}^2(C)$.

Theorem 6 (cf. [LT], §.7). $\forall d \geq 2$,

$\overline{M}_{0,0}(X, d\ell) = \mathcal{R}_d \cup \mathcal{N}_d$: union of two components of dim $2d$

- general $f \in \mathcal{R}_d$ is birational

- any $f \in \mathcal{N}_d$ is a multiple cover of an H -line of deg d

In particular, $\forall d \geq 2$, $\exists! \mathcal{R}_d \subset \overline{M}_{0,0}(X, d\ell)$: Manin component

References

- [BLRT] R. Beheshti et al. *Moduli spaces of rational curves on Fano threefolds*. Adv. Math., 408 Part A:Article 108557, 2022.
- [LT] B. Lehmann and S. Tanimoto. *Geometric Manin's conjecture and rational curves*. Compos. Math., 155(5):833–862, 2019.
- [Tan] S. Tanimoto. *An introduction to Geometric Manin's conjecture*. arXiv:2110.06660, 2021.