

Geometric Characterizations of Standard Normal Distribution

- Two Types of Differential Equations, Relationships with Square and Circle, and Their Similar Characterizations -

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1. Introduction

We confirm various kinds of geometric characterizations of a standard normal distribution with Pearson's finding probabilistic point 0.612003 and its cumulative probability 0.2702678 on a standard normal distribution [1]. After Pearson proposed that, their numbers were studied by Kelley [2, 3], Mosteller [4], Cox [5] and many researchers [6-12]. Our studies [13-17] are also some of them. Sclove [6, 7] explained Cox's proposal to us as the clustering of normal distribution. Nakamori *et. al.* [8] also mentioned that Cox's proposal was one of the original papers of K-means Algorithms. Kelley's formulation, $\phi(\lambda) = 2\lambda\Phi(-\lambda)$, is also known as the 27 percent rule [9-12]. We found the other characterizations about 0.612003 from the parabolic curves of the cost of the repetitions game of coin tossing [13] and the square on a standard normal distribution with the probabilistic point 0.612003 [14-17].

In this paper, we confirm several characterizations based on this paper title. First, we show the geometric characterizations both Bernoulli differential equations [18] of inverse Mills ratios [19, 20] as the hazard functions and survival functions on standard normal distributions. At the same time, we find that Pearson's 0.612003 should be also an important role in their relations. Second, we clarify that variable coefficient second-order linear homogeneous differential equations [21] are related to a standard normal distribution. Especially, Sturm-Liouville form [22] with a probability density function of a standard normal distribution is shown concisely. Moreover, we illustrate that the value 0.612003 shows the curve which is the integral form of the cumulative distribution has the tangent lines on essential key points.

Furthermore, we also find the similar tendencies from European and Oriental historical cultures about the relations close to theology between circles and squares. Coincidentally, we get a practical approximated value 0.612004 from an equilateral triangle on the geometric characterization between circles, squares and a standard normal distribution instead of Pearson's finding true value 0.612003.

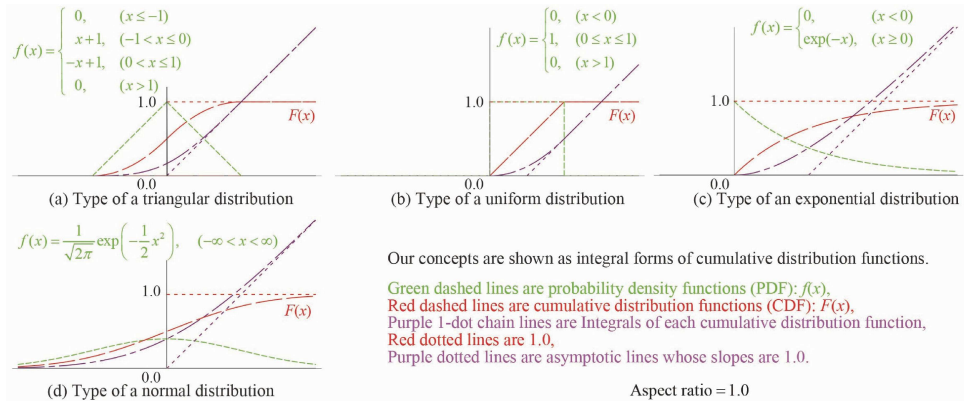


Figure 1 Concepts of integral forms of various kinds of cumulative distribution functions and their probability density functions. (a) Triangular distribution, (b) Uniform distribution, (c) Exponential distribution and (d) Standard normal distribution.

2. Our Concepts of This Study

Prof. Sandoh who was one of two referees of the first author's doctoral thesis [14] advised us to reconfirm the calculations practically about our previous research [13, 14] such as a general integral form [23]. We reconsider that as the following equation

$$E(U|U < -\lambda) = \frac{\int_{-\infty}^{-\lambda} u\phi(u)du}{\int_{-\infty}^{-\lambda} \phi(u)du} = -\frac{\phi(\lambda)}{\Phi(-\lambda)} = -2\lambda \quad (1)$$

as well as using the regression analysis [13, 14]. $\phi(u)$ is a probability density function of a standard normal distribution on

Especially, our concept about a negative return is similar to the meaningful insists of both Prof. Watanabe's website [26] and Prof. Tanioka's textbook [27] as the fee of gambling. On the other hand, the other about a positive return is similar to the general thinking about that a capital return is greater than its growth return historically such as Piketty's proposal [28].

3. Bernoulli Differential Equations for Inverse Mills Ratio

In section 2, we confirm that the equilibrium points on a standard normal distribution are shown based on $\lambda = 0.612003$. According to Isa's research [20], we rethink of the following formulations as Bernoulli differential equation with a standard normal distribution when we consider the tendencies of an inverse Mills ratio and a hazard function as the special cases on λ . The equation such as Bernoulli differential equation with a standard normal distribution is also mentioned in Prof. Kijima's special lecture [18]. However, since we are interested in the geometric characterizations of that, we propose two types of homogeneous Bernoulli differential equations. That is

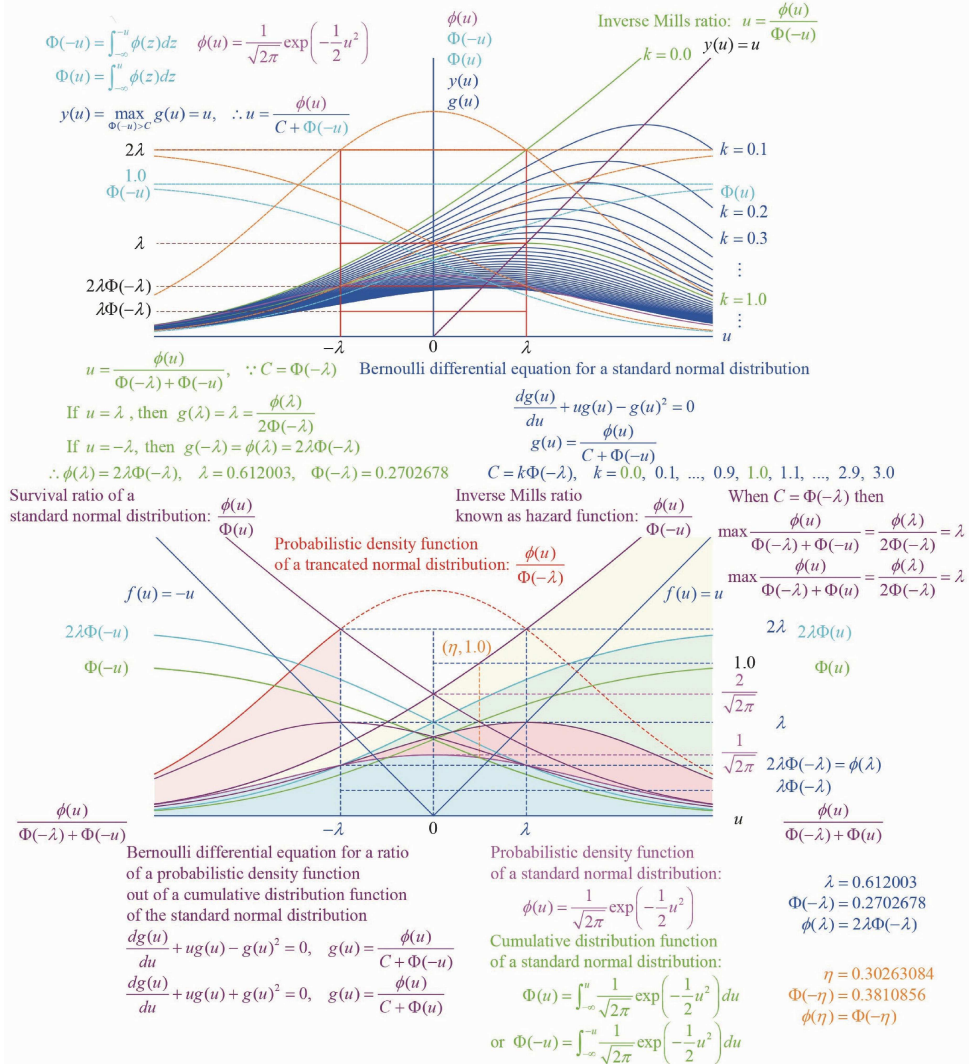


Figure 3 Bernoulli differential equation of the ratios of probability density function of standard normal distribution out of its cumulative distribution function with thinking of its truncated normal distribution on the probabilistic point $\lambda = 0.612003$.

$$\frac{dg(u)}{du} + ug(u) - g(u)^2 = 0, \quad g(u) = \frac{\phi(u)}{C + \Phi(-u)}, \quad (3)$$

$$\frac{dg(u)}{du} + ug(u) + g(u)^2 = 0, \quad g(u) = \frac{\phi(u)}{C + \Phi(u)}. \quad (4)$$

Based on the ideas of asymptotic lines by Gordon [19] and Isa [20], we illustrate the Bernoulli differential equations as the geometric curves in the upper chart of Figure 3 with $\lambda = 0.612003$. At this time, a constant C is changed by k from 0.0 to 3.0 by 0.1 to illustrate those curves clearly. Especially, we confirm that their tendencies are shown as the special types with $\lambda = 0.612003$ when $k = 0.0$ or 1.0. Moreover, 0.30263084 that is also a probabilistic point with the same values of both its cumulative distribution function and probability density function is another important point because the inverse Mills ratio is equal to 1.0. Under the condition of $u = 0$, we find the inverse Mills ratio is $2\phi(0)(= 2/\sqrt{2\pi})$ as 2 times of $\phi(0)$. The lower chart also illustrates the geometric tendencies about Equations (3) and (4) divided by $u = 0$ symmetrically.

4. Sturm-Liouville Differential Equations for a Standard Normal Distribution

In section 2, we mentioned our research concepts as the integral forms of a cumulative distribution of a standard normal distribution. We searched for their several characterizations [29, 30] and their differential equations about normal distribution [30, 31]. From our investigations, we propose the variable coefficient second-order linear homogeneous differential equation in Figure 4.

Sturm-Liouville differential equation: $\phi(u)(h_2'(u) - uh_2'(u) - h_2(u)) = 0$

Sturm-Liouville form: $(\phi(u)h_2'(u))' - (\phi(u)h_2(u)) = 0$

Variable coefficient second-order linear homogeneous differential equation for

a standard normal distribution: $\frac{d^2 h_2(u)}{du^2} - u \frac{dh_2(u)}{du} - h_2(u) = 0,$

$h_2(u)$

$f(u)$

$\phi(u)$

$\psi(u)$

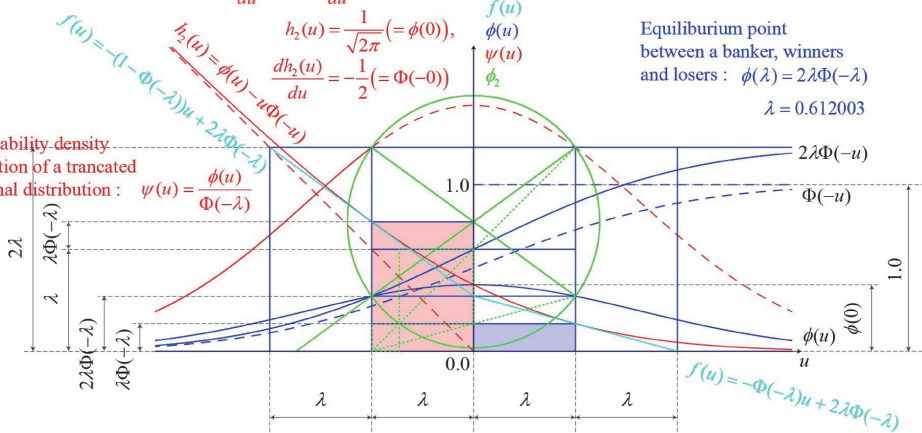
ϕ_2

$$\frac{dh_2(u)}{du} = -\Phi(-u) \quad \frac{d^2 h_2(u)}{du^2} = \phi(u)$$

Equilibrium point
between a banker, winners
and losers: $\phi(\lambda) = 2\lambda\Phi(-\lambda)$

$$\lambda = 0.612003$$

Probability density
function of a truncated
normal distribution: $\psi(u) = \frac{\phi(u)}{\Phi(-\lambda)}$



$$\text{Tangential equation: } f(u) = \begin{cases} -(1 - \Phi(-\lambda))u + 2\lambda\Phi(-\lambda) & (-2 \leq u \leq 0) \\ -\Phi(-\lambda)u + 2\lambda\Phi(-\lambda) & (0 \leq u \leq 2\lambda) \end{cases}$$

Probability density function of a standard normal distribution: $\phi(u)$

Cumulative distribution function of a standard normal distribution: $\Phi(-u)$

Equilibrium point

of an inverse

$$\text{Mills ratio: } \frac{\phi(\lambda)}{\Phi(-\lambda)} = 2\lambda$$

$$\text{Intercept form of a linear equation for winners: } -\frac{1}{\lambda}u + \frac{1}{\lambda\Phi(-\lambda)}\phi_2 = 1$$

$$\text{Intercept form of a linear equation for losers: } -\frac{1}{\lambda}u + \frac{1}{\lambda(1 + \Phi(-\lambda))}\phi_2 = 1$$

$$\text{Intercept form of a linear equation for a banker: } -\frac{1}{\lambda}u + \frac{1}{\lambda}\phi_2 = 1$$

Utility function for winners:

$$U_w(t) = (\phi(\lambda) - \lambda\Phi(-\lambda))\sqrt{t} = \lambda\Phi(-\lambda)\sqrt{t}$$

Figure 4 Geometric relationships between Sturm-Liouville differential equation and standard normal distribution with a circle, squares, meaningful rectangles, their diagonals, and special tangent lines under $\lambda = 0.612003$.

That is

$$\frac{d^2 h_2(u)}{du^2} - u \frac{dh_2(u)}{du} - h_2(u) = 0. \quad (5)$$

The way of an expression of $h_2(u)$ in Equation (5) is based on the descriptions of our previous research [14]. The formulations of $h_2(u)$ are shown as the following equation

$$h_2(u) = A(\phi(u) - u\Phi(-u)) + B(\phi(u) + u\Phi(u)). \quad (6)$$

If we consider the conditions $dh_2(u)/du = 1/\sqrt{2\pi}$ and $h_2(u) = 1/2$ in Equation (5), we get the answer with $A = 1.0$ and $B = 0.0$ in Equation (6). Therefore, we also confirm that the special curve in Figure 4 passes through several fundamental points with $\lambda\Phi(-\lambda)$ and $\lambda(1 + \Phi(-\lambda))$ [17]. At the same time, we find their points also have the crucial tangent lines shown in Figure 4. Moreover, we illustrate a true circle with a square and a standard normal distribution in Figure 4 [17]. The center of this circle is an intersection of meaningful rectangle diagonals. Without passing through the intersection, we find the other important line to distinguish into the separated squares by the proportion $\Phi(-\lambda): 1 - \Phi(-\lambda)$.

We reconfirm the geometric characterizations about $\lambda = 0.612003$ with considering a square, a circle and a normal distribution. And their tendencies have a special differential equation as the following Sturm-Liouville form [22]

$$(\phi(u)h_2'(u))' + (\phi(u)h_2(u)) = 0. \quad (7)$$

At this time, to investigate the tendencies of the combinations between A and B in Equation (6), A and B are changed by the values from 0.0 to 1.0 by 0.25 respectively in Figure 5.

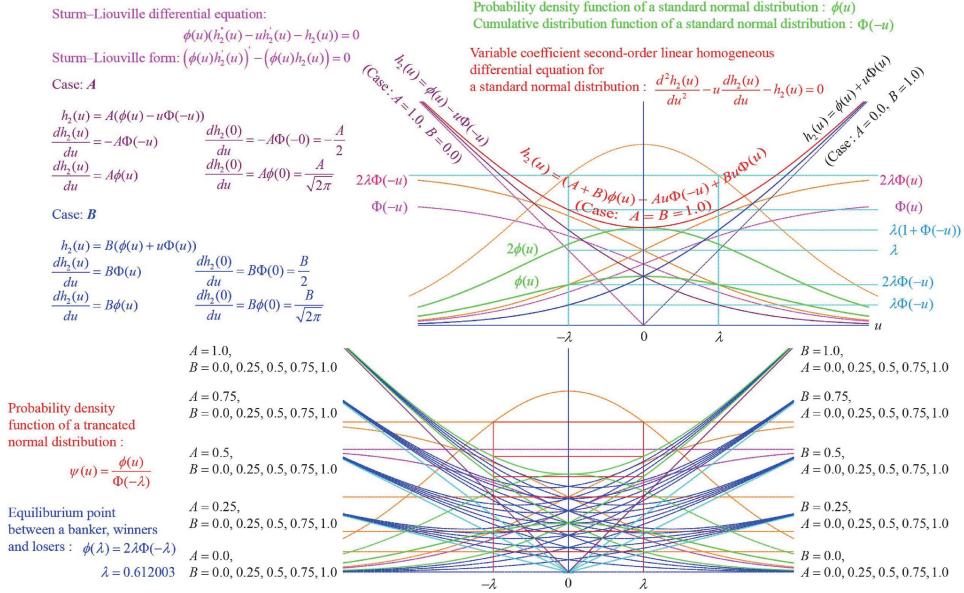


Figure 5 Various geometric characterizations of Sturm-Liouville differential equations with standard normal distribution.

5. Square and Circle on Standard Normal Distribution and Approximated Value about Polygons

Although we confirm that there is a relationship between a circle, a square and a normal distribution, we cannot get a correct value λ as well as drawing an equilateral triangle and a regular hexagon from their characterizations which are similar to the problem of squaring the circle [32, 33]. Instead of that, we can get the practical approximated Pearson's finding value. That is

$$\lambda \cong \frac{1 + \sqrt{3}}{1 + 2\sqrt{3}} = 0.612004. \quad (8)$$

This approximated value is estimated by drawing equilateral triangles and regular hexagons shown in Figure 6.

By the way, there are a lot of thinking about circles and squares for a long time all over the world religiously and historically [32-39]. We also show one of their tendencies with a standard normal distribution based on $\lambda = 0.612003$. Coincidentally,

を記します。

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