

Parametrized Thue Equations — A Survey

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Abstract

We consider families of parametrized Thue equations

$$F_a(X, Y) = \pm 1, \quad a \in \mathbb{N},$$

where $F_a \in \mathbb{Z}[a][X, Y]$ is a binary irreducible form with coefficients which are polynomials in some parameter a .

We give a survey on known results.

1 Thue Equations

Let $F \in \mathbb{Z}[X, Y]$ be a homogeneous, irreducible polynomial of degree $n \geq 3$ and m be a nonzero integer. Then the Diophantine equation

$$F(X, Y) = m \tag{1}$$

is called a *Thue equation* in honour of A. Thue, who proved in 1909 [57]:

Theorem 1 (Thue). (1) has only a finite number of solutions $(x, y) \in \mathbb{Z}^2$.

Thue's proof is based on his approximation theorem: Let α be an algebraic number of degree $n \geq 2$ and $\epsilon > 0$. Then there exists a constant $c_1(\alpha, \epsilon)$, such that for all $p \in \mathbb{Z}$ and $q \in \mathbb{N}$

$$\left| \alpha - \frac{p}{q} \right| \geq \frac{c_1(\alpha, \epsilon)}{q^{n/2+1+\epsilon}}.$$

Since this approximation theorem is not effective, Thue's theorem is neither effective.

2 Number of Solutions

We call a solution (x, y) to $F(x, y) = m$ primitive, if x and y are coprime integers. The problem of giving upper bounds (depending on m and the degree n) for the number of primitive solutions goes back to Siegel. Such a bound has first been given by Evertse [14]. An improved version has been given by Bombieri and Schmidt [6]:

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Theorem 2 (Bombieri-Schmidt [6]). *There is an absolute constant c_2 such that for all $n \geq c_2$ the Diophantine equation $F(X, Y) = m$ has at most $215 \cdot n^{1+\omega(m)}$ primitive solutions, where $\omega(m)$ denotes the number of prime factors of m and solutions (x, y) and $(-x, -y)$ are regarded as the same.*

At least for $m = \pm 1$, this result is best possible (up to the constant 215), since the equation

$$X^n + (X - Y)(2X - Y) \dots (nX - Y) = \pm 1$$

has at least the $n + 1$ solutions $\pm\{(1, 1), \dots, (1, n), (0, 1)\}$.

Sharper bounds have been obtained for special classes of Thue equations.

If only k coefficients of $F(X, Y)$ are nonzero, the number of solutions depends on k and m only (and not on n). For $k = 3$, this is proved by Mueller and Schmidt [41]: There are at most $O(m^{2/n})$ solutions. The general case $k \geq 3$ is proved in Mueller and Schmidt [42]: There are at most $O(k^2 m^{2/n} (1 + \log m^{1/n}))$ solutions. Thomas [56] gives absolute upper bounds for the number of solutions for $m = 1$ and $k = 3$: If $n \geq 38$, then there are at most 20 solutions (x, y) with $|xy| \geq 2$, where solutions (x, y) and $(-x, -y)$ are only counted once. For smaller n , similar bounds are given.

If only 2 coefficients of $F(X, Y)$ are nonzero, we arrive at the special case $ax^n - by^n = \pm 1$ and we consider only the case $ab \neq 0$, $x > 0$, $y > 0$. This equation has been studied by many authors, starting with Delone [11] and Nagell [43], who proved that there is at most one solution for $n = 3$. Several authors have contributed to this question. Finally, Bennett [4] could prove that there is at most one solution (x, y) .

We now consider cubic Thue equations $F(X, Y) = 1$. If the discriminant of F is negative, there are at most 5 solutions, and the cases of 4 and 5 solutions can be listed explicitly. This has been shown independently by Delaunay [10] and Nagell [44] in the 1920's. If the discriminant is positive, there are at most 10 solutions, as it has been proved by Bennett [3]. Okazaki [47] proves that if the discriminant is at least $5.65 \cdot 10^{65}$, then there are at most 7 solutions. It is conjectured by Nagell [45], Pethő [48], and Lippok [35] that there are at most 5 solutions except for five equations (modulo equivalence) which have 6 or 9 solutions. We note that there are two families of cubic Thue equations which have exactly five solutions, cf. items 2 and 3 in the list in Section 4.1.

Okazaki [46] considers the analogous problem for quartic Thue equations $F(X, Y) = \pm 1$. If all roots of $F(x, 1)$ are real and the discriminant is larger than a computable constant c_3 , this equation has at most 14 solutions, where solutions (x, y) and $(-x, -y)$ are counted once.

3 Algorithmic Solution of Single Thue Equations

Studying linear forms in logarithms of algebraic numbers, A. Baker could give an effective upper bound for the solutions of such a Thue equation in 1968 [1]:

Theorem 3 (Baker). *Let $\kappa > n + 1$ and $(x, y) \in \mathbb{Z}^2$ be a solution of (1). Then*

$$\max\{|x|, |y|\} < c_4 e^{\log^\kappa |m|},$$

where $c_4 = c_4(n, \kappa, F)$ is an effectively computable number.

Since that time, these bounds have been improved; Bugeaud and Győry [7] give the following bound:

Theorem 4 (Bugeaud-Győry). *Let $B \geq \max\{|m|, e\}$, α be a root of $F(X, 1)$, $K := \mathbb{Q}(\alpha)$, $R := R_K$ the regulator of K and r the unit rank of K . Let $H \geq 3$ be an upper bound for the absolute values of the coefficients of F .*

Then all solutions $(x, y) \in \mathbb{Z}^2$ of (1) satisfy

$$\max\{|x|, |y|\} < \exp\left(c_5 \cdot R \cdot \max\{\log R, 1\} \cdot (R + \log(HB))\right)$$

and

$$\max\{|x|, |y|\} < \exp\left(c_6 \cdot H^{2n-2} \cdot \log^{2n-1} H \cdot \log B\right),$$

with $c_5 = 3^{r+27}(r+1)^{7r+19}n^{2n+6r+14}$ and $c_6 = 3^{3(n+9)}n^{18(n+1)}$.

The bounds for the solutions obtained by Baker's method are rather large, thus the solutions practically cannot be found by simple enumeration. For a similar problem Baker and Davenport [2] proposed a method to reduce drastically the bound by using continued fraction reduction. Pethő and Schulenberg [50] replaced the continued fraction reduction by the LLL-algorithm and gave a general method to solve (1) for the totally real case with $m = 1$ and arbitrary n . Tzanakis and de Weger [61] describe the general case. Finally, Bilu and Hanrot [5] were able to replace the LLL-algorithm by the much faster continued fraction method and solve Thue equations up to degree 1000.

4 Families of Thue Equations

We study families of Thue equations

$$F_a(X, Y) = \pm 1, \quad a \in \mathbb{N} \quad (2)$$

where $F_a \in \mathbb{Z}[a][X, Y]$ is an irreducible binary form of degree of at least 3 with coefficients which are integer polynomials in a . In the investigation of such families usually only two types of solutions appear: Firstly, there are *polynomial solutions* $X(a), Y(a) \in \mathbb{Z}[a]$ which satisfy (2) in $\mathbb{Z}[a]$, and secondly, there occur (sometimes) single solutions for a few small values of the parameter a . However, Lettl [30] points out that the family $X^6 - (a-1)Y^6 = a^2$ does not have any polynomial solution, but there are sporadic solutions for infinitely many values of the parameter a .

The first infinite parametrized families of Thue equations were considered by Thue [58] himself: He proved that the equation

$$(a+1)X^n - aY^n = 1, \quad X > 0, Y > 0 \quad (3)$$

has only the solution $x = y = 1$ for a suitably large in relation to prime $n \geq 3$. For $n = 3$, the equation (3) has only this solution for $a \geq 386$. Of course, Bennett's result [4] cited in Section 2 implies that this is true for all $n \geq 3$ and $a \geq 1$.

For a description of the techniques used to solve families of Thue equations, we refer to Heuberger [20]. Some automated procedures are presented in [26].

4.1 Families of Fixed Degree

In 1990, Thomas [53] investigated for the first time a parametrized family of cubic Thue equations of positive discriminant. Since 1990, the following particular families of Thue equations have been studied:

1. $X^3 - (a-1)X^2Y - (a+2)XY^2 - Y^3 = 1$.

Thomas [53] and Mignotte [36] proved that for $a \geq 4$, the only solutions are $(0, -1)$, $(1, 0)$ and $(-1, +1)$, while for the cases $0 \leq a \leq 4$ there exist some nontrivial solutions, too, which are given explicitly in [53]. For the same form $F_a(X, Y)$, all solutions of the Thue inequality $|F_a(X, Y)| \leq 2a+1$ have been found by Mignotte, Pethő, and Lemmermeyer [39].

2. $X^3 - aX^2Y - (a+1)XY^2 - Y^3 = X(X+Y)(X-(a+1)Y) - Y^3 = 1$.

Lee [29] and independently Mignotte and Tzanakis [40] proved that for $a \geq 3.33 \cdot 10^{23}$ there are only the solutions

$$(1, 0), (0, -1), (1, -1), (-a-1, -1), (1, -a).$$

Mignotte [37] could prove the same result for all $a \geq 3$.

3. Wakabayashi [66] proved that for $a \geq 1.35 \cdot 10^{14}$, the equation $X^3 - a^2XY^2 + Y^3 = 1$ has exactly the five solutions $(0, 1)$, $(1, 0)$, $(1, a^2)$, $(\pm a, 1)$.
4. Togbe [60] considered the equation $X^3 - (n^3 - 2n^2 + 3n - 3)X^2Y - n^2XY^2 - Y^3 = \pm 1$. If $n \geq 1$, the only solutions are $(\pm 1, 0)$ and $(0, \pm 1)$.
5. Wakabayashi [64]: $|X^3 + aXY^2 + bY^3| \leq a + |b| + 1$ for arbitrary b and $a \geq 360b^4$ as well as for $b \in \{1, 2\}$ and $a \geq 1$. He uses Padé approximations.
6. Thomas [55]: Let b, c be nonzero integers such that the discriminant of $t^3 - bt^2 + ct - 1$ is negative, $\Delta = 4c - b^2 > 0$, and $c \geq \min\{4.2 \times 10^{41} \times |b|^{2.32}, 3.6 \times 10^{41} \times \Delta^{1.1582}\}$. Then the Thue equation $X^3 - bX^2Y + cXY^2 - Y^3 = 1$ only has the trivial solutions $(1, 0)$, $(0, -1)$.
7. $X(X - a^{d_2}Y)(X - a^{d_3}Y) \pm Y^3 = 1$.

This family was investigated by Thomas [54]. He proved that for $0 < d_2 < d_3$ and

$$a \geq (2 \cdot 10^6 \cdot (d_2 + 2d_3))^{4.85/(d_3 - d_2)}$$

nontrivial solutions cannot exist. He also investigated this family with a^{d_1} and a^{d_2} replaced by monic polynomials in a of degrees d_1 and d_2 , respectively (see Theorem 5).

8. $X^4 - aX^3Y - X^2Y^2 + aXY^3 + Y^4 = X(X - Y)(X + Y)(X - aY) + Y^4 = \pm 1$.
This quartic family was solved by Pethő [49] for large values of a ; Mignotte, Pethő, and Roth [38] solved it completely: The only solutions are $\pm\{(0, 1), (1, 0), (1, 1), (1, -1), (a, 1), (1, -a)\}$ for $|a| \notin \{2, 4\}$. If $|a| = 4$, four more solutions exist. If $|a| = 2$, the family is reducible.
9. $X^4 - aX^3Y - 3X^2Y^2 + aXY^3 + Y^4 = \pm 1$ has been solved for $a \geq 9.9 \cdot 10^{27}$ by Pethő [49].
10. $|bX^4 - aX^3Y - 6bX^2Y^2 + aXY^3 + bY^4| \leq N$.
For $b = 1$ and $N = 1$, this equation has been solved completely by Lettl and Pethő [31]; Chen and Voutier [9] solved it independently by using the hypergeometric method. For the same form binary form $F_{a,b}(X, Y)$, Lettl, Pethő and Voutier [33] proved that $|F_a(X, Y)| \leq 6a + 7$ has only trivial primitive solutions for $a \geq 58$, if $b = 1$. Furthermore, $x^2 + y^2 \leq \max\{25a^2/(64b^2), 4N^2/a\}$ if $a > 308b^4$, cf. Yuan [67].
11. Togbé [59] gives all solutions to $X^4 - a^2X^3Y - (a^3 + 2a^2 + 4a + 2)X^2Y^2 - a^2XY^3 + Y^4 = 1$ for $a \geq 1.191 \cdot 10^{19}$ and $a, a + 2, a^2 + 4$ squarefree.
12. $|X^4 - a^2X^2Y^2 + Y^4| = |X^2(X - a)(X + a) + Y^4| \leq a^2 - 2$
This family of Thue inequalities has only trivial solutions with $|y| \leq 1$ for $a \geq 8$ (Wakabayashi [62]).
13. $|X^4 + 4aX^3Y + 6aX^2Y^2 + 4a^2XY^3 + a^2Y^4| \leq a^2$ has been solved for $a \geq 205$ by Chen and Voutier [8].
14. Dujella and Jadrijević [12], [13] prove that $|X^4 - 4cX^3Y + (6c + 2)X^2Y^2 + 4cXY^3 + Y^4| \leq 6c + 4$ has only trivial solutions for all $c \geq 3$.
15. $X(X - Y)(X - aY)(X - bY) - Y^4 = \pm 1$.
All solutions of this two-parametric family are known for $10^{2 \cdot 10^{28}} < a + 1 < b \leq a(1 + (\log a)^{-4})$, cf. Pethő and Tichy [51]. The case of $b = a + 1$ has been considered by Heuberger, Pethő and Tichy [23], where all solutions could be determined for all $a \in \mathbb{Z}$.
16. Jadrijević [27] proves that for every $0.5 < s \leq 1$, there is an effectively computable constant $P(s)$ such that if $a \neq 0$ and $\max\{|a|, |b|\} \geq P(s)$ and $\gcd(a, b) \geq \max\{|a|^s, |b|^s\}$, then the equation $X^4 - 2abX^3Y + 2(a^2 - b^2 + 1)X^2Y^2 + 2abXY^3 + Y^4 = 1$ only has trivial solutions. In particular, $P(0.999) = 10^{27}$ and $P(0.501) = 10^{36836}$.

17. Wakabayashi [63] found all solutions of $|X^4 - a^2X^2Y^2 - bY^4| \leq a^2 + b - 1$ for $a \geq 5.3 \cdot 10^{10}b^{6.22}$.
18. $X(X^2 - Y^2)(X^2 - a^2Y^2) - Y^5 = \pm 1$.
For $a > 3.6 \cdot 10^{19}$, all solutions have been found by Heuberger [18].
19. Gaál and Lettl [15] investigated the family $X^5 + (a-1)X^4Y - (2a^3 + 4a + 4)X^3Y^2 + (a^4 + a^3 + 2a^2 + 4a - 3)X^2Y^3 + (a^3 + a^2 + 5a + 3)XY^4 + Y^5 = \pm 1$ and found all solutions for $|a| \geq 3.3 \cdot 10^{15}$. The remaining cases have been solved in Gaál and Lettl [16].
20. Levesque and Mignotte [34] found all solutions of the equation $X^5 + 2X^4Y + (a+3)X^3Y^2 + (2a+3)X^2Y^3 + (a+1)XY^4 - Y^5 = \pm 1$ for sufficiently large a .
21. $X^6 - 2aX^5Y - (5a+15)X^4Y^2 - 20X^3Y^3 + 5aX^2Y^4 + (2a+6)XY^5 + Y^6 \in \{\pm 1, \pm 27\}$ was investigated by Lettl, Pethő, and Voutier. They found all solutions for $a \geq 89$ by hypergeometric methods [33] and all solutions for $a < 89$ by using Baker's method [32]. In [33], they also proved that $|F_a(X, Y)| \leq 120a + 323$ (for the form $F_a(X, Y)$ considered) has only trivial primitive solutions for $a \geq 89$.
22. $X^8 - 8nX^7Y - 28X^6Y^2 + 56nX^5Y^3 + 70X^4Y^4 - 56nX^3Y^5 - 28nX^2Y^6 + 8nXY^7 + Y^8 = \pm 1$ has only trivial solutions for $n \in \{a \in \mathbb{Z} : a + b\sqrt{2} = (1 + \sqrt{2})^{2k+1}, k \in \mathbb{N}\}$ with $n \geq 6.71 \cdot 10^{32}$. (Heuberger, Togbé and Ziegler [26]).

A more detailed survey on cubic families is contained in Wakabayashi [65].

4.2 Families of Relative Thue Equations

A few families of relative Thue equations have also been solved, i.e., families where the parameters and the solutions are elements of the same imaginary quadratic number field.

So let $D > 0$ be an integer, $k := \mathbb{Q}(\sqrt{-D})$, $\mathfrak{o}_k$ its ring of algebraic integers, and μ a root of unity in $\mathfrak{o}_k$.

1. For $t \in \mathfrak{o}_k$ with $|t| \geq 3.03 \cdot 10^9$, the only solutions $(x, y) \in \mathfrak{o}_k^2$ to $X^3 - (t-1)X^2Y - (t+2)XY^2 - Y^3 = \mu$ satisfy $\max\{|x|, |y|\} \leq 1$ and can be listed explicitly (Heuberger, Pethő, and Tichy [24]).
2. For $t \in \mathfrak{o}_k$ with $|t| > 2.88 \cdot 10^{33}$, the only solutions $(x, y) \in \mathfrak{o}_k^2$ to $X^3 - tX^2Y - (t+1)XY^2 - Y^3 = \mu$ satisfy $\min\{|x|, |y|\} \leq 1$ and can be listed explicitly (Ziegler [68]).
3. For $s, t \in \mathfrak{o}_k$ with $|t| \geq 5.3 \cdot 10^{10}|s|^{12.44}$ or $s = 1$ and $|t| > \sqrt{550}$, all solutions $(x, y) \in \mathfrak{o}_k^2$ to $|X^4 - t^2X^2Y^2 + s^2Y^4| \leq |t|^2 - |s|^2 - 2$ are explicitly known (Ziegler [69]).

4.3 Families of Arbitrary Degree

Moreover, some general families of arbitrary degree have been considered. Apart from (3), the investigated general families are of the shape

$$F_a(X, Y) := \prod_{i=1}^n (X - p_i(a)Y) - Y^n = \pm 1, \quad (4)$$

where $p_1, \dots, p_n \in \mathbb{Z}[a]$ are polynomials, which have been called *split families* by E. Thomas [54]. For $i = 1, \dots, n$ it can easily be seen that $(X, Y) \in \{\pm(p_i, 1), (\pm 1, 0)\}$ are solutions. Thomas conjectured that if

$$p_1 = 0, \quad \deg p_2 < \dots < \deg p_n$$

and the polynomials are monic, there are no further solutions for sufficiently large values of the parameter a . In [54] he proved this conjecture for $n = 3$ under some technical hypothesis:

Theorem 5. Let $u = \pm 1$, $a(t), b(t) \in \mathbb{Z}[t]$ be monic polynomials and $a := \deg a(t)$, $b := \deg b(t)$ with $0 < a < b$. We write $A(t) := a(t)/t^a - 1$ and $B(t) := b(t)/t^b - 1$ and define for $n \geq 1$

$$W(n) := \sum_{j=1}^{\infty} \frac{(-1)^{j+1}}{j} (b \cdot A(n)^j - a \cdot B(n)^j),$$

which can be written in powers of $1/n$ as $W(n) = \sum_{j=1}^n w_j n^{-j}$. Further we define $J := \min\{j \in \mathbb{N} : w_j \neq 0\}$.

If $J \neq b - a$ or $J = b - a \wedge 3w_J + 2b + a \neq 0 \wedge 3w_J - 2(b - a) \neq 0$, then there is an effectively computable constant c_7 depending on the coefficients of $a(t)$ and $b(t)$ such that for $n \geq c_7$ the family of Thue equations

$$X(X - a(n)Y)(X - b(n)Y) + uY^3 = \pm 1$$

only has the solutions

$$\pm\{(1, 0), (0, u), (a(n)u, u), (b(n)u, u)\}.$$

Halter-Koch, Lettl, Pethő and Tichy [17] considered (4) for $p_1 = 0, p_2 = d_2, \dots, p_{n-1} = d_{n-1}$ and $p_n = a$, where $d_2, \dots, d_{n-1}$ are fixed distinct integers. They found all solutions for sufficiently large values of a assuming a conjecture of Lang and Waldschmidt [28]—which is a very sharp bound for linear forms in logarithms of algebraic numbers—:

Theorem 6. Let $n \geq 3$, $p_1 = 0, p_2 = d_2, \dots, p_{n-1} = d_{n-1}$ be distinct integers and $p_n = a$. Let $\alpha = \alpha(\alpha)$ be a zero of $P(x) = \prod_{i=1}^n (x - p_i) - d$ with $d = \pm 1$ and suppose that the index I of $(\alpha - d_1, \dots, \alpha - d_{n-1})$ in $\mathfrak{D}^\times$, the group of units of $\mathfrak{D} := \mathbb{Z}[\alpha]$, is bounded by a constant $J = J(d_1, \dots, d_{n-1}, n)$ for every a from some subset $\Omega \in \mathbb{Z}$. Assume further that the Lang-Waldschmidt conjecture is true. Then for all but finitely many values of $a \in \Omega$ the Diophantine equation

$$\prod_{i=1}^n (x - p_i y) - dy^n = \pm 1$$

has only solutions $(x, y) \in \mathbb{Z}^2$ with $|y| \leq 1$, except for the cases of $n = 3$ and $|d_2| = 1$ or $n = 4$ and $(d_2, d_3) \in \{(1, -1), (\pm 1, \pm 2)\}$, where it has exactly one more solution for every value of a .

If $\mathbb{Q}(\alpha)$ is primitive over $\mathbb{Q}$ — especially if n is prime — then there exists a bound $J = J(d_1, \dots, d_{n-1}, n)$ for the index I by lower bounds for the regulator of $\mathfrak{D}$ (cf. Pohst and Zassenhaus [52], chapter 5.6, (6.22)). Applying the theory of Hilbertian fields and results on thin sets, primitivity is proved for almost all choices (in the sense of density) of the parameters, cf. [17].

The two exceptional families are those considered under 2 and 8 in the list in Section 4.1.

A similar family has been considered by Heuberger in [19], however, in this case, the result is unconditionally true:

Theorem 7. Let $n \geq 4$ be an integer, $d_2, \dots, d_{n-1}$ pairwise distinct integers and a an integral parameter. Furthermore we assume

$$d_2 + \dots + d_{n-1} \neq 0 \quad \text{or} \quad d_2 \cdots d_{n-1} \neq 0.$$

Let

$$F_a(X, Y) := (X + aY)(X - d_2Y)(X - d_3Y) \cdots (X - d_{n-1}Y)(X - aY) - Y^n.$$

Then there exists a (computable) constant c_8 depending only on the degree n and $d_2, \dots, d_{n-1}$, such that for all $a \geq c_8$, the only solutions $(x, y) \in \mathbb{Z}^2$ of the Diophantine equation

$$F_a(X, Y) = \pm 1$$

are $\pm\{(1, 0), (-a, 1), (d_2, 1), (d_3, 1), \dots, (d_{n-1}, 1), (a, 1)\}$.

In [25], Heuberger and Tichy considered a multivariate version of (4):

Theorem 8. Let $n \geq 4$, $r \geq 1$, $p_i \in \mathbb{Z}[A_1, \dots, A_r]$ for $1 \leq i \leq n$. We make the following assumptions on the polynomials p_i :

$$\deg p_1 < \dots < \deg p_{n-2} < \deg p_{n-1} = \deg p_n,$$

$$\text{LH}(p_n) = \text{LH}(p_{n-1}), \text{ but } p_n \neq p_{n-1}.$$

Furthermore we suppose that for $p \in \{p_1, \dots, p_n, p_n - p_{n-1}\}$, there exist positive constants t_p, c_p such that

$$|(\text{LH}(p))(a_1, \dots, a_r)| \geq c_p \cdot (\min_k a_k)^{\deg p} \quad \text{for } a_1, \dots, a_r \geq t_p.$$

Let

$$F_{a_1, \dots, a_r}(X, Y) := \prod_{i=1}^n (X - p_i(a_1, \dots, a_r)Y) - Y^n.$$

For every constant $C > 1$ there is a constant t_0 such that for all integers $a_1, \dots, a_r$ satisfying $t_0 \leq \min_k a_k$ and

$$\max_k a_k \leq C \cdot \min_k a_k,$$

the Diophantine equation

$$F_{a_1, \dots, a_r}(x, y) = \pm 1$$

considered for $x, y \in \mathbb{Z}$ only has the solutions $\{(\pm 1, 0)\} \cup \{\pm(p_i(a_1, \dots, a_r), 1) : 1 \leq i \leq n\}$.

In Heuberger [21] Thomas' conjecture is proved under some technical hypothesis:

Theorem 9. Let $n \in \mathbb{N}$, $n \geq 3$ and $p_i \in \mathbb{Z}[a]$ be monic polynomials for $i = 1, \dots, n$. We write

$$p_i(a) = a^{d_i} + k_i a^{d_i-1} + \text{terms of lower degree}, \quad i = 2, \dots, n,$$

allow $p_1 = 0$ and assume

$$d_1 < d_2 < \dots < d_{n-1} < d_n \quad \text{and} \quad n + d_2 \geq 4.$$

Let

$$\delta_i := \begin{cases} 1 & \text{if } d_i - d_{i-1} = 1, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad e := \sum_{i=2}^n d_i.$$

If $\delta_4 = 1$ or

$$(e - d_2 + 2d_3)(k_2 - \delta_2) + (-e - 2d_2 + d_3)k_3 + (d_3 - d_2) \sum_{i=4}^n k_i \notin \{2\delta_3, -(e + d_3)\delta_3\}, \quad (5)$$

then there is a (computable) constant $c_9 = c_9(p_1, \dots, p_n)$ depending on the coefficients of the polynomials p_i such that for all integers $a \geq c_9$ the Diophantine equation

$$F_a(X, Y) := \prod_{i=1}^n (X - p_i(a)Y) - Y^n = \pm 1$$

only has the solutions

$$(\pm 1, 0) \text{ and } \pm(p_i(a), 1), 1 \leq i \leq n.$$

In [21], there is also a version with a stronger technical hypothesis than that in (5). For $n = 3$, that version improves Theorem 5.

Especially there are only trivial solutions if

$$\max(\deg p_1, 0) < \deg p_2 < \deg p_3 < \dots < \deg p_n$$

$$\max(\deg p_1, 0) + \deg p_2 + \dots + \deg p_n < 15.$$

In Heuberger [22], an explicit constant c_9 for Theorem 9 is given:

$$c_9 = \exp \left(1.01(n+1)(n-1)!(n-1)^{n-2} \exp(1.04(n-2)(nd_n - n + 3)) \binom{nd_n - 1}{n-3} (2P+1)^{nd_n} \right),$$

where $d_j = \deg p_j$ and P is an upper bound for the absolute values of the coefficients of the p_j , $j = 1, \dots, n$.

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