

An extension of existence and mean approximation of fixed points of generalized hybrid non-self mappings in Hilbert spaces

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Abstract

In this paper we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore we prove a mean convergence theorem of Baillon's type for widely more generalized hybrid non-self mappings in Hilbert spaces.

1 Introduction

Let H be a real Hilbert space and let C be a non-empty subset of H . In 2010, Kocourek, Takahashi and Yao [14] defined a class of nonlinear mappings in a Hilbert space. A mapping T from C into H is said to be generalized hybrid if there exist real numbers α and β such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for any $x, y \in C$. We call such a mapping an (α, β) -generalized hybrid mapping. We observe that the class of the mappings covers the classes of well-known mappings. For example, an (α, β) -generalized hybrid mapping is nonexpansive [19] for $\alpha = 1$ and $\beta = 0$, that is, $\|Tx - Ty\| \leq \|x - y\|$ for any $x, y \in C$. It is nonspreading [16] for $\alpha = 2$ and $\beta = 1$, that is, $2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. It is also hybrid [20] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, that is, $3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$ for any $x, y \in C$. They proved fixed point theorems for such mappings; see also Kohsaka and Takahashi [15] and Iemoto and Takahashi [9]. Moreover they proved a nonlinear ergodic theorem. Furthermore they defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be super hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + (1 - \alpha + \gamma)\|x - Ty\|^2 \\ & \leq (\beta + (\beta - \alpha)\gamma)\|Tx - y\|^2 + (1 - \beta - (\beta - \alpha - 1)\gamma)\|x - y\|^2 \\ & \quad + (\alpha - \beta)\gamma\|x - Tx\|^2 + \gamma\|y - Ty\|^2 \end{aligned}$$

for any $x, y \in C$. A generalized hybrid mapping with a fixed point is quasinonexpansive. However a super hybrid mapping is not quasi-nonexpansive generally even if it has a fixed point. Very recently, the author [12] also defined a class of nonlinear mappings in a Hilbert space which covers the class of contractive mappings and the class of generalized hybrid mappings. A mapping T from C into H is said to be widely generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon$ and ζ such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \max\{\varepsilon\|x - Tx\|^2, \zeta\|y - Ty\|^2\} \leq 0$$

for any $x, y \in C$. Furthermore the author [13] defined a class of nonlinear mappings in a Hilbert space which covers the class of super hybrid mappings and the class of widely generalized hybrid mappings. A mapping T from C into H is said to be widely more generalized hybrid if there exist real numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and η such that

$$\alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0$$

for any $x, y \in C$. Then we prove fixed point theorems for such new mappings in a Hilbert space. Furthermore we prove nonlinear ergodic theorems of Baillon's type in a Hilbert space. It seems that the results are new and useful. For example, using our fixed point theorems, we can directly prove Browder and Petryshyn's fixed point theorem [5] for strictly pseudocontractive mappings and Kocourek, Takahashi and Yao's fixed point theorem [14] for super hybrid mappings. On the other hand, Hojo, Takahashi and Yao [8] defined a more broad class of nonlinear mappings than the class of generalized hybrid mappings. A mapping T from C into H is said to be extended hybrid if there exist real numbers α, β and γ such that

$$\begin{aligned} & \alpha(1 + \gamma)\|Tx - Ty\|^2 + (1 - \alpha(1 + \gamma))\|x - Ty\|^2 \\ & \leq (\beta + \alpha\gamma)\|Tx - y\|^2 + (1 - (\beta + \alpha\gamma))\|x - y\|^2 \\ & \quad - (\alpha - \beta)\gamma\|x - Tx\|^2 - \gamma\|y - Ty\|^2 \end{aligned}$$

for any $x, y \in C$. Furthermore they proved a fixed point theorem for generalized hybrid non-self mappings by using the extended hybrid mapping.

In this paper we prove a fixed point theorem for widely more generalized hybrid non-self mappings in Hilbert spaces. Furthermore we prove a mean convergence theorem of Baillon's type for widely more generalized hybrid non-self mappings in a Hilbert space.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$ and

let C be a non-empty subset of H . We denote by $\overline{\text{co}}C$ the closure of the convex hull of C . In a Hilbert space it is known that

$$\|(1 - \lambda)x + \lambda y\|^2 = (1 - \lambda)\|x\|^2 + \lambda\|y\|^2 - (1 - \lambda)\lambda\|x - y\|^2$$

for any $x, y \in H$ and for any $\lambda \in \mathbb{R}$; see [19]. Furthermore in a Hilbert space we obtain that

$$2\langle x - y, z - w \rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2$$

for any $x, y, z, w \in H$. Let T be a mapping from C into H . We denote by $F(T)$ the set of fixed points of T . A mapping T from C into H with $F(T) \neq \emptyset$ is said to be quasi-nonexpansive if $\|x - Ty\| \leq \|x - y\|$ for any $x \in F(T)$ and for any $y \in C$. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping T is closed and convex; see Ito and Takahashi [10]. It is not difficult to prove such a result in a Hilbert space; see, for instance, [22]. Let C be a non-empty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y \in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. The mapping P_C is said to be the metric projection from H onto C . It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u \rangle \geq 0$$

for any $x \in H$ and for any $u \in C$; see [19] for more details. For proving a mean convergence theorem, we also need the following lemma proved by Takahashi and Toyoda [21].

Lemma 2.1. *Let C be a non-empty closed convex subset of H . Let P be the metric projection from H onto C . Let $\{u_n\}$ be a sequence in H . If $\|u_{n+1} - u\| \leq \|u_n - u\|$ for any $u \in C$ and for any $n \in \mathbb{N}$, then $\{Pu_n\}$ converges strongly to some $u_0 \in C$.*

Let ℓ^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(\ell^\infty)^*$ (the dual space of ℓ^∞). Then we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$. Sometimes we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on ℓ^∞ is said to be a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is said to be a Banach limit on ℓ^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on ℓ^∞ . If μ is a Banach limit on ℓ^∞ , then for $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$,

$$\liminf_{n \rightarrow \infty} x_n \leq \mu_n(x_n) \leq \limsup_{n \rightarrow \infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in \ell^\infty$ and $x_n \rightarrow a \in \mathbb{R}$, then we obtain $\mu(f) = \mu_n(x_n) = a$. See [18] for the proof of existence of a Banach limit and its other elementary properties. Using means and the Riesz theorem, we have the following result; see [17] and [18].

Lemma 2.2. *Let H be a Hilbert space, let $\{x_n\}$ be a bounded sequence in H and let μ be a mean on ℓ^∞ . Then there exists a unique point $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$ such that*

$$\mu_n \langle x_n, y \rangle = \langle z_0, y \rangle$$

for any $y \in H$.

The author [13] proved by Lemma 2.2 the following fixed point theorem.

Theorem 2.1. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\varepsilon + \eta \geq 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

As a direct consequence of Theorem 2.1, we obtain the following.

Theorem 2.2. *Let H be a real Hilbert space, let C be a bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into itself which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$ and $\zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$ and $\varepsilon + \eta \geq 0$.

Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

Using Theorem 2.2, we prove a fixed point theorem for widely more generalized hybrid non-self mappings in a Hilbert space; see [11].

Theorem 2.3. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $(\alpha + \beta)\lambda + \zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $(\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0$.

Suppose that for any $x \in C$ there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1 - \lambda)m \leq 1$ and $Tx = x + m(y - x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

Moreover, for proving a mean convergence theorem of Baillon's type in a Hilbert space, we need the following lemmas and theorems; see [11].

Lemma 2.3. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition:*

$$\alpha + \gamma + \varepsilon + \eta > 0, \text{ or } \alpha + \beta + \zeta + \eta > 0.$$

Then $F(T)$ is closed.

Lemma 2.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0 \text{ and } \alpha + \gamma + \varepsilon + \eta > 0;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0 \text{ and } \alpha + \beta + \zeta + \eta > 0.$$

Then $F(T)$ is convex.

Lemma 2.5. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0 \text{ and } \varepsilon + \eta \geq 0;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0 \text{ and } \zeta + \eta \geq 0.$$

Then T is quasi-nonexpansive.

Moreover we obtain the following.

Lemma 2.6. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta.$$

Then $(1 - \lambda)T + \lambda I$ is quasi-nonexpansive.

Now, using the technique developed by Takahashi [17], by Lemmas 2.3, 2.4, 2.5 and 2.6 we obtain the following mean convergence theorems for widely more generalized hybrid non-self mappings in a Hilbert space.

Theorem 2.4. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):*

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \gamma > 0 \text{ and } \varepsilon + \eta \geq 0;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \alpha + \beta > 0 \text{ and } \zeta + \eta \geq 0.$$

Then for any $x \in C(T; 0) = \{z \mid T^n z \in C, \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P T^n x$.

Theorem 2.5. Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which has a fixed point and satisfies the condition (1) or (2):

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta.$$

Then for any $x \in C(T; \lambda) = \{z \mid ((1 - \lambda)T + \lambda I)^n z \in C, \forall n \in \mathbb{N} \cup \{0\}\}$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$.

Moreover we obtain the following.

Theorem 2.6. Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which satisfies the following condition (1) or (2):

$$(1) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta;$$

$$(2) \quad \alpha + \beta + \gamma + \delta \geq 0, \text{ and there exists } \lambda \in \mathbb{R} \text{ such that } 0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta.$$

Suppose that for any $x \in C$, there exist $m \in \mathbb{R}$ and $y \in C$ such that $0 \leq (1 - \lambda)m \leq 1$ and $Tx = x + m(y - x)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$.

3 Fixed point theorem

Let H be a real Hilbert space and let C be a non-empty subset of H . A mapping T from C into H was said to be widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha\|Tx - Ty\|^2 + \beta\|x - Ty\|^2 + \gamma\|Tx - y\|^2 + \delta\|x - y\|^2 \\ & + \varepsilon\|x - Tx\|^2 + \zeta\|y - Ty\|^2 + \eta\|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned}$$

for any $x, y \in C$; see Introduction. Such a mapping T is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid; see [13]. An $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping is generalized hybrid in the sense of Kocourek, Takahashi and Yao [14] if $\alpha + \beta = -\gamma - \delta = 1$ and $\varepsilon = \zeta = \eta = 0$. Moreover it is an extension of widely generalized hybrid mappings in the sense of Kawasaki and Takahashi [12]; see also [11].

Theorem 3.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which satisfies the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \gamma + \varepsilon + \eta > 0$, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $(\alpha + \beta)\lambda + \zeta + \eta \geq 0$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, $\alpha + \beta + \zeta + \eta > 0$, and there exists $\lambda \in \mathbb{R}$ such that $\lambda \neq 1$ and $(\alpha + \gamma)\lambda + \varepsilon + \eta \geq 0$.

Let

$$M = \begin{cases} [0, \infty), & \text{if } \lambda < 1, \\ (-\infty, 0], & \text{if } \lambda > 1. \end{cases}$$

Suppose that for any $x \in C$ there exist $m \in M$ and $y \in C$ such that $Tx = x + m(y - x)$. Then T has a fixed point. In particular, a fixed point of T is unique in the case of $\alpha + \beta + \gamma + \delta > 0$ on the conditions (1) and (2).

As a direct consequence of Theorem 3.1, we obtain the following fixed point theorem for generalized hybrid mappings as an extension of [8, Theorem 3.4].

Theorem 3.2. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an (α, β) -generalized hybrid mapping from C into H . Suppose that for any $x \in C$ there exist $m \in [0, \infty)$ and $y \in C$ such that $Tx = x + m(y - x)$. Then T has a fixed point.*

Example 3.1. Let $H = \mathbb{R}$, let $C = [0, \frac{\pi}{2}]$, let $Tx = (1 + 2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H . Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $(\alpha + \beta)\lambda + \zeta + \eta = \frac{\pi-3}{1+\pi} \geq 0$. Let $m = 1 + \pi$ and $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 3.1 T has a unique fixed point.

Example 3.2. Let $H = \mathbb{R}^2$, let $C = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, let

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1 + 2\sqrt{x_1 x_2}) \cos \sqrt{x_1 x_2} - 2x_1 x_2 \\ (1 + x_1 + x_2) \cos \frac{x_1 + x_2}{2} - \frac{(x_1 + x_2)^2}{2} \end{pmatrix}$$

and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H . Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $(\alpha + \beta)\lambda + \zeta + \eta = \frac{\pi-3}{1+\pi} \geq 0$. Let $m = 1 + \pi$ and

$$y = \begin{pmatrix} x_1 + \frac{(1+2\sqrt{x_1 x_2}) \cos \sqrt{x_1 x_2} - 2x_1 x_2 - x_1}{1+\pi} \\ x_2 + \frac{(1+x_1+x_2) \cos \frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} - x_2}{1+\pi} \end{pmatrix}$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 3.1 T has a unique fixed point.

4 Mean convergence theorem

In this section we prove a mean convergence theorem of Baillon's type in a Hilbert space.

Theorem 4.1. *Let H be a real Hilbert space, let C be a non-empty bounded closed convex subset of H and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H which satisfies $F(T) \neq \emptyset$ and the following condition (1) or (2):*

- (1) $\alpha + \beta + \gamma + \delta \geq 0$, and there exists $\lambda \in \mathbb{R}$ such that $0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta < \alpha + \gamma + \varepsilon + \eta$;
- (2) $\alpha + \beta + \gamma + \delta \geq 0$, and there exists $\lambda \in \mathbb{R}$ such that $0 \leq (\alpha + \beta)\lambda + \zeta + \eta < \alpha + \beta + \zeta + \eta$.

Let

$$M = \begin{cases} [0, \infty), & \text{if } \lambda < 1, \\ (-\infty, 0], & \text{if } \lambda > 1. \end{cases}$$

Suppose that for any $x \in C$ there exist $m \in M$ and $y \in C$ such that $Tx = x + m(y - x)$. Then for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1 - \lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$.

As a direct consequence of Theorem 4.1, we obtain the following mean convergence theorem for super hybrid mappings [8, Theorem 4.2].

Theorem 4.2. *Let H be a real Hilbert space, let C be a non-empty closed convex subset of H and let T be an (α, β, γ) -super hybrid mapping from C into itself with $F(T) \neq \emptyset$. Suppose that $\gamma \geq 0$. Then for any $x \in C$,*

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} \left(\frac{1}{1+\gamma} T + \frac{\gamma}{1+\gamma} I \right)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P \left(\frac{1}{1+\gamma} T + \frac{\gamma}{1+\gamma} I \right)^n x$.

Example 4.1. Let $H = \mathbb{R}$, let $C = [0, \frac{\pi}{2}]$, let $Tx = (1+2x)\cos x - 2x^2$ and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H . Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta = \frac{\pi-3}{1+\pi} < 1 = \alpha + \gamma + \varepsilon + \eta$. Let $m = 1 + \pi$ and $y = x + \frac{(1+2x)(\cos x - x)}{1+\pi}$ for any $x \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 4.1 for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P((1-\lambda)T + \lambda I)^n x$.

Example 4.2. Let $H = \mathbb{R}^2$, let $C = [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}]$, let

$$T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (1+2\sqrt{x_1 x_2}) \cos \sqrt{x_1 x_2} - 2x_1 x_2 \\ (1+x_1+x_2) \cos \frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} \end{pmatrix}$$

and let $\alpha = 1$, $\beta = \gamma = 11$, $\delta = -22$, $\varepsilon = \zeta = -12$ and $\eta = 1$. Then T is an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid mapping from C into H . Moreover $\alpha + \beta + \gamma + \delta = 1 \geq 0$ and $\alpha + \gamma + \varepsilon + \eta = 1 > 0$ hold. Let $\lambda = \frac{2+3\pi}{3(1+\pi)}$. Then we obtain $0 \leq (\alpha + \gamma)\lambda + \varepsilon + \eta = \frac{\pi-3}{1+\pi} < 1 = \alpha + \gamma + \varepsilon + \eta$. Let $m = 1 + \pi$ and

$$y = \begin{pmatrix} x_1 + \frac{(1+2\sqrt{x_1 x_2}) \cos \sqrt{x_1 x_2} - 2x_1 x_2 - x_1}{1+\pi} \\ x_2 + \frac{(1+x_1+x_2) \cos \frac{x_1+x_2}{2} - \frac{(x_1+x_2)^2}{2} - x_2}{1+\pi} \end{pmatrix}$$

for any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in C$. Then we obtain $y \in C$ and $Tx = x + m(y - x)$. Therefore by Theorem 4.1 for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} ((1-\lambda)T + \lambda I)^k x$$

is weakly convergent to a fixed point p of T , where P is the metric projection from H onto $F(T)$ and $p = \lim_{n \rightarrow \infty} P((1 - \lambda)T + \lambda I)^n x$.

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