

Birational-integral extensions

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We shall study extensions of rings, especially, integral extensions of rings. Let R be an integral domain and let A be a ring which is birational and integral over R . For this purpose, the seminormalization ${}^+_A R$ of R in A , defined as follows, plays an important role.

Definition. (i) ${}^+_A R = \{\alpha \in A \mid \alpha \in R_{\mathfrak{p}} + J(A_{\mathfrak{p}}) \text{ for any } \mathfrak{p} \in \text{Spec } R\}$, where $J(A_{\mathfrak{p}})$ is the Jacobson radical of $A_{\mathfrak{p}}$.
(ii) If $R = {}^+_A R$, then we call R seminormal in A .
(iii) $\widetilde{{}_A R} = \{\alpha \in A \mid \alpha^{p^n} \in R_{\mathfrak{p}} + J(A_{\mathfrak{p}}) \text{ for some non-zero integer } n, \text{ where } p \text{ is the characteristic of the residue field } k(\mathfrak{p}), \mathfrak{p} \in \text{Spec } R\}$ is called the weak normalization of R in A .

The seminormalization and the weak normalization are characterized by the following proposition.

Proposition 1. ${}^+_A R$ (resp. $\widetilde{{}_A R}$) is the greatest subring R' of A such that $R' \supseteq R$, and

(i) for any $\mathfrak{p} \in \text{Spec } A$, there is exactly one $\mathfrak{p}' \in \text{Spec } R'$ lying over $\mathfrak{p}$, that is, the canonical map

$$\text{Spec } R' \longrightarrow \text{Spec } R$$

is injective (especially, since R' is integral over R , this map is an isomorphism).

(ii) the canonical homomorphism

$$k(\mathfrak{f}) \longrightarrow k(\mathfrak{f}')$$

is an isomorphism (resp. gives a pure inseparable extension),
where $k(\mathfrak{f})$ is the residue field of R .

Corollary 2. We have

(i) ${}_A^+({}_A^+R) = {}_A^+R$, that is, ${}_A^+R$ is seminormal in A .

(ii) Let $R \subseteq B \subseteq {}_A^+R$. Then ${}_B^+R = B$.

The proof of this corollary is easy.

Definition. Let $\mathfrak{f}_1, \dots, \mathfrak{f}_t$ be prime ideals of R and let P_{ij} , $1 \leq j \leq e_i$, be all prime ideals of A lying over $\mathfrak{f}_i$. We call a subring B of A the gluing of A with respect to $\{\mathfrak{f}_1, \dots, \mathfrak{f}_t\}$ if

$$B = \{\alpha \in A \mid \alpha(P_{i1}) = \dots = \alpha(P_{ie_i}) \in k(\mathfrak{f}_i) \text{ for all } i\},$$

where $\alpha(P_{ij})$ is the residue class of α with respect to P_{ij} .

Put $P_i = \{\alpha \in B \mid \alpha(P_{i1}) = \dots = \alpha(P_{ie_i}) = 0\}$. Then $P_i = P_{ij} \cap B$ for all j , P_i is the only prime ideal of B lying over $\mathfrak{f}_i$, and we have $k(P_i) = k(\mathfrak{f}_i)$.

Proposition 3. If B is given by the gluing of A with respect to $\{\mathfrak{f}_1, \dots, \mathfrak{f}_t\} \subset \text{Spec } R$, then B is seminormal in A and $\text{Ass}_B(A/B) \subseteq \{P_1, \dots, P_t\}$, where $\text{Ass}_B(A/B)$ is the set

of all the associated prime divisors of the B - module A/B .

Conversely, we have:

Theorem 4. If B is seminormal in A , then B is given by the gluing of A with respect to $\text{Ass}_B(A/B)$.

The proof of the above results is shown in [2] and [3].

Therefore, we know that if A/R is an integral extension, then ${}^+_A R$ is given the gluing of A with respect to $\text{Ass}_{({}^+_A R)}(A/{}^+_A R)$. Hence we ask what is the extension ${}^+_A R/R$. For convenience, put $C = {}^+_A R$. Then ${}^+_C R = C$, that is, the seminormalization of R in C is equal to C , itself.

Let $\bar{\varphi}_A$ be a homomorphism of A to $(A \otimes_R A)_{\text{red}}$ over R , such that

$$\begin{aligned} \bar{\varphi}_A: A &\longrightarrow A \otimes_R A \longrightarrow (A \otimes_R A)_{\text{red}} \\ \alpha &\longmapsto \alpha \otimes 1 - 1 \otimes \alpha \longmapsto \overline{\alpha \otimes 1 - 1 \otimes \alpha}. \end{aligned}$$

M. Manaresi proved in [1] the following:

Proposition 5. $\text{Ker } \bar{\varphi}_A = {}^W_A R$, that is, the kernel of $\bar{\varphi}_A$ is equal to the weak normalization of R in A .

In our situation, since $C = {}^+_A R$, we have $C = {}^W_C R$. Hence we have $\bar{\varphi}_C(C) = (0)$, i.e., $\bar{\varphi}_C$ is the trivial map. Let I_C be the kernel of the canonical homomorphism

$$\gamma_C: C \otimes_R C \longrightarrow C.$$

Then I_C is generated by $\{\alpha \otimes 1 - 1 \otimes \alpha \mid \alpha \in C\}$. Since $\bar{\varphi}_C(C) = (0)$, that is, $\alpha \otimes 1 - 1 \otimes \alpha$ is nilpotent and I_C is a two-sided ideal of $C \otimes_R C$, we have that I_C is nilpotent, say

$I_C^{q+1} = (0)$. Therefore we see that the q -th differential module

Ω_C^q is isomorphic to $I_C / I_C^{q+1} = I_C$. Hence there exists a

canonical q -th derivation of C over R such that

$$\begin{aligned} \Delta_q: C &\longrightarrow I_C \cong \Omega_C^q \\ \alpha &\longmapsto \alpha \otimes 1 - 1 \otimes \alpha \end{aligned}$$

and $\Delta_q^{-1}(0)$ is a subring of C containing R .

More generally, we have:

Proposition 6. Let N be a $C \otimes_R C$ -submodule of Ω_C^q

(for example, I_C^t , where t is an integer). Then $\Delta_q^{-1}(N)$ is an intermediate ring between R and C .

Proof. Let α, β be any elements of $\Delta_q^{-1}(N)$. Then

$$\Delta_q(\alpha + \beta) = \Delta_q(\alpha) + \Delta_q(\beta) \in N \quad \text{and}$$

$$\Delta_q(\alpha\beta) = \alpha\beta \otimes 1 - 1 \otimes \alpha\beta = (\alpha \otimes 1)(\beta \otimes 1 - 1 \otimes \beta) + (1 \otimes \beta)(\alpha \otimes 1 - 1 \otimes \alpha)$$

$\in N$.

Therefore we have $\alpha + \beta, \alpha\beta \in \Delta_q^{-1}(N)$.

In the paper [4], Lipman introduced the following notion:

For a ring A and a subring B of A , we call

$$*_A B = \left\{ \alpha \in A \mid \alpha \otimes 1 = 1 \otimes \alpha \text{ in } A \otimes_B A \right\}$$

the strict closure of B in A . If $B = *_A B$ then we say that B is strictly closed in A .

Using these notations, we have:

Proposition 7. In the above notation, $\Delta_q^{-1}(N)$ is strictly closed in C .

Remark. If D is a high order derivation, then $B = \{\alpha \in C \mid \alpha D = D\alpha\}$ is a subring of C and strictly closed. Especially, if D is a derivation (of 1 - st order) of C over R such that $D: C \longrightarrow M$, where M is a C -module, then $D^{-1}(0) = \{\alpha \in C \mid D(\alpha) = 0\} = \{\alpha \in C \mid D\alpha = \alpha D\}$ is a subring and strictly closed in C .

Proposition 8. Let A be a ring containing a field k and let $\mathfrak{m}$ be a maximal ideal of A and assume that $A/\mathfrak{m} \cong k$. Let Q be a primary ideal belonging to $\mathfrak{m}$ and assume that $Q \supseteq \mathfrak{m}^{q+1}$. Then

$$\begin{array}{ccc} \Delta: A & \longrightarrow & \mathfrak{m}/Q \\ \psi & & \\ \alpha & \longmapsto & \overline{\alpha - \alpha(\mathfrak{m})} \end{array},$$

where $\alpha(\mathfrak{m})$ is the residue class of α and we regard $\alpha(\mathfrak{m})$ as an element of k , hence of A .

Then Δ is a q -th order derivation and we have

$$R = \left\{ \alpha \in A \mid \alpha \Delta = \Delta \alpha \right\} = k + Q.$$

Remark. Let A be a noetherian domain containing a field k and $\mathfrak{p}$ be a prime ideal of A . Then the completion $\hat{A}$ of A has a coefficient field. Let K be the coefficient field and let q be a primary ideal belonging to $\mathfrak{p}$. Hence we have a canonical high order derivation

$$\Delta: A \longrightarrow A_{\mathfrak{p}} \longrightarrow \hat{A}_{\mathfrak{p}} \longrightarrow \hat{\mathfrak{p}}/\hat{q}.$$

On the other hand, the following results are well known (see [5]).

Proposition 9. (i) ${}^*B = A$ if and only if the extension A/B is epimorphic ($\stackrel{def}{\iff}$ if $B \xrightarrow{i} A \xrightarrow[f_2]{f_1} D$ and $f_1 \cdot i = f_2 \cdot i$, then $f_1 = f_2$).

(ii) A/B is epimorphic and integral if and only if $A = B$.

For convenience, let $C = C_0$ and $C_1 = \Delta_q^{-1}(0)$. Then we have:

Proposition 10. If $C_0 \not\supseteq R$, then $C_0 \not\supseteq C_1 \supseteq R$.

Proof. If $C_0 = C_1$, then we have ${}^*R = C$, hence $C_0 = C = R$, because the extension C/R is epimorphic and integral.

Continuing this process, we have:

Theorem 11. There exist a sequence of invariant subrings with respect to some high order derivations and an integer d such that

$$C_0 \supsetneq C_1 \supsetneq \dots \supsetneq C_d = R.$$

Remark. In the above C_{i+1} is an invariant subring of C_i with respect to the canonical high order derivation over R , $\Delta_i: C_i \longrightarrow \Omega_R(C_i)$.

Lemma. If $I_C^t = I_C^s$ for $t > s$, then we have $I_C^s = (0)$.

Proof. Indeed, for $\ell \gg 0$, we have $I_C^\ell = (0)$. Hence

$$I_C^s = I_C^t = I_C^{t-s+s} = I_C^{2t-s} = \dots = (0).$$

Therefore there

$$I_C \supsetneq I_C^2 \supsetneq \dots \supsetneq I_C^q \supsetneq I_C^{q+1} = (0).$$

For the canonical high order derivations

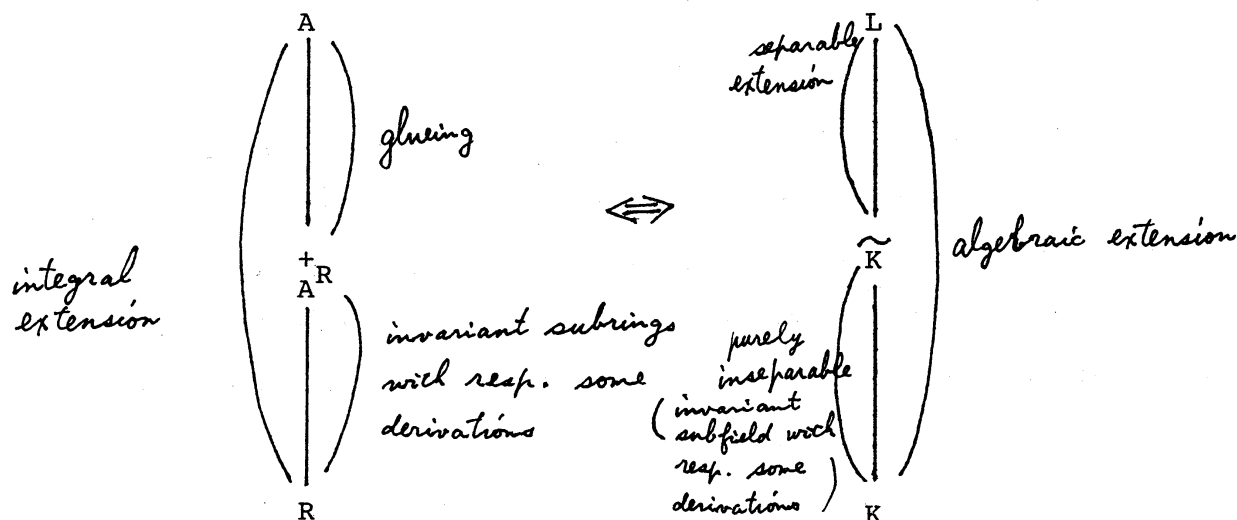
$$\Delta_i: C \longrightarrow I_C / I_C^i = \Omega_C^{i-1},$$

let $B_i = \Delta_i^{-1}(0)$. Then $B_i = \Delta_q^{-1}(I_C^i)$, and B_i is the invariant subring of B_{i-1} with respect to $\Delta_i|_{B_{i-1}}$. Since $\Delta_i|_{B_{i-1}}$ is a (1-st order) derivation, we have the following:

Theorem 12. There exists a sequence of invariant subrings with respect to some derivations and an integer g such that

$$C = C_0' \supsetneq C_1' \supsetneq \dots \supsetneq C_g' = R.$$

Therefore we say that the integral extensions are similar to the algebraic extensions of fields.



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