

ON REGION UNKNOTTING NUMBERS

AYAKA SHIMIZU

ABSTRACT. A region crossing change at a region of a knot diagram is the crossing changes at all the crossing points on the boundary of the region. In this paper, we show that for any knot diagram and any region R , we can make any crossing change by a sequence of region crossing changes except at R . We also discuss about region unknotting numbers of 3-braids.

1. INTRODUCTION

A *region crossing change* at a region R of a link diagram D on S^2 is the crossing changes at all the crossing points on the boundary of R [3]. For example, we obtain the diagram D' from the knot diagram D by the region crossing change at the region R in Figure 1.

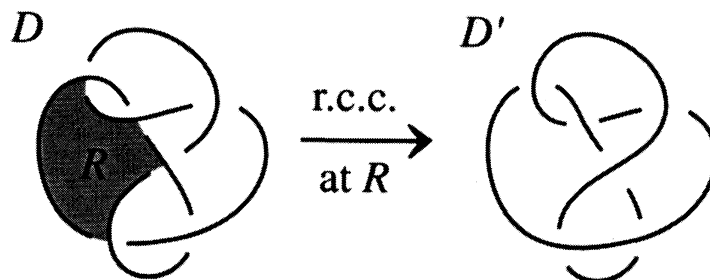


FIGURE 1

We remark that K. Kishimoto proposed a region crossing change at a seminar at Osaka City University, and asked whether a region crossing change is an unknotting operation. To give the positive answer to this question, the following theorem is shown in [3]:

Theorem 1.1 ([3]). *For any knot diagram D , we can make any crossing change on D by a sequence of region crossing changes.*

Since a crossing change is an unknotting operation, a region crossing change on a knot diagram is also an unknotting operation. Moreover, we have the following theorems:

Theorem 1.2. *Let D be a knot diagram and let R be a region of D . We can make any crossing change on D by a sequence of region crossing changes at regions of D except R .*

Theorem 1.3. *Let D be a reduced knot diagram. For any region R of D , there exists a region $S \neq R$ of D such that we can make any crossing change on D by a sequence of region crossing changes at regions of D except R and S .*

The proofs are given in Section 2. For example, for the diagram D and the region R in Figure 2, the region S satisfies the above condition: We can change the crossing at c_1 (resp. c_2) by region crossing changes at T_1 and T_3 (resp. T_1, T_2 and T_3).

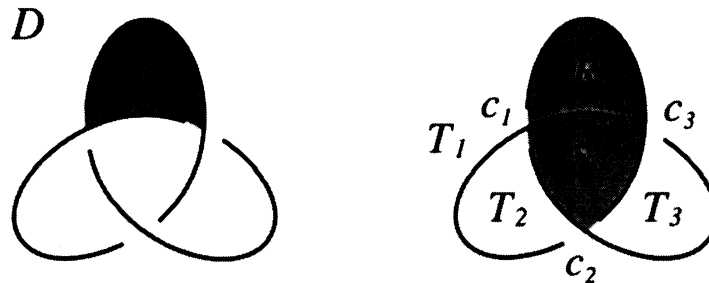


FIGURE 2

The *region unknotting number* $u_R(D)$ of a knot diagram D is the minimal number of region crossing changes which are needed to obtain a diagram of the trivial knot (without Reidemeister moves) [3]. For example, the diagram D in Figure 1 has the region unknotting number one. The *region unknotting number* $u_R(K)$ of a knot K is the minimal $u_R(D)$ for all minimal crossing diagrams D of K [3]. We have $u_R(D) \leq c(D)/2 + 1$ for any reduced knot diagram D , and hence we have $u_R(K) \leq c(K)/2 + 1$ for any knot K , and we have $u_R(K) = m$ for the $(2, 4m \pm 1)$ -torus knot K ($m = 1, 2, \dots$) [3].

We will discuss about region unknotting numbers of the standard diagrams of $(3, n)$ -torus knots in Section 3.

The rest of this paper is organized as follows: In Section 2, we prove Theorem 1.2 and Theorem 1.3. In Section 3, we discuss about region unknotting numbers of closed 3-braid diagrams.

2. PROOF OF THEOREM 1.2

In this section, we prove Theorem 1.2 after proving Theorem 1.3. The following lemmas are shown in [3]:

Lemma 2.1 ([3]). *For a reduced knot diagram D and the set B of all the black-colored regions of D with a checkerboard coloring, we obtain D from D by region crossing changes at B .*

Lemma 2.2 ([3]). *Let D be a reduced knot diagram, and let B be the set of all the black-colored regions of D with a checkerboard coloring. Let P be a subset of B . Then we obtain the same diagram from D by the region crossing changes at P and the region crossing changes at $B - P$.*

We prove Theorem 1.3.

Proof of Theorem 1.3. Let B (resp. W) be the set of all the black-colored (resp. white-colored) regions of D with a checkerboard coloring. If $R \in B$ (resp. $R \in W$), we can take any white-colored (resp. black-colored) region as S . By Lemma 2.2, the region crossing change at R is equivalent to the region crossing changes at $B - R$, and the region crossing change at S is equivalent to the region crossing changes at $W - S$. By Theorem 1.1, we can make any crossing change on D by region crossing changes at regions of D except R and S . $\square$

From Theorem 1.3, we have the following corollaries:

Corollary 2.3. *Let D be a reduced knot diagram. For any two regions R and S of D which are adjacent to each other, we can make any crossing change on D by a sequence of region crossing changes except at R and S .*

Corollary 2.4. *Let T be a one-string tangle diagram. We can make any crossing change by a sequence of region crossing changes at regions of T except the outer region.*

Now we prove Theorem 1.2.

Proof of Theorem 1.2. It is enough to show that for any knot diagram D on $\mathbb{R}^2$ and any crossing point c , we can make the crossing change at c by region crossing changes at regions of D except the outer region of D . If D is a knot diagram which has only one reducible crossing as c as shown in Figure 3, we can change the crossing at c by region crossing changes as follows: We splice D at c , and apply the checkerboard coloring to the knot diagram corresponding to A in Figure 3 so that the outer region of the knot diagram is colored white. Then, if we apply region crossing changes at all the regions of D corresponding to the black-colored regions, the crossing of only c is changed. This theorem also holds for reduced knot diagrams

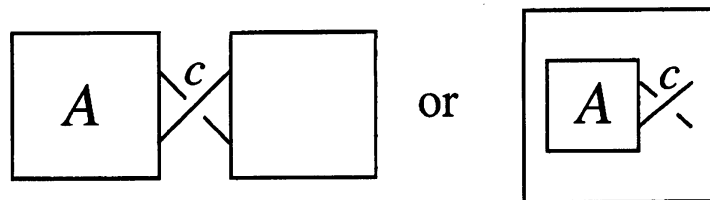


FIGURE 3

by Theorem 1.3. For other cases, we can prove by an induction on the number of reducible crossings as shown in Figure 4. $\square$

3. REGION UNKNOTTING NUMBERS OF CLOSED 3-BRAID DIAGRAMS

In this section we discuss about region unknotting numbers of closed 3-braid diagrams. For standard diagrams of $(3, m)$ -torus knots, we have the following proposition:

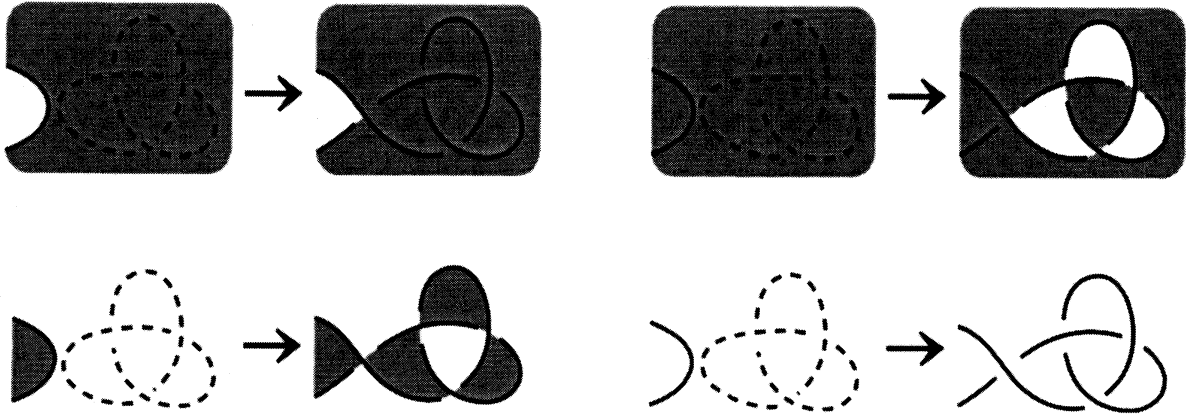


FIGURE 4

Proposition 3.1. *Let $D_{3,m}$ be the standard diagram of the $(3, m)$ -torus link ($m = 1, 2, 3, \dots$). We have $u_R(D_{3,3n+1}) \leq n$ and $u_R(D_{3,3n+2}) \leq n + 1$ ($n = 0, 1, 2, \dots$).*

Proof. We have $u_R(D_{3,1}) = 0$ and $u_R(D_{3,2}) = 1$. Since we can deform the braid diagram of $(\sigma_2\sigma_1)^3$ into a braid diagram which represents the trivial 3-braid by one region crossing change (see for example Figure 5), we have the inequalities. $\square$

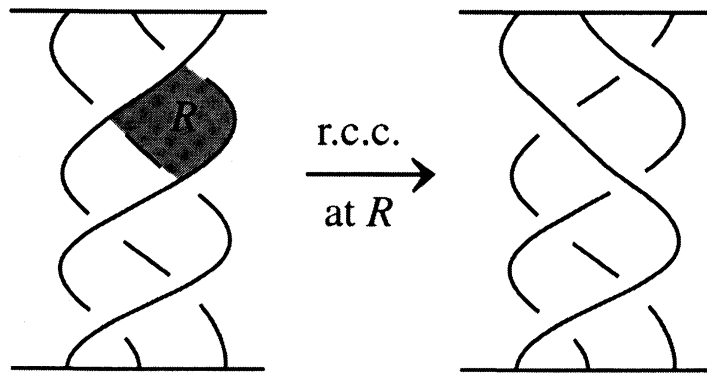


FIGURE 5

From Proposition 3.1, we have the following corollary:

Corollary 3.2. *The closed braid diagram of $(\sigma_2^{-1}\sigma_1)^{3n+1}$ has the region unknotting number less than or equal to $n+1$, and the closed braid diagram of $(\sigma_2^{-1}\sigma_1)^{3n+2}$ has the region unknotting number less than or equal to $n+2$ ($n = 0, 1, 2, \dots$).*

Proof. We can obtain $D_{3,m}$ from the closed braid diagram of $(\sigma_2^{-1}\sigma_1)^m$ by one region crossing change (Figure 6). $\square$

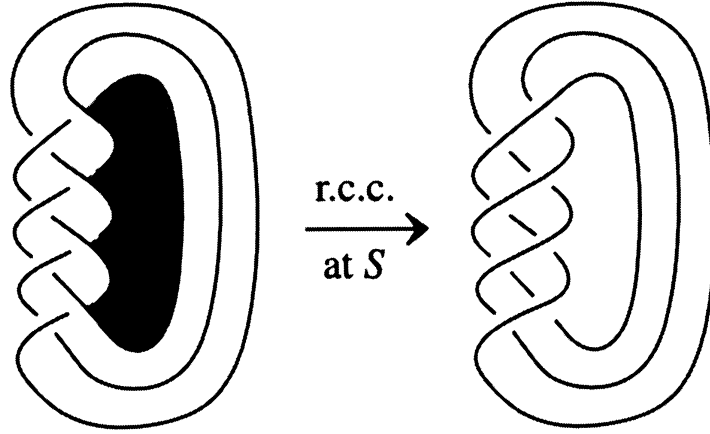


FIGURE 6

Remark. 3.3. Z. Cheng and H. Gao showed in [1] that a region crossing change on a diagram of a 3-component link such that the linking number of each two components is even is an unknotting operation. For example, a region crossing change on the closed braid diagram of $(\sigma_2^{-1}\sigma_1)^{3n}$ is an unknotting operation. As shown in Figure 5, we can obtain a trivial link diagram from $D_{3,3n}$ ($n = 0, 1, 2, \dots$) by at most n region crossing changes, i.e., a region crossing change on $D_{3,3n}$ is also an unknotting operation.

For a 3-braid $\beta = \sigma_1^{n_1}\sigma_2^{n_2}\sigma_1^{n_3}\dots\sigma_2^{n_m}$, let β_1 and β_2 be the 3-braids defined to be $\beta_1 = \sigma_2^{-n_m}\dots\sigma_1^{-n_3}\sigma_2^{-n_2}\sigma_1^{-n_1}$ and $\beta_2 = \sigma_1^{-n_m}\dots\sigma_2^{-n_3}\sigma_1^{-n_2}\sigma_2^{-n_1}$ ($n_1, n_2, \dots, n_m \in \mathbb{Z}$). K. Kishimoto pointed out that each closed 3-braid diagram of the following $A_1, A_2, \dots$ or B_3 can be deformed into a diagram

of a trivial link by one region crossing change:

$$\begin{aligned} A_1 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^3, \\ A_2 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^3\sigma_2^{-1}, \\ A_3 &= \beta(\sigma_1^{-1}\sigma_2)^3\beta_1(\sigma_2^{-1}\sigma_1)^4, \\ B_1 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2\sigma_2^{-1}\sigma_1\sigma_2^{-1}, \\ B_2 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2(\sigma_2^{-1}\sigma_1)^2, \\ B_3 &= \beta\sigma_2\sigma_1^{-1}\sigma_2\beta_2(\sigma_2^{-1}\sigma_1)^2\sigma_2^{-1}, \end{aligned}$$

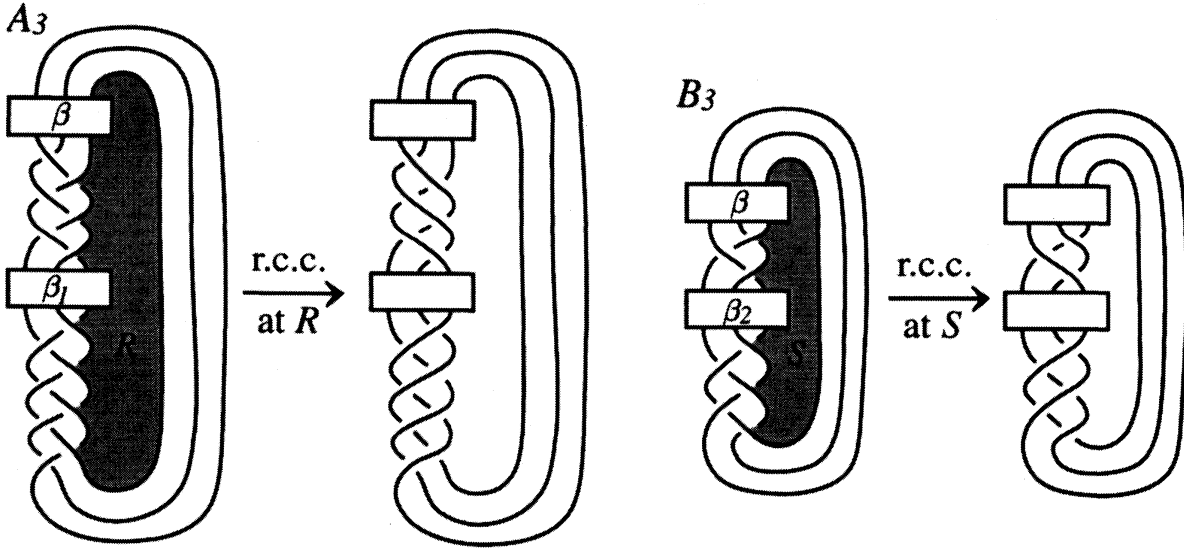


FIGURE 7

where β is a 3-braid, and A_3 and B_3 are illustrated in Figure 7.

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OSAKA CITY UNIVERSITY ADVANCED MATHEMATICAL INSTITUTE, 3-3-138 SUGIMOTO SUMIYOSHI-KU OSAKA 558-8585, JAPAN
E-mail address: shimizu1984@gmail.com