

LENS SPACE SURGERIES ALONG TWO COMPONENT LINKS AND REIDEMEISTER-TURAEV TORSION

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1. INTRODUCTION

This article is a short survey of the core part of the authors' joint work "Lens space surgeries along certain 2-component links, Park's rational blow down, and Reidemeister-Turaev torsion" [KTY, preprint]. In [KTY], certain two families of 2-component links, denoted by $A_{m,n}$ and $B_{p,q}$ are focused, and the main result is the decision of the coefficient(s) of the knotted component yielding a lens space by Dehn surgery. The links are related to rational homology 4-ball used in J. Park's (generalized) rational blow down in 4-dimensional topology (see [3, 12]). Concrete calculus on the links $A_{m,n}$ and $B_{p,q}$ was important. The results made the contrast between $A_{m,n}$ (hyperbolic) and $B_{p,q}$ (Seifert) clear, which was one of the purpose of [KTY] (see also [17]).

In this article, we focus another importance, the method itself to get some necessary conditions on the lens space surgery coefficients of a given link, by using Alexander polynomial and Reidemeister torsion. Our method satisfies that a result on a link L always extends to the links whose Alexander polynomials are same with that of L .

We will compare the Reidemeister torsion of the result M of Dehn surgery along a given link and that of a lens space $L(p, q)$ (in Example 3.4). Some necessary conditions are obtained from the value $\tau^{\psi_d}(M)$ of the Reidemeister torsion in the d -th cyclotomic field $\mathbb{Q}(\zeta_d)$ by d -norm, where $d(\geq 2)$ is a divisor of p . From the sequence of the equalities on $\tau^{\psi_d}(M)$ s in $\mathbb{Q}(\zeta_d)$ for all divisors d of p (with a fixed combinatorial Euler structure of M), we take an *identity on symmetric Laurent polynomials*, as a lift of the equalities. We regard the identity as an equation of the surgery coefficient for M to be a lens space.

In the next section, we start with some definitions of the Reidemeister torsion. In Section 3, we review surgery formulae. In Section 4, we will study d -norms in the d -th cyclotomic field, and show a certain uniqueness of a symmetric polynomial as a lift of the sequence of the equalities in $\mathbb{Q}(\zeta_d)$ s. In Section 5, we will explain the method to get some necessary condition of lens space surgery coefficients of a given link. In Section 6, as a demonstration, we will apply our method to Berge's link, which is one of the most famous targets in lens space surgery ([1]).

2. REIDEMEISTER TORSION

For a precise definition of the Reidemeister torsion, the reader refer to V. Turaev [14, 15]. Let X be a finite CW complex and $\pi : \tilde{X} \rightarrow X$ its maximal abelian covering. Then $\tilde{X}$ has a CW structure induced by that of X and π , and the cell chain complex C_* of $\tilde{X}$ has a

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E_L	the complement of L .
m_i, l_i	a meridian and a longitude of the i -th component K_i .
$[m_i], [l_i]$	their homology classes.
$\Delta_L(t_1, \dots, t_\mu)$	the Alexander polynomial of L , where t_i is represented by $[m_i]$.
$(L; r_1, \dots, r_\mu)$	the result of Dehn surgery along L , where $r_i \in \mathbb{Q} \cup \{\infty, \emptyset\}$ is the surgery coefficient of K_i .
V_i	the solid torus attached along K_i in the Dehn surgery.
$m'_i, [m'_i]$	a meridian of V_i , and its homology class.
$l'_i, [l'_i]$	an oriented core curve of V_i , and its homology class.

TABLE 1. Notations (for manifolds)

$\mathbb{Z}[H]$ -module structure, where $H = H_1(X; \mathbb{Z})$ is the first homology of X . For an integral domain R and a ring homomorphism $\psi : \mathbb{Z}[H] \rightarrow R$, “the chain complex of $\tilde{X}$ related with ψ ”, denoted by $\mathbf{C}_*^\psi$, is constructed as $\mathbf{C}_* \otimes_{\mathbb{Z}[H]} Q(R)$, where $Q(R)$ is the quotient field of R . The Reidemeister torsion of X related with ψ , denoted by $\tau^\psi(X)$, is calculated from $\mathbf{C}_*^\psi$, and is an element of $Q(R)$ determined up to multiplication of $\pm\psi(h)$ ($h \in H$). If $R = \mathbb{Z}[H]$ and ψ is the identity map, then we denote $\tau^\psi(X)$ by $\tau(X)$. We note that $\tau^\psi(X)$ is not zero if and only if $\mathbf{C}_*^\psi$ is acyclic.

Notation (for manifolds and homologies) Let $L = K_1 \cup \dots \cup K_\mu$ be an oriented μ -component link in S^3 . We will use the notations in Table 1.

Notation (for algebra) For a pair of elements A, B in $Q(R)$, if there exists an element $h \in H$ such that $A = \pm\psi(h)B$, then we denote the equality by $A \doteq B$. We will often take a field F and a ring homomorphism $\psi : \mathbb{Z}[H_1(M)] \rightarrow F$. We mainly use the d -th cyclotomic fields $\mathbb{Q}(\zeta_d)$ as F , where ζ_d is a primitive d -th root of unity.

3. SURGERY FORMULAE

Let E be a compact 3-manifold whose boundary ∂E consists of tori (E is possibly not E_L for a link L). We study the 3-manifold $M = E \cup V_1 \cup \dots \cup V_n$ obtained by attaching solid tori V_i s to E by attaching maps $f_i : \partial V_i \rightarrow \partial E$ ($\text{Im}(f_i) \cap \text{Im}(f_j) = \emptyset$ for $i \neq j$). By l'_i we denote the core of V_i . We let $\iota : E \hookrightarrow M$ denote the natural inclusion.

Lemma 3.1. (Surgery formula I) *If $\psi([l'_i]) \neq 1$ for every $i = 1, \dots, n$, then*

$$\tau^\psi(M) \doteq \tau^{\psi'}(E) \prod_{i=1}^n (\psi([l'_i]) - 1)^{-1},$$

where $\psi' = \psi \circ \iota_*$ (ι_* is a ring homomorphism induced by ι).

For the case of the complement E_L of a μ -component link L in S^3 as in Table 1. The Reidemeister torsion is closely related with the Alexander polynomial.

Lemma 3.2. (Milnor [11]) *Let $\Delta_L(t_1, \dots, t_\mu)$ be the Alexander polynomial of a μ -component link $L = K_1 \cup \dots \cup K_\mu$ in S^3 , where a variable t_i is represented by the meridian*

of K_i ($i = 1, \dots, \mu$).

$$\tau(E_L) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & (\mu = 1), \\ \Delta_L(t_1, \dots, t_\mu) & (\mu \geq 2). \end{cases}$$

Next, we study the result of Dehn surgery $M = (L; p_1/q_1, \dots, p_\mu/q_\mu)$ along L . We take integers r_i and s_i satisfying $p_i s_i - q_i r_i = -1$.

Lemma 3.3. (Surgery formula II; T. Sakai [13], V. G. Turaev [14])

- (1) In the case $M = (K; p/q)$ ($|p| \geq 2$), we have $H = H_1(M) \cong \langle T \mid T^p = 1 \rangle \cong \mathbb{Z}/|p|\mathbb{Z}$, where T is represented by the meridian $[m]$. For a divisor d (≥ 2) of p , we define a ring homomorphism $\psi_d : \mathbb{Z}[H] \rightarrow \mathbb{Q}(\zeta_d)$ by $\psi_d(T) = \zeta_d$. Then we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod{p}$.

- (2) In the case $M = (L; p_1/q_1, \dots, p_\mu/q_\mu)$ ($\mu \geq 2$). Let F be a field and $\psi : \mathbb{Z}[H_1(M)] \rightarrow F$ a ring homomorphism. If $\psi([m_i]^{r_i}[l_i]^{s_i}) \neq 1$ for every $i = 1, \dots, \mu$, then we have

$$\tau^\psi(M) \doteq \Delta_L(\psi([m_1]), \dots, \psi([m_\mu])) \prod_{i=1}^{\mu} (\psi([m_i]^{r_i}[l_i]^{s_i}) - 1)^{-1}.$$

Example 3.4. The lens space $L(p, q)$ is obtained as $-p/q$ -surgery along the unknot. By Lemma 3.3 (1), for a divisor $d \geq 2$ of p , we have

$$\tau^{\psi_d}(L(p, q)) \doteq (\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1},$$

where $q\bar{q} \equiv 1 \pmod{p}$.

4. CYCLOTOMIC FIELD AND POLYNOMIAL

4.1. d -norm.

About algebraic fields, the reader refer to L. C. Washington [16] for example.

For an element x in the d -th cyclotomic field $\mathbb{Q}(\zeta_d)$, the d -norm of x is defined as

$$N_d(x) = \prod_{\sigma \in \text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})} \sigma(x),$$

where $\text{Gal}(\mathbb{Q}(\zeta_d)/\mathbb{Q})$ is the Galois group ($\cong (\mathbb{Z}/d\mathbb{Z})^\times$) related with a Galois extension $\mathbb{Q}(\zeta_d)$ over $\mathbb{Q}$. The following is well-known.

Proposition 4.1.

- (1) If $x \in \mathbb{Q}(\zeta_d)$, then $N_d(x) \in \mathbb{Q}$. The map $N_d : \mathbb{Q}(\zeta_d) \setminus \{0\} \rightarrow \mathbb{Q} \setminus \{0\}$ is a group homomorphism.
(2) If $x \in \mathbb{Z}[\zeta_d]$, then $N_d(x) \in \mathbb{Z}$.

By easy calculations, we have the following.

Lemma 4.2.

- (1) $N_d(\pm\zeta_d) = \begin{cases} \pm 1 & (d = 2), \\ 1 & (d \geq 3). \end{cases}$
(2) $N_d(1 - \zeta_d) = \begin{cases} \ell & (d \text{ is a power of a prime } \ell \geq 2), \\ 1 & (\text{otherwise}). \end{cases}$

About applications of d -norms, for example, see [5, 6, 7, 8, 9, 10].

4.2. Reidemeister–Turaev torsion.

Let M be a homology lens space with $H = H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then the Reidemeister torsion $\tau^{\psi_d}(M)$ of M related with ψ_d is determined up to multiplication of $\pm \zeta_d^m$ ($m \in \mathbb{Z}$), where $d \geq 2$ is a divisor of p and ψ_d is the same ring homomorphism as in Lemma 3.3 (1). Once we fix a basis of a cell chain complex for the maximal abelian covering of M as a $\mathbb{Z}[H] = \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ -module, the value $\tau^{\psi_d}(M)$ is uniquely determined as an element of $\mathbb{Q}(\zeta_d)$ for every d . The choice of the basis up to “base change equivalence” is called a *combinatorial Euler structure* of M (cf. Turaev [15]). The Reidemeister torsion of a manifold with a fixed combinatorial Euler structure is said the *Reidemeister–Turaev torsion*.

We consider the sequence of the values $\tau^{\psi_d}(M)$ in $\mathbb{Q}(\zeta_d)$ of the Reidemeister–Turaev torsion for every divisor $d \geq 2$ of p , and regard them as a value sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ defined as below.

Definition 4.3. We define that a sequence of values $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ is a *value sequence* (of degree p) if $x_d \in \mathbb{Q}(\zeta_d)$ for every d . Two value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ are *equal* ($\mathbf{x} = \mathbf{y}$) if $x_d = y_d$ for every d . We are mainly concerned with the value sequence of type $\mathbf{x} = \{F(\zeta_d)\}_{d|p, d \geq 2}$ for a rational function $F(t) \in \mathbb{Q}(t)$. In such a case, we say that $\mathbf{x}$ is *induced by* $F(t)$ and that $F(t)$ is a *lift* of $\mathbf{x}$. A *control* of $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ by a trivial unit $u = \eta t^m \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ is defined by

$$u\mathbf{x} = \{\eta \zeta_d^m x_d\}_{d|p, d \geq 2},$$

where $\eta = 1$ or -1 (constant) and $m \in \mathbb{Z}$. Two value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ are *control equivalent* if there is a trivial unit $u \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$ such that $\mathbf{y} = u\mathbf{x}$. A value sequence $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ is a *real value sequence* if x_d is a real number for every d .

Example 4.4. A value sequence $\mathbf{x}$ of degree 12 is in the form $\mathbf{x} = \{x_2, x_3, x_4, x_6, x_{12}\}$. The following two value sequences $\mathbf{x}, \mathbf{y}$ of degree 12 are not equal, but control equivalent for $u = t^6$.

$$\mathbf{x} = \{2, -1, -2, -1, 1\}, \quad \mathbf{y} = \{2, -1, 2, 1, -1\}.$$

In fact, $\mathbf{x}$ and $\mathbf{y}$ is induced by $t^2 + t^{-2}$ and $t^4 + t^{-4}$, respectively.

Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then a sequence $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ of the Reidemeister torsions of M with a combinatorial Euler structure is a value sequence of degree p . We say the value sequence a *torsion sequence* of M .

Lemma 4.5.

- (1) Let M and M' be homeomorphic homology lens spaces with $H_1(M) \cong H_1(M') \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then torsion sequences $\{\tau^{\psi_d}(M)\}_{d|p, d \geq 2}$ and $\{\tau^{\psi'_d}(M')\}_{d|p, d \geq 2}$ related with the corresponding ring homomorphisms ψ_d and ψ'_d (i.e., $\psi_d = \psi'_d \circ h_*$, where h_* is the induced homomorphism of the homeomorphism) are control equivalent.
- (2) Let M be a homology lens space with $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ ($p \geq 2$). Then we can control a torsion sequence of M into a real value sequence.

Proof. (1) It is easy to see.

(2) Here we let ζ denote any d -th primitive root (ζ_d) of unity. Since M is obtained by p/q -surgery along a knot K in a homology 3-sphere for some q (cf. [2]), and we can also

apply Lemma 2.5 (1) for the case, we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

where $q\bar{q} \equiv 1 \pmod{p}$. By the duality of the Alexander polynomial (cf. [11, 14, 15]), we may assume

$$\Delta_K(t) = \Delta_K(t^{-1}).$$

This is also a control of the combinatorial Euler structure of the exterior of K , which induces a control of a torsion sequence of M . We take an odd integer lift of $\bar{q}$. Then

$$\zeta^{\frac{1+\bar{q}}{2}} \Delta_K(\zeta)(\zeta - 1)^{-1}(\zeta^{\bar{q}} - 1)^{-1}$$

is a real number for every d . □

Lemma 4.6. *If two real value sequences $\mathbf{x} = \{x_d\}_{d|p, d \geq 2}$ and $\mathbf{y} = \{y_d\}_{d|p, d \geq 2}$ of degree p are control equivalent satisfying $\mathbf{y} = u\mathbf{x}$ for a trivial unit $u = \eta t^m \in \mathbb{Z}[t, t^{-1}]/(t^p - 1)$, where $\eta = \pm 1$ and $m \in \mathbb{Z}$, then the possibility of u is restricted as follows:*

- (i) *If p is odd, then $u = 1$ or -1 .*
- (ii) *If p is even, then $u = 1, -1, t^{p/2}$ or $-t^{p/2}$.*

Proof. Since the ratio $\zeta_p^m = \pm y_p/x_p$ is a real number, we have (i) $m \equiv 0 \pmod{p}$ if p is odd, and (ii) $m \equiv 0$ or $p/2 \pmod{p}$ if p is even. □

Definition 4.7. (Symmetric Laurent polynomial) A Laurent polynomial $F(t) \in \mathbb{Z}[t, t^{-1}]$ is *symmetric* if it is of the form

$$F(t) = a_0 + \sum_{i=1}^{\infty} a_i(t^i + t^{-i}),$$

where a_i is an integer for all $i = 1, 2, \dots$ and $a_i = 0$ for every sufficiently large i . Note that, if $F(t)$ is a symmetric Laurent polynomial, the induced value sequence $\{F(\zeta_d)\}_{d|p, d \geq 2}$ is a real value sequence. We are concerned with symmetric Laurent polynomials that are lifts (in $\mathbb{Z}[t, t^{-1}]$) of a polynomial in the quotient ring $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$. We say that $F(t)$ (as above) is *reduced* if $a_i = 0$ for all $i > [p/2]$. We often *reduce* the symmetric polynomials by using $t^i + t^{-i} = t^{p+i} + t^{-(p+i)}$ modulo $(t^p - 1)$. We let $\text{red}(F(t))$ denote the reduction of $F(t)$ (i.e., $\text{red}(F(t))$ is reduced and $\text{red}(F(t)) = F(t)$ in $\mathbb{Z}[t, t^{-1}]/(t^p - 1)$). We will use a notation $\langle t^i \rangle = t^i + t^{-i}$, for short.

For a Laurent polynomial $F(t) \in \mathbb{Z}[t, t^{-1}]$, the *span* of $F(t)$ is the difference of the maximal degree of $F(t)$ and the minimal degree of $F(t)$, and we denote it by $\text{span}(F(t))$.

Lemma 4.8. *Let $N \geq 2$ be an integer. Let $F(t), G(t)$ be symmetric Laurent polynomials and $\mathbf{x} = \{F(\zeta_d)\}_{d|N, d \geq 2}$, $\mathbf{y} = \{G(\zeta_d)\}_{d|N, d \geq 2}$ the induced real value sequences, respectively. If $\mathbf{x}$ and $\mathbf{y}$ are control equivalent, i.e., $u\mathbf{x} = \mathbf{y}$ for a trivial unit u (here, $u = 1$ or -1 if N is odd, $u = 1, -1, t^{N/2}$ or $-t^{N/2}$ if N is even, by Lemma 4.6), and $F(1) = G(1) = 0$, then we have a congruence*

$$uF(t) \equiv G(t) \pmod{t^N - 1}$$

Furthermore, assuming $\text{span}(G(t)) \leq 2[N/2]$,

- (i) *In the case that $u = 1$ or -1 and $\text{span}(F(t)) \leq N - 1$, we have an identity $uF(t) = G(t)$ in $\mathbb{Z}[t, t^{-1}]$.*

- (ii) Otherwise (in the case that N is even and $u = \eta t^{N/2}$ with $\eta = 1$ or -1), we have $\text{red}(t^{N/2}F(t)) = \eta G(t)$ in $\mathbb{Z}[t, t^{-1}]$.

Proof. By Chinese Remainder Theorem, we have a ring isomorphism:

$$\mathbb{Q}[t, t^{-1}]/(t^N - 1) \cong \bigoplus_{d|N, d \geq 1} \mathbb{Q}(\zeta_d),$$

where $f(t)$ in the left-hand side maps to the value sequences $\{f(\zeta_d)\}_{d|N, d \geq 2}$ in the right-hand side. The isomorphism implies the required congruence. $\square$

Note that $F(t)$ and $t^{N/2}F(t)$ induce the control equivalent real value sequences by $u = t^{N/2}$, but $\text{red}(t^{N/2}F(t)) \neq F(t)$ in general, see Example 4.4. Thus we have to care the case (ii) in the lemma. Here, we study relation between the coefficients of $F(t)$ and those of $\text{red}(t^{N/2}F(t))$.

Lemma 4.9. *Let N be an even integer.*

$$\text{If } F(t) = a_0 + \sum_{i=1}^{N/2} a_i(t^i + t^{-i}), \text{ then } \text{red}(t^{N/2}F(t)) = b_0 + \sum_{i=1}^{N/2} b_i(t^i + t^{-i})$$

with

$$b_0 = 2a_{N/2}, \quad b_{N/2} = a_0/2 \text{ and } b_j = a_{N/2-j} \quad (j = 1, 2, \dots, N/2 - 1).$$

Proof. It is because

$$t^{N/2}(t^j + t^{-j}) = t^{N/2+j} + t^{N/2-j} \equiv t^{(N/2-j)} + t^{-(N/2-j)} \pmod{t^N - 1}.$$

$\square$

5. METHOD

Let $L = K_1 \cup K_2 \cup \dots \cup K_\mu$ be a link. We let M simply denote the result $(L; r_1, \dots, r_\mu)$ of the Dehn surgery. We use the notations in Table 1.

Step 1 Study the first homologies (the generators and relations), from the exterior E_L of L (Of course, $H_1(E_L; \mathbb{Z}) \cong \bigoplus_{i=1}^\mu \mathbb{Z}[m_i]$) to the result M , by attaching solid tori V_i one by one.

The first (obvious) necessary condition for the result M of Dehn surgery to be a lens space $L(p, q)$ is

$$H_1(M; \mathbb{Z}) \cong \mathbb{Z}/p\mathbb{Z}.$$

Step 2 Calculate the Alexander polynomial $\Delta_L(t_1, \dots, t_\mu)$ of L . Using Lemma 3.2 and Lemma 3.3, calculate the Reidemeister torsion $\tau^\psi(M)$ related with a ring homomorphism $\psi: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$, where $d (\geq 2)$ is a divisor of p .

If M is homeomorphic to a lens space $L(p, q)$ (with undecided q), then their Reidemeister torsions are equal to each other. By Example 3.4, there exists integers i, j coprime to p with $0 < i, j < p$ (they are lifts of $(\mathbb{Z}/p\mathbb{Z})^\times / \{\pm 1\}$) such that

$$(1) \quad \tau^\psi(M) \doteq \frac{1}{(\zeta_d^i - 1)(\zeta_d^j - 1)} \quad \text{in } \mathbb{Q}(\zeta_d),$$

for each divisor $d (\geq 2)$ of p . We can assume $i + j$ is even by retaking $p - j$ instead of j .

Step 3 Using d -norm in $\mathbb{Q}(\zeta_d)$, studied in Subsection 4.1, to the equality (1), we have a necessary condition on the coefficient of lens space surgery.

We fix a combinatorial Euler structure (multiple of trivial unit $\pm\zeta_d^k$), deform both hand-sides of the equality (1) into real values by Lemma 4.5(2). If M is homeomorphic to $L(p, q)$, we have a control equivalence between the real value sequence:

$$\{\tau^\psi(M)\}_{d|p, d \geq 2} = u \{ \zeta_d^{\frac{i+j}{2}} (\zeta_d^i - 1)^{-1} (\zeta_d^j - 1)^{-1} \}_{d|p, d \geq 2},$$

where u is a trivial unit ± 1 , or $\pm t^{p/2}$ (only in the case p is even). By Lemma 4.8, we have, via a congruence mod $(t^p - 1)$, an identity between symmetric Laurent polynomials. We regard the identity as an equation (on (i, j)) of the coefficients of lens space surgery.

Step 4 By the equation, we have a necessary condition on the coefficient(s) of lens space surgery.

6. DEMONSTRATION

We call the link in Figure 1 *Berge's link BL*. The compliment is a hyperbolic 3-manifold, known as Berge's manifold in [1]. The component K_1 is the famous pretzel knot $P(-2, 3, 7)$. The link, regarded as a knot in a solid torus (the exterior of the component K_2), admits two surgery coefficients yielding solid torus itself, and it is proved that such a hyperbolic link is unique [1]. We demonstrate our method in Section 5 to Berge's link,

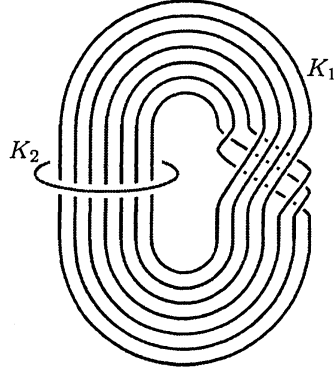


FIGURE 1. Berge link BL

to study lens space surgeries $M := (BL; r, 0)$, where $r = \alpha/\beta$ ($\alpha, \beta \in \mathbb{Z}, \gcd(\alpha, \beta) = 1$). We assume that $\beta \geq 1$.

(Step 1)

$$H_1(M) \cong \langle [m_1], [m_2] \mid [l_1] = [m_2]^7, [l_2] = [m_1]^7, [m_1]^\alpha [l_1]^\beta = 1, [m_1]^7 = 1 \rangle.$$

It is finite cyclic $\mathbb{Z}/p\mathbb{Z}$ if and only if $\gcd(\alpha, 7) = 1$, and then we have $p = 7^2\beta = 49\beta$. An element $T = [m_1]^{\gamma'} [m_2]^{\delta'}$ with $\alpha\delta' - 7\beta\gamma' = -1$ is a generator: $T^{49\beta} = 1$. We also have $[l'_1] = [m_1]^\gamma [l_1]^\delta$ with $\alpha\delta - \beta\gamma = -1$, and

$$[m_1] = T^{7\beta}, [m_2] = [l'_2] = T^{-\alpha}, [l'_1] = T^7.$$

(Step 2) The Alexander polynomial of Berge's link is

$$\Delta_{BL}(t, x) \doteq 1 + t^3x + t^5x^2 + t^8x^3 + t^{11}x^4 + t^{13}x^5 + t^{16}x^6 = \sum_{i=0}^6 t^{s_i} x^i,$$

where we define a sequence $(s_0, s_1, \dots, s_6) = (0, 3, 5, 8, 11, 13, 16)$. This is not periodic, but we regard it as "Periodicity is broken a little". We let $M_1 = E_{BL} \cup V_1 = (BL; \alpha/\beta, -)$. We have, up to the ambiguity (multiplication $\pm T^k$),

$$\tau(M_1) \doteq \Delta_{BL}(T^{7\beta}, T^{-\alpha})(T^7 - 1)^{-1} = \left(\sum_{i=0}^6 T^{7\beta s_i - \alpha i} \right) (T^7 - 1)^{-1}.$$

We take a divisor $d = 7$ of $p = 49\beta$ and let ζ denote a primitive 7-th root of unity. We use deformations

$$T^{7\beta s_i - \alpha i} = T^{-\alpha i} (T^{7\beta s_i} - 1) + T^{-\alpha i}, \quad \frac{T^{7\beta s_i} - 1}{T^7 - 1} = 1 + T^7 + T^{14} + \dots + T^{7(\beta s_i - 1)}.$$

For a ring homomorphism ψ satisfying $\psi(T) = \zeta$ with $\xi = \zeta^{-\alpha}$ (then ξ is still a primitive unity, since $\gcd(\alpha, 7) = 1$),

$$\begin{aligned} \tau^\psi(M) &\doteq \left\{ \beta(\zeta^{-\alpha} - 1) \left(\sum_{i=0}^6 s_i \zeta^{-\alpha i} \right) - \alpha \right\} (\zeta^{-\alpha} - 1)^{-2} \\ &= \left\{ \beta(\xi - 1) \left(\sum_{i=0}^6 s_i \xi^i \right) - \alpha \right\} (\xi - 1)^{-2}. \end{aligned}$$

In the 7-th cyclotomic field $\mathbb{Q}(\zeta_7)$, using the equalities $\xi^7 = 1$ and $1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6 = 0$,

$$\begin{aligned} (\xi - 1) \sum_{i=0}^6 s_i \xi^i &= -3\xi - 2\xi^2 - 3\xi^3 - 3\xi^4 - 2\xi^5 - 3\xi^6 + 16 \\ &= -3\xi - 2\xi^2 - 3\xi^3 - 3\xi^4 - 2\xi^5 - 3\xi^6 + 16 \\ &\quad + 3(1 + \xi + \xi^2 + \xi^3 + \xi^4 + \xi^5 + \xi^6) \\ &= 19 + \xi^2 + \xi^5 \\ &= 19 + \xi^2 + \xi^{-2}. \end{aligned}$$

The Reidemeister–Turaev torsion of Dehn surgery $M = (BL; \alpha/\beta, 0)$ is

$$(2) \quad \tau^\psi(M) \doteq \{ \beta(\xi^2 + \xi^{-2}) - (\alpha - 19\beta) \} (\xi - 1)^{-2}.$$

Now, suppose that M is a lens space $L(p, q)$ with $p = 49\beta$ (by Step 1) and undecided q . Then there exist integers i, j coprime to p with $0 < i, j < p$ such that

$$(3) \quad \tau^\psi(M) \doteq (\xi^i - 1)^{-1} (\xi^j - 1)^{-1}.$$

We can assume $i + j$ is even. We treat with $i, j \bmod 7$ ($i, j \in \{1, 2, 3, 4, 5, 6\}$), since $d = 7$.

(Step 3) Using Lemma 4.2 on d -norm with $d = 7$ on (2) and (3), we have a necessary condition for the Dehn surgery $M = (BL; \alpha/\beta, 0)$ to be a lens space:

$$N_d(\beta(\xi^2 + \xi^{-2}) - (\alpha - 19\beta)) = 1.$$

Roughly, it means $r = \alpha/\beta$ is near 19.

(Step 4) We set $\alpha' = \alpha - 19\beta$. By (2) and (3), we have

$$\xi \{ \beta(\xi^2 + \xi^{-2}) - \alpha' \} (\xi - 1)^{-2} = \pm \xi^{(i+j)/2} (\xi^i - 1)^{-1} (\xi^j - 1)^{-1}.$$

We regard it as an equality between real value sequence. Without loss of generality, we assume $0 < i < d/2$ (i.e., $i = 1, 2$ or 3), $i \leq j$, and define $f = (i+j)/2$, $e = (j-i)/2$. The equality lifts as an identity of symmetric Laurent polynomial

$$(4) \quad (\beta \langle t^2 \rangle - \alpha') (\langle t^f \rangle - \langle t^e \rangle) = \pm (\langle t \rangle - 2),$$

in $\mathbb{Z}[t, t^{-1}]/(t^7 - 1)$, where $\langle t^i \rangle = t^i + t^{-i}$, as in Definition 4.7. The left-hand side $F(t)$ is expanded to

$$\beta \langle t^{f+2} \rangle + \beta \langle t^{f-2} \rangle - \alpha' \langle t^f \rangle - \beta \langle t^{e+2} \rangle - \beta \langle t^{e-2} \rangle + \alpha' \langle t^e \rangle.$$

We regard the identity (4) as an equation on (f, e) : It is a necessary condition on (α', β) for the equation to have a solution (f, e) . Since $f \neq e$ is obvious and $\langle t^4 \rangle = \langle t^3 \rangle$, $\langle t^5 \rangle = \langle t^2 \rangle \pmod{(t^7 - 1)}$, we only have to consider six cases

$$(f, e) = (1, 0), (2, 0), (3, 0), (2, 1), (3, 1), (3, 2).$$

Note that $\langle t^{-x} \rangle = \langle t^x \rangle$ and $\langle t^0 \rangle = 2$.

(f, e)	$F(t)$	(α', β)
$(1, 0)$	$\beta \langle t^3 \rangle - 2\beta \langle t^2 \rangle - (\alpha' - \beta) \langle t^1 \rangle + 2\alpha'$	No
$(2, 0)$	$\beta \langle t^3 \rangle - (\alpha' + 2\beta) \langle t^2 \rangle + 2(\alpha' + \beta)$	No
$(3, 0)$	$-\alpha' \langle t^3 \rangle - \beta \langle t^2 \rangle + \beta \langle t^1 \rangle + 2\alpha'$	No
$(2, 1)$	$-\alpha' \langle t^2 \rangle + (\alpha' - \beta) \langle t^1 \rangle + 2\beta$	$(\alpha', \beta) = (0, 1)$
$(3, 1)$	$-(\alpha' + \beta) \langle t^3 \rangle + \beta \langle t^2 \rangle + \alpha' \langle t^1 \rangle$	No
$(3, 2)$	$-(\alpha' + \beta) \langle t^3 \rangle + (\alpha' + \beta) \langle t^2 \rangle + \beta \langle t^1 \rangle - 2\beta$	$(\alpha', \beta) = (-1, 1)$

Since $\alpha' = \alpha - 19\beta$, $(\alpha', \beta) = (0, 1)$ (and $(-1, 1)$, respectively) corresponds to $\alpha/\beta = 19$ (and 18). We have the required conclusion (pointed out in [1]):

Berge's link BL yields a lens space as $(BL; r, 0)$ only if $r = 19$ or $r = 18$.

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