

MULTI-DIMENSIONAL QUANTUM AZÉMA MARTINGALES

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ABSTRACT. We introduce a multi-dimensional generalization of the quantum Azéma martingale.

1. INTRODUCTION

We will study the process defined by the quantum stochastic differential equations

$$(1.1) \quad dX_i(t) = A_i^+(t) + \sum_{j,k,\ell=1}^n \int_0^t C_{j\ell}^{ik} X_j(s) d\Lambda_k^\ell(s),$$

$$(1.2) \quad dV^i(t) = A_i(t) + \sum_{j,k,\ell=1}^n \int_0^t \overline{C_{jk}^{i\ell}} V^j(s) d\Lambda_k^\ell(s),$$

where A_i^+ , A_i , Λ_j^i are the creation, annihilation, and conversation processes on the boson Fock space $\Gamma(L^2(\mathbb{R}_+, \mathbb{C}^n))$,

$$A_i^+(t) = A^+(\mathbf{1}_{[0,t]} \otimes e_i), \quad A_i(t) = A(\mathbf{1}_{[0,t]} \otimes e_i), \quad \Lambda_j^i(t) = \Lambda(\mathbf{1}_{[0,t]} \otimes |e_i\rangle\langle e_j|),$$

and $C_{i\ell}^{jk}$ are complex coefficients.

The special case $n = 1$ and $C_{11}^{11} = q-1$ leads to the quantum Azéma martingales defined by Parthasarathy [Par90] to get a quantum version of the classical Azéma martingales introduced by Emery in [Eme89]. Emery proved that the classical Azéma martingales have the chaotic representation property, i.e. the iterated integrals

$$Q^{(0)}(t) = 1, \quad Q^{(1)}(t) = M_t, \quad Q^{(n)}(t) = \int_0^t Q^{(n-1)}(s) dM_s, \quad n = 2, 3, \dots$$

are total in $L^2(\Omega, \mathcal{F}, P)$, where $(\Omega, \mathcal{F}, P)$ is the probability space on which $(M_t)_{t \in \mathbb{R}_+}$ is defined and $\mathcal{F}$ the σ -algebra generated by the process.

Other martingales having this property are, e.g., the Brownian motion and the compensated Poisson process. The Azéma martingale provided the first example of a martingale having the chaotic representation property that is not a Lévy process. The problem of the classification of all normal martingales having the chaotic representation property is still open.

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Parthasarathy's [Par90] construction of the quantum Azéma martingales lead to a new proof of their chaotic representation property.

It was shown by Schürmann [Sch91, Sch93] that the quantum Azéma martingales arise as a component of a Lévy process on an involutive bialgebra.

The goal of this paper is to construct multi-dimensional analogues of the quantum Azéma martingales and to show that they lead to classical processes having the chaotic representation property. For some results on classical multi-dimensional Azéma martingales, see [AÉ94, AÉ96].

Roughly speaking, Parthasarathy's proof of the chaotic representation property relies on three properties of the quantum Azéma martingale defined by

$$\begin{aligned} X(t) &= A^+(t) + \int_0^t (q-1)X(s)d\Lambda_s, \\ V(t) &= A(t) + \int_0^t (q-1)V(s)d\Lambda_s. \end{aligned}$$

He shows that the symmetric linear combinations

$$Z(t) = X(t) + V(t), \quad t \in \mathbb{R}_+,$$

are self-adjoint for $-1 \leq q \leq 1$ (even bounded for $-1 \leq q < 1$). Furthermore, he shows that they commute for all times, i.e.

$$[Z(t), Z(s)] = 0$$

for all $s, t \in \mathbb{R}_+$. These two properties imply that there exists a classical process $(\tilde{Z}_t)_{t \in \mathbb{R}_+}$ having the same distribution as the joint spectral density of the commuting family of self-adjoint operators $(Z(t))_{t \in \mathbb{R}_+}$ evaluated in the vacuum state. Then he shows that the iterated integrals of $Z(t)$ generate the same space from the vacuum as the quantum Brownian motion $Q_t = A^+(t) + A(t)$. Since we know that the Brownian motion has the chaotic representation property, this allows one to show that the classical process $(\tilde{Z}_t)_{t \in \mathbb{R}_+}$ also has this property.

In this paper we will look for the conditions on the coefficients $C_{j\ell}^{ik}$ in the quantum stochastic differential equations (1.1) and (1.2), that guarantee that the multi-dimensional process defined by

$$Z_i(t) = X_i(t) + V^i(t)$$

has similar properties. In the end we get a multi-dimensional version of Parthasarathy's result by combining these conditions.

Note that the $Z_i(t)$ can be defined directly by one single quantum stochastic differential equation

$$(1.3) \quad Z_i(t) = A_i^+(t) + A_i(t) + \sum_{j,k,\ell=1}^n \int_0^t C_{j\ell}^{ik} Z_j(s) d\Lambda_k^\ell(s),$$

if the coefficients satisfy

$$C_{j\ell}^{ik} = \overline{C_{jk}^{i\ell}}$$

for all $1 \leq i, j, k, \ell \leq n$.

2. MULTI-DIMENSIONAL AZÉMA PROCESSES AS NON-COMMUTATIVE LÉVY PROCESSES

Let $\mathcal{B}$ be the free algebra generated by $x_i, a_j^i, i, j = 1, \dots, n$. Setting

$$\Delta x_i = x_j \otimes a_i^j + 1 \otimes x_i, \quad \Delta a_j^i = a_k^i \otimes a_j^k, \quad \varepsilon(x_i) = 0, \quad \varepsilon(a_j^i) = \delta_{ij},$$

and extending $\Delta : \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{B}, \varepsilon : \mathcal{B} \rightarrow \mathbb{C}$ as algebra homomorphisms, we get a bialgebra. Taking the free $*$ -algebra $\tilde{\mathcal{B}}$ generated by these elements, and extending $\Delta : \tilde{\mathcal{B}} \rightarrow \tilde{\mathcal{B}} \otimes \tilde{\mathcal{B}}, \varepsilon : \tilde{\mathcal{B}} \rightarrow \mathbb{C}$ as algebra homomorphisms, we get an involutive bialgebra $\tilde{\mathcal{B}}$. For the adjoints $v^i = (x_i)^*$ and $b_i^j = (a_j^i)^*$ we have

$$\Delta v^i = v^j \otimes b_j^i + 1 \otimes v^i, \quad \Delta b_i^j = b_i^k \otimes b_k^j, \quad \varepsilon(v^i) = 0, \quad \varepsilon(b_i^j) = \delta_{ij}.$$

Recall the definition of a Lévy process on an involutive bialgebra.

Definition 2.1. [Sch93] Let $(\mathcal{B}, \Delta, \varepsilon)$ be a an involutive bialgebra. A quantum stochastic process $\{j_{st}\}_{0 \leq s \leq t \leq T}$ on $\mathcal{B}$ over some quantum probability space $(\mathcal{A}, \Phi)$ is called a *Lévy process on the involutive bialgebra $\mathcal{B}$* , if the following four conditions are satisfied.

1. (Increment property) We have

$$\begin{aligned} j_{rs} \star j_{st} &= j_{rt} \quad \text{for all } 0 \leq r \leq s \leq t \leq T, \\ j_{tt} &= \varepsilon \mathbf{1}_{\mathcal{A}} \quad \text{for all } 0 \leq t \leq T. \end{aligned}$$

2. (Independence of increments) The family $\{j_{st}\}_{0 \leq s \leq t \leq T}$ is independent (w.r.t. Φ), i.e.

- (i) $\Phi(j_{s_1 t_2}(b_1) \cdots j_{s_n t_n}(b_n)) = \Phi(j_{s_1 t_1}(b_1)) \cdots \Phi(j_{s_n t_n}(b_n))$ for all $b_1, \dots, b_n \in \mathcal{B}$, and all $0 \leq s_1 \leq t_1 \leq s_2 \leq \dots \leq t_n \leq T$ and
- (ii) $[j_{st}(b_1), j_{s't'}(b_2)] = 0$ for all $b_1, b_2 \in \mathcal{B}$ and all s, t, s', t' with $0 \leq s \leq t \leq T, 0 \leq s' \leq t' \leq T$, and $]s, t[\cap]s', t'[= \emptyset$.

3. (Stationarity of increments) The distribution $\varphi_{st} = \Phi \circ j_{st}$ of j_{st} depends only on the difference $t - s$.

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4. (Weak continuity) The quantum random variables j_{st} converge to j_{ss} in distribution for $t \searrow s$.

If $\{j_{st}\}_{0 \leq s \leq t \leq T}$ is a Lévy process on an involutive bialgebra, then the marginal distributions $\varphi_{t-s} = \varphi_{st} = \Phi \circ j_{st}$ form a convolution semi-group of states, i.e. $\varphi_0 = \varepsilon$ and $\varphi_s \star \varphi_t = \varphi_{s+t}$ for all s, t . It can be shown that there exists a unique hermitian linear functional $\psi : \mathcal{B} \rightarrow \mathbb{C}$ such that $\varphi_t = \exp_\star t\psi = \varepsilon + t\psi + \frac{t^2}{2}\psi \star \psi + \dots$. Furthermore, ψ is conditionally positive, i.e. positive on $\mathcal{B}^0 = \ker \varepsilon$, and kills the unit of $\mathcal{B}$, i.e. $\psi(1_{\mathcal{B}}) = 0$. Such functionals are called generators.

Conversely, given a generator $\psi : \mathcal{B} \rightarrow \mathbb{C}$, we can define a convolution semi-group of states by $\varphi_t = \exp_\star t\psi$ and reconstruct a Lévy process on $\mathcal{B}$ from it.

Let us briefly describe how one can construct a realization of the Lévy process on a boson Fock space. By a GNS-type construction one can complete a generator ψ to a Schürmann triple (ρ, η, ψ) , i.e. a triple consisting of a unital $\ast$ -representation ρ of $\mathcal{B}$ on some pre-Hilbert space D , a linear map $\eta : \mathcal{B} \rightarrow D$ that satisfies

$$(2.1) \quad \eta(ab) = \rho(a)\eta(b) + \eta(a)\varepsilon(b), \quad \text{for all } a, b \in \mathcal{B},$$

and a hermitian linear functional $\psi : \mathcal{B} \rightarrow \mathbb{C}$ that satisfies

$$(2.2) \quad \psi(ab) = \varepsilon(a)\psi(b) + \langle \eta(a^\ast), \eta(b) \rangle + \psi(a)\varepsilon(b), \quad \text{for all } a, b \in \mathcal{B}.$$

The realization of the Lévy process associated to ψ on the boson Fock space $\Gamma(L^2(\mathbb{R}_+, \overline{D}))$ is then given as solution of the quantum stochastic differential equations

$$(2.3) \quad j_{st}(b) = \varepsilon(b)\text{id} + \left(\int_s^t j_{s\tau} \otimes dI_\tau \right) \Delta(b), \quad \text{for all } b \in \mathcal{B},$$

where the integrator dI is given by

$$dI_t(b) = d\Lambda_t(\rho(b) - \varepsilon(b)\text{id}) + dA_t^+(\eta(b)) + dA_t(\eta(b^\ast)) + \psi(b)dt.$$

For details and proofs see [Sch93].

Set

$$R_{k\ell}^{ij} = C_{k\ell}^{ij} + \delta_{ij}\delta_{k\ell}$$

and define a Schürmann triple (ρ, η, ψ) on $\tilde{\mathcal{B}}$, acting on $\mathbb{C}^n$, by

$$\begin{aligned} \rho(a_j^i)e_\ell &= \sum_{k=1}^n R_{j\ell}^{ik}e_k, & \rho(x_i) &= 0, \\ \eta(a_j^i) &= \eta(b_j^i) = 0, & \eta(x_i) &= e_i, & \eta(v^i) &= 0, \\ \psi(a_j^i) &= 0, & \psi(x_i) &= 0, \end{aligned}$$

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where $e_1, \dots, e_n$ are a basis of $\mathbb{C}^n$. There exists a unique Schürmann triple having these values on the generators of $\tilde{\mathcal{B}}$. The quantum stochastic differential equations for the associated Lévy process are

$$\begin{aligned} j_{st}(a_j^i) &= \delta_{ij} \text{id} + \int_s^t \sum_{k,p,q} j_{s\tau}(a_k^i) \left(R_{jq}^{kp} - \delta_{kj} \delta_{pq} \right) d\Lambda_p^q(\tau), \\ j_{st}(x_i) &= A_i^+(t) - A_i^+(s) + \int_s^t \sum_j j_{s\tau}(x_j) \left(R_{iq}^{jp} - \delta_{ij} \delta_{pq} \right) d\Lambda_p^q(\tau), \end{aligned}$$

Comparing the second equation with (1.1), we see that we have

$$j_{0t}(x_i) = j_t(x_i) = X_i(t), \quad j_{0t}(v^i) = j_t(v^i) = V^i(t).$$

The theory of Lévy processes now implies that the domains of the $X_i(t)$, $V_i(t)$ obtained by solving the quantum stochastic differential equations (1.1) and (1.2) contain $\bigcap_{\alpha \in \mathbb{R}_+} \text{dom } \alpha^N$, where N denotes the number operator on the boson Fock space.

Note finally that we can also define the same Lévy process as a solution of the backwards quantum stochastic differential equations

$$(2.4) \quad j_{st}(a_j^i) = \delta_{ij} \text{id} + \int_s^t \sum_{k,p,q} (R_{kq}^{ip} - \delta_{ik} \delta_{pq}) d\Lambda_p^q(\tau) j_{\tau t}(a_j^k),$$

$$(2.5) \quad j_{st}(x_i) = \int_s^t \sum_j dA_i^+(\tau) j_{\tau t}(a_i^j),$$

cf. [Sch93].

3. WHEN ARE OUR MULTI-DIMENSIONAL AZÉMA MARTINGALES COMMUTATIVE?

In this section we find conditions on the coefficients that guarantee that the processes commute. In the one-dimensional case [Par90], no conditions were necessary, but the commutativity already followed from the form of the quantum stochastic differential equations.

Proposition 3.1. (a): *The two-sided ideal $\mathcal{I}$ of $\mathcal{B}$ generated by the elements*

$$\begin{array}{ll} x_i x_j - x_j x_i, & i, j = 1, \dots, n \\ a_i^k x_j + x_i a_j^k - a_j^k x_i - x_j a_i^k, & i, j, k = 1, \dots, n \\ a_i^k a_j^\ell - a_j^\ell a_i^k, & i, j, k, \ell = 1, \dots, n \end{array}$$

is a coideal.

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(b): The two-sided ideal $\tilde{\mathcal{I}}$ of $\tilde{\mathcal{B}}$ generated by the elements

$$\begin{aligned}
& (x_i + v^i)(x_j + v^j) - (x_j + v^j)(x_i + v^i), & i, j = 1, \dots, n \\
& a_i^k x_j + x_i a_j^k - a_j^k x_i - x_j a_i^k + a_i^k v^j + v^i a_j^k - a_j^k v^i - v^j a_i^k, & i, j, k = 1, \dots, n \\
& b_k^i x_j + x_i b_k^j - b_k^j x_i - x_j b_k^i + b_k^i v^j + v^i b_k^j - b_k^j v^i - v^j b_k^i, & i, j, k = 1, \dots, n \\
& a_i^k a_j^\ell - a_j^\ell a_i^k, \quad a_i^k b_j^\ell - a_j^\ell a_i^k, \quad b_k^i a_j^\ell - b_j^\ell b_k^i, \quad b_k^i b_j^\ell - b_j^\ell b_k^i, & i, j, k, \ell = 1, \dots, n
\end{aligned}$$

is a coideal.

Proof. We only show (a), the proof of (b) is similar.

We have to show

$$\Delta(\mathcal{I}) \subseteq \mathcal{I} \otimes \mathcal{B} + \mathcal{B} \otimes \mathcal{I}.$$

and $\varepsilon(\mathcal{I}) = \{0\}$. It is sufficient, to verify this on the algebraic generators of $\mathcal{I}$.

We get

$$\begin{aligned}
& \Delta(x_i x_j - x_j x_i) \\
&= (x_k \otimes a_i^k + 1 \otimes x_i)(x_\ell \otimes a_j^\ell + 1 \otimes x_j) - (x_\ell \otimes a_j^\ell + 1 \otimes x_j)(x_k \otimes a_i^k + 1 \otimes x_i) \\
&= x_k x_\ell \otimes (a_i^k a_j^\ell - a_j^\ell a_i^k) + x_k \otimes (a_i^k x_j + x_i a_j^k - a_j^k x_i - x_j a_i^k) + 1 \otimes (x_i x_j - x_j x_i),
\end{aligned}$$

similarly

$$\begin{aligned}
\Delta(a_i^k x_j + x_i a_j^k - a_j^k x_i - x_j a_i^k) &= a_\ell^k x_m \otimes (a_i^\ell a_j^m - a_j^m a_i^\ell) + x_m a_i^k \otimes (a_i^m a_j^\ell - a_j^\ell a_i^m) \\
&\quad + a_\ell^k \otimes (a_i^\ell x_j + x_i a_j^\ell - a_j^\ell x_i - x_j a_i^\ell),
\end{aligned}$$

and finally

$$\Delta(a_i^k a_j^\ell - a_j^\ell a_i^k) = a_r^k a_s^\ell \otimes (a_i^r a_j^s - a_j^s a_i^r).$$

□

Corollary 3.2. (a): If the coefficients $R_{j\ell}^{ik}$ satisfy

$$R_{j\ell}^{ik} = R_{\ell j}^{ik}$$

and

$$R_{j\ell}^{ks} R_{j\ell}^{ik} = R_{j\ell}^{ik} R_{j\ell}^{ks}$$

for all i, j , then we have

$$[X_i(s), X_j(t)] = 0$$

(b): If the coefficients $R_{j\ell}^{ik}$ satisfy

$$R_{j\ell}^{ik} = R_{\ell j}^{ik},$$

and

for all i, j , then we have

$$[Z_i(s), Z_j(t)] = 0$$

Remark 3.3. Expressed in terms of the coefficients $C_{j\ell}^{ik}$ appearing in the quantum stochastic differential equations (1.1) and (1.2) the conditions become

..

and

..

Proof.

□

4. WHEN ARE OUR MULTI-DIMENSIONAL AZÉMA MARTINGALES BOUNDED?

Lemma 4.1. Suppose that there exists a constant $0 \leq M < 1$ such that

$$\left\| \left(\sum_{k,\ell} \bar{a}_k b_\ell R_{j\ell}^{ik} \right)_{1 \leq k, \ell \leq n} \right\| \leq M \|a\| \|b\|$$

for all $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n$. Then the operators $j_{st}(a_j^i)$ are contractions.

Proof. Using the first fundamental lemma [Par92, Proposition 25.1], we get the integral equations

$$\begin{aligned} \langle \mathcal{E}(f), j_t(a_j^i) \mathcal{E}(g) \rangle &= \delta_{ij} \langle \mathcal{E}(f), \mathcal{E}(g) \rangle \\ &+ \int_0^t \sum_{k,q,p} \langle \mathcal{E}(f), j_s(a_k^i) \mathcal{E}(g) \rangle (R_{jq}^{kp} - \delta_{kj} \delta_{pq}) \overline{f_p(t)} g_q(t) ds \end{aligned}$$

for the matrix elements of the operators $j_{st}(a_j^i)$, where $f, g \in L^2(\mathbb{R}_+, \mathbb{C}^n)$. If we set

$$A_{f,g}(t) = \langle \mathcal{E}(f), j_t(a_j^i) \mathcal{E}(g) \rangle$$

and

$$\begin{aligned} R_{f,g}(t) &= \left(\sum_{p,q} R_{jq}^{ip} \overline{f_p(t)} g_q(t) \right)_{1 \leq i,j \leq n}, \\ \bar{R}_{f,g}(t) &= \left(\sum_{p,q} (R_{jq}^{ip} - \delta_{ij} \delta_{pq}) \overline{f_p(t)} g_q(t) \right)_{1 \leq i,j \leq n}, \end{aligned}$$

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then we get the matrix-valued differential equation

$$\frac{d}{dt}A_{f,g}(t) = A_{f,g}(t)\overline{R}_{f,g}(t),$$

with the initial condition

$$A_{f,g}(0) = \left(\langle \mathcal{E}(f), \mathcal{E}(g) \rangle \delta_{ij} \right)_{1 \leq i, j \leq n}.$$

The unique solution of this equation is

$$A_{f,g}(t) = A_{f,g}(0) \exp \left(\int_0^t \overline{R}_{f,g}(s) ds \right),$$

and therefore we get

$$(4.1) \quad \langle \mathcal{E}(f), j_{st}(a_j^i) \mathcal{E}(g) \rangle = \left(\exp \left(\int_s^t \overline{R}_{f,g}(s) ds \right) \right)_{ij} \exp \left(\int_0^s \langle f(\tau), g(\tau) \rangle d\tau + \int_t^\infty \langle f(\tau), g(\tau) \rangle d\tau \right)$$

Using this identity, we get the desired estimate. $\square$

Proposition 4.2. *Suppose that there exists a constant $0 \leq M < 1$ such that*

$$\left\| \left(\sum_{k,\ell} \overline{a_k} b_\ell R_{j\ell}^{ik} \right)_{1 \leq k, \ell \leq n} \right\| \leq M \|a\| \|b\|$$

for all $a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \in \mathbb{C}^n$. Then the operators $X_i(t)$, $V^i(t)$ and $Z_i(t) = X_i(t) + V^i(t)$ are bounded for all $t \leq t$.

Proof. Using the first fundamental lemma [Par92, Proposition 25.1] and the backward stochastic differential equation (2.5) for $X_i(t) = j_t(x_i)$, we get

$$\langle \mathcal{E}(f), X_i(t) \mathcal{E}(g) \rangle = \int_0^t \langle \mathcal{E}(f), j_{st}(a_i^j) \mathcal{E}(g) \rangle \overline{f_j(s)} ds$$

Using (4.1), this implies that the $X_i(t)$ are bounded, and therefore also the $V_i(t)$ and the $Z_i(t)$. $\square$

[FSS01]

5. CHAOS COMPLETENESS

Define the iterated integrals

$$Q_{i_1, \dots, i_r}(t) = \int_{0 \leq t_1 \leq \dots \leq t_r \leq t} dQ_{i_1} \cdots dQ_{i_r},$$

where $Q_i(t) = A_i^+(t) + A_i(t)$, and

$$I_{i_1, \dots, i_r}(t) = \int_{0 \leq t_1 \leq \dots \leq t_r \leq t} dZ_{i_1} \cdots dZ_{i_r}$$

then we have

$$Q_{i_1, \dots, i_r}(t)\Omega = I_{i_1, \dots, i_r}(t)\Omega,$$

for all $r \in \mathbb{N}$, $i_1, \dots, i_r \in \{1, \dots, n\}$, and $t \geq 0$.

This implies that we can copy Parthasarathy's proof of the chaos completeness whenever the Z_i define a bounded and commutative self-adjoint operator process.

Theorem 5.1. *Suppose that the coefficients satisfy ...*

Then there exists a classical process $\left((\tilde{Z}_1(t), \dots, \tilde{Z}_n(t))\right)_{t \in \mathbb{R}_+}$ having the same joint distribution as the quantum stochastic process $\left((Z_1(t), \dots, Z_n(t))\right)_{t \in \mathbb{R}_+}$ and havin the chaotic representation property, i.e. the iterated integrals

$$\tilde{I}_{i_1, \dots, i_r}(t) = \int_{0 \leq t_1 \leq \dots \leq t_r \leq t} d\tilde{Z}_{i_1} \cdots d\tilde{Z}_{i_r}$$

with $r \in \mathbb{N}$ and $i_1, \dots, i_r \in \{1, \dots, n\}$ are total in the L^2 -space over the underlying probability space (where we assume that the σ -algebra is generated by the process).

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