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Operators and operator algebras in Krein Spaces I.

Spectral analysis in Pontrjagin space.

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Introduction

Krein space is an infinite dimensional generalization of Minkowsky space, and more than fourty years ago several physists already awared the importance of studying such spaces. For instance one old idea is that a Hamiltonian in relativestic quantum mechaniques must be a selfadjoint operator in a Krein space.

In any sense the theory of operators and operator algebras in Krein spaces must be founded by developping the spectral analysis of selfadjoint operators in such spaces.

This paper is devoted to founding spectral analysis of self-adjoint operators in Pontrjagin spaces.

Chapter 1. Notations and definitions

§1.0. Preliminary images of Krein space and Pontrjagin space.

Consider the product space $L := H_1 \times H_2$ of Hilbert spaces H_1 and H_2, and define a sequilinear form $\langle x,y \rangle$ in $L \times L$ by

$$\langle x,y \rangle = (x_1|y_1) - (x_2|y_2),$$

where $x := (x_1,x_2)$ $y := (y_1,y_2)$ are elements of L. If H_1 and H_2 are of finite dimensional. L is known as a Minkowsky space. If H_1 and H_2 are Hilbert spaces, L has been called a Krein space, or an indefinite metric space. If either H_1 or H_2 are of finite dimensional, L has been called a Pontrjagin space.

In the following sections we shall introduce other definitions of Krein and Pontrjagin spaces, as we shall start from defining indefinite innerproduct space.

§1.1. Indefinite innerproduct space.

In what follows "linear" always means" complex linear", C denotes the complex numbers, and R the real numbers.

1.1.1. (Definition) A linear space L is called an indefinite innerproduct space if L has an indefinite innerproduct $\langle x,y \rangle$ satisfying the following conditions I.1 and I.2.

I.1. $\langle x,y \rangle$ is a symmetric sequilinear form on $L \times L$. Namely, it is a complex valued functional on $L \times L$ satisfying

$$\langle x,y\rangle = \overline{\langle y,x\rangle} ,$$

$$\langle \alpha x_1+\beta x_2,y\rangle = \alpha\langle x_1,y\rangle + \beta\langle x_2,y\rangle .$$

$$\langle x,\alpha y_1+ y_2\rangle = \overline{\alpha}\langle x,y_1\rangle + \overline{\beta}\langle x,y_2\rangle .$$

I.2. There is a certain innerproduct-norm $\| \ \|$ (ie., a norm $\|x\| := (x|x)$ defined by an innerproduct $(x|y)$ of L) which is selfpolar. Namely, it satisfies

$$\|x\| = \sup_{\|y\|\leq 1} |\langle x,y\rangle| .$$

A norm $\| \ \|$ satisfying I.2. in L is called a finite unitary norm of L.

1.1.2. (Definition). An indefinite innerproduct space L is called a Krein space if it becomes a Hilbert space under its certain finite unitary norm. The topology of L is that of the Hilbert space.

1.1.3. (Definition). The metrical completion L(p) of an indefinite innerproduct space L under a certain finite unitary norm p of L is a Krein space, which we call the Krein completion of L under p.

1.1.4. We must beware that the space L may have finite unitary norms which are not generally mutually equivalent on L, so that Krein completions of L under different norms cannot generally be identified to each other.

1.2. #-innerproducts in Krein spaces.

Let L be a Krein space. Then $\langle x,y\rangle$ denotes the indefinite innerproduct of L, and it is also called the #-innerproduct of L, to distinguish it from the *-innerproduct which is defined by a fixed anxiliary finite unitary norm * of L.

In what follows L and M denote Krein spaces.

1.2.1. (Definition). Let x and y belong to L. x is called #-orthogonal to y if $\langle x,y\rangle = 0$ holds. Let $\mathcal{M}$ and $\mathcal{N}$ be subsets of L. $\mathcal{M}$ is called #-orthogonal if $\langle x,y\rangle = 0$ holds for $x \in \mathcal{M}$ and $y \in \mathcal{N}$. The elements of $\mathcal{M}$ which are #-orthogonal to is denoted by $\mathcal{M}^{\#\perp}$. If $\mathcal{M}$ is a closed linear subspace of L, $\mathcal{M}^{\#\perp}$ is called the #-orthogonal complement of $\mathcal{M}$.

1.2.2. (Definition). $X : L \to M$ denotes a mapping whose domain $\mathcal{D}(X)$ is in L, and range $R(X)$ is in M. X is called densely defined if $\mathcal{D}(X)$ is everywhere dense in L. X is called invertiblly densely defined of X and its inverse mapping $X^{-1} : M \to L$ are densely defined.

1.2.3. (Definition). Let $X : L \to M$ be a linear mapping. The #-adjoint, or the Krein adjoint, of X means a linear mapping $X^{\#} : M \to L$ which is determined by the following rule : Take any $x \in L$ and $y \in M$. $X^{\#}y$ is defined and $x = X^{\#}y$ holds iff it satisfies

$$\langle x,z\rangle = \langle y,Xz\rangle \quad \text{for all} \quad z \in \mathcal{D}(X).$$

The #-adjoint $X^{\#}$ exists iff X is densely defined.

1.2.4. (Definition) (a). An operator X in L is called #-selfadjoint if $X = X^{\#}$ holds.

(b). An operator X in L is called #-positive if it is #-selfadjoint and satisfies $\langle Xx,x\rangle \geq 0$ on $\mathcal{D}(X)$.

(c). An operator X in L is called a #-projection if it satisfies $X = X^{\#} = X^2$.

(d). A linear mapping $X : L \to M$ is called #-unitary if $X^{\#} = X^{-1}$ holds.

(e). A linear mapping $X : L \to M$ is called #-imaginary if $X^{\#} = -X^{-1}$ holds.

(f) A linear mapping $x : L \to M$ is called #-selforthogonal if $X^{\#}X$ vanishes on $\mathcal{D}(X)$.

1.2.5. Every #-selfadjoint operator is closed and densely defined, and so are every #-positive operator and #-projection. Every #-unitary mapping and every #-imaginary. Mapping are closed and invertiblly densely defined. Indeed, for instance, if $X : L \to M$ is #-unitary, $X^{\#} = X^{-1}$ means that X is densely defined and the inverse mapping X^{-1} is closed. Then X is closed and hence $X^{\#}(=X^{-1})$ is densely defined. A mapping $X : L \to M$ is #-selforthogonal iff X is densely defined and has the #-selforthogonal range. Indeed, X is selforthogonal iff $X^{\#}$ is defined and satisfies $\langle Xx,Xy\rangle = 0$ for all x, y in $\mathcal{D}(X)$.

1.2.6. (Remark). We must beware that #-projections and #-unitary mappings in our definition are not generally continuous.

§1.3. The *-norm in Krein space.

1.3.1. We consider a certain fixed finite unitary norm of L which makes L a Hilbert space. We call this norm the *-norm and denote it by $\| \ \|$ or by &. The innerproduct associate with the norm * is denoted by $(x|y)$, and is called the *-innerproduct of L. The definitions in the Section 1.2 are also applied to the *-innerproducts of the spaces L and M.

1.3.2. (Definition). Let x, y belong to L. x is *-orthogonal to y if $(x|y) = 0$ holds. Let $\mathcal{M}$, $\mathcal{N}$ be subsets of L. $\mathcal{M}$ is *-orthogonal to $\mathcal{N}$ if $(x|y) = 0$ holds for $x \in \mathcal{M}$ and $y \in \mathcal{N}$.

Let X be an operator in L. X is *-selfadjoint if $X = X^*$ holds. X is *-positive if X is *-selfadjoint and satisfies $(Xx|x) \geq 0$ on $\mathcal{D}(X)$. X is a *-projection if $X = X^* = X^2$ holds. A linear mapping $X : L \to M$ is *-unitary if $X^* = X^{-1}$ holds.

A subset $\mathcal{M} \neq \phi$ of L becomes *-selforthogonal iff $\mathcal{M} = 0$ holds. *-projections and *-unitary mappings are continuous. *-imaginary mapping $X : L \to M$ does not exist. A densely defined linear mapping $X : L \to M$ is *-selforthogonal iff X vanishes on the domain.

1.3.3. (Definition). An operator X in L is called biselfadjoint if it is #-selfadjoint and *-selfadjoint. X is called a biprojection if it is a #-projection and a *-projection. A linear mapping $X : L \to M$ is called biunitary if it is #-unitary and *-unitary.

§1.4. Defining operator of the *-norm.

The #-innerproduct $\langle x,y\rangle$ is a continuous sequilinear form on $L \times L$ treating L as a Hilbert space under the *-norm. Then

1.4.1. (Proposition). There is a certain continuous linear operator J in L satisfying

$$\langle x,y\rangle = (Jx|y),$$

$$J = J^* = J^{-1}.$$

Indeed the last identity immediately follws from

$$\langle x,y\rangle = \overline{\langle y,x\rangle}$$

$$\|x\| = \sup_{\|y\|\leq 1} |\langle x,y\rangle|.$$

1.4.2. (Definition). The operator J in L is called the defining operator of the *-norm. If necessary, we also denote J by J_L.

The Definition 1.4.2 is based on the next theorem.

1.4.3. (Theroem). The defining operator J of the *-norm of L is a continuous #-positive #-unitary operator in L, and the *-norm of L is determined by

$$\|x\|^2 = \langle Jx, x\rangle.$$

It is also clear that :

1.4.4. (Proposition). Let $X : L \to M$ be a densely defined linear mapping. Then

$$X^{\#} = J_L X^* J_M, \quad X^* = J_L X^{\#} J_M.$$

Chapter 2. Geometry in Krein space.

In Krein space, we may consider two types of geometries. One is the classical geometry which investigates the invariants of the group of continuous #-unitary operators in L. Another is the investigations of geometrical aspect of unbounded #-unitary operators in L. In this chapter we shall sketch certain aspects of the latter geometry based on the concept of the Cartan decomposition.

§2.1. Cartan decomposition of #-unitary operators.

The Cartan decomposition of Lorentz groups in Minkowsky spaces is generalized to the case of #-unitary operators in Krein spaces.

2.1.1. (Theorem). Let $X : L \to M$ be a #-unitary mapping Then X is represented in the form

$$X = K_2 U = U K_1.$$

$U : L \to M$ is a biunitary mapping. K_1 and K_2 are *-positive #-unitary operators in L and M respectively.

The above representation of X is called the Cartan decomposition of X.

Proof. $X = K_2U = UK_1$ is the V. Neumann's polarization of the mapping $X : L \to M$ regarding L and M as Hilbert spaces. $U : L \to M$ is a *-unitary mapping, and K_1 and K_2 are *-positive operators in L and M, satisfying

$$K_1^2 = X^*X, \quad K_2^2 = XX^*.$$

We show that X^*X and K_1 are #-unitary. Indeed, remarking that $X^{-1} = X^{\#} = JX^*J$, we have

$$J(X^*X)J = (JX^*J)(JXJ) = X^{-1}X^{-1*} = (X^*X)^{-1},$$

and $JK_1^2J = K_1^{-2}$. From this we have $JK_1J = K_1^{-1}$ and K_1 is #-unitary. Similarly we find that K_2 is #-unitary, and U satisfies

$$\langle K_1x, K_1y\rangle = \langle x,y\rangle = \langle Xx, Xy\rangle$$

$$= \langle UK_1x, UK_1y\rangle.$$

Since U is bijective, it is #-unitary, and the Theorem is proved.

§2.2. Unitary norms and #-unitary operators.

2.2.1. (Definition). Let $X : L \to M$ be a linear mapping. Then X induces a (unbounded) seminorm $p_X : x \to \|Xx\|$ defined on L, where we set $\|Xx\| = +\infty$ if Xx is not defined. If X is injective, p_X is called a (unbounded) norm in L.

Now we characterize unbounded norms in L induced from #-unitary mappings.

2.2.2. (Definition). A functional p on L is called a unitary norm of L if the following U0, U1, U2 are satisfied :

U0. $0 \leqq p(x) \leqq +\infty$.

U1. The finite part $\mathcal{D}(p) := \{x \in L : p(x) < +\infty\}$ of p is a linear subspace of L, and p is an innerproduct norm on it. Namely, $p(x)^2 = (x|x)_p$ holds for a certain innerproduct $(x|y)_p$ in $\mathcal{D}(p)$. The innerproduct $(x|y)_p$ is called the p-innerproduct in L.

U2. The norm p is selfpolar on L. Namely, it satisfies

$$p(x) = \sup_{p(y)\leqq 1} |\langle x,y\rangle|.$$

2.2.3. (Theorem). A functional p on L is a unitary norm of L iff p is induced from a certain #-unitary mapping $X : L \to K$, where K is a certain Krein space.

Proof. Suppose taht p is induced from a certain #-unitary mapping $X : L \to K$. Then the finite part of p is identical to the domain $\mathcal{D}(X)$ of X and p is an innerproduct norm on it. p is self-polar. Indeed,

$$\|Xx\| = \|X^{-1\#}x\| = \sup_{\|y\|\leqq 1} |\langle x,X^{-1}y\rangle|$$

$$= \sup_{\|Xz\|\leqq 1} |\langle x,z\rangle|.$$

The identity is valid even if Xx is not defined. Then p is a unitary norm of L.

Conversely, suppose that p is a unitary norm of L. p is an innerproduct norm of the finite part $\mathcal{D}(p)$ of p, and $\mathcal{D}(p)$ has an indefinite innerproduct $\langle x,y\rangle$ and a finite unitary norm p. Let $L(p)$ be the Krein completion of $\mathcal{D}(p)$ under the norm p, and $X : x \to x$, $L(p) \to L$ be the identity mapping on $\mathcal{D}(p)$. X is injective, densely defined, and $X^{\#} \supseteq X^{-1}$. We show that $X^{\#} \subseteq X^{-1}$. Namely, if a pair $x \in L$, $y \in L(p)$ satisfies

$$\langle x, Xz\rangle = \langle y, z\rangle \quad \text{for} \quad z \in \mathcal{D}(p),$$

then $x = y$ holds. If x and y are as above, then the norm $p(y)$ of y in $L(p)$ is determined by

$$\begin{aligned} p(y) &= \sup_{p(z)\leq 1, z\in\mathcal{D}(p)} |\langle y, z\rangle| \\ &= \sup_{p(z)\leq 1} |\langle x, z\rangle| \\ &= p(x). \end{aligned}$$

Therefore x belongs to $\mathcal{D}(p)$, and remarking

$$\langle x, z\rangle = \langle y, z\rangle \quad \text{for} \quad z \in \mathcal{D}(p),$$

we have $x = y$. Then X is #-unitary and induces p.

2.2.4. (Theorem). For every unitary norm p of L there is a unique *-positive #-unitary oeprator K in L which induces p.

Proof. We take a certain #-unitary mapping $X : L \to M$ which induces the norm p, and consider the Cartan decomposition

$X := UK$ of X. $U : L \to M$ is biunitary. K is a x-positive #-unitary operator in L and induces p. Suppose also that p is induced from another *-positive #-unitary operator S in L. Then $\|K<x\| = \|Sx\|$ holds on $\mathcal{D}(p) := \{x \in L : p(x) < +\infty\}$. Therefore $K = S$ holds, and the Theorem is proved.

§2.3. Defining operator of unitary norm.

Proposition 1.4.1 is now generalized as follows :

2.3.1. (Theorem). Let p be a unitary norm of L. Then there is a unique #-positive #-unitary operator J_p in L satisfying

$$p(x)^2 = \langle J_p x, x \rangle.$$

for all x in $\mathcal{D}(J_p)$.

Proof. p is induced from a certain *-positive #-unitary operator K in L. We set

$$J_p = JK^2.$$

Then J_p is #-unitary. We show that J_p is #-positive. Indeed, it is #-selfadjoint and

$$\langle J_p x, x \rangle = (K^2 x | x) = \|Kx\|^2 = p(x)^2,$$

and we have the Theorem. (For the uniqueness of J_p, refer the next Theorem 2.3.3).

2.3.2. (Definition). The operator J_p in Theorem 2.3.1 is called the defining operator of the norm p.

2.3.3. (Theorem). Let S be a #-positive #-unitary operator in L, then S defines a certain unitary norm p in L.

Proof. JS is a *-positive #-unitary operator in L. Indeed, it is *-selfadjoint, #-unitary and

$$(JSx|x) = \langle Sx,x\rangle \geqq 0.$$

holds on $\mathcal{D}(S)$. Let $K = (JS)^{\frac{1}{2}}$. Then K is also *-positive #-unitary and $S = JK^2$ holds, and induces the norm p_K.

§2.4. Canonical partition and canonical quasi-partition.

2.4.1. (Definition). Let L_1 and L_2 be subspaces of L such that :

(a). L_1 and L_2 are the #-orthogonal complements of each other.

(b). If $0 \neq x \in L_1$ then $\langle x,x\rangle > 0$. If $0 \neq y \in L_2$ then $\langle y,y\rangle < 0$.

(c). The linear sum of L_1 and L_2 is everywhere dense in L.

Then the representation of L : $L = L_1+L_2$ is called a canonical quasi-partition of L. Inparticular, if $L = L_1 + L_2$ holds, the representation $L = L_1 + L_2$ is called a canonical partition of L.

2.4.2. (Definition). Let p be a unitary norm of L. Let $L^+(p)$ be the elements x of L satisfying $p(x)^2 = \langle x,x\rangle$,

and $L^-(p)$ the elements x of L satisfying $p(x)^2 = -\langle x,x\rangle$. We call the representation $L := [L^+(p) + L^-(p)]$ the p-quasi-partition of L.

2.4.3. (Theorem). Let p be a unitary norm of L. Then the p-quasi-partition $L := [L^+(p) + L^-(p)]$ is a canonical quasi-partition of L. Let J_p be the defining operator of p. Then $L^+(p)$ and $L^-(p)$ are eigen-spaces for eigenvalues 1 and -1 of J_p, and we have

$$\mathcal{D}(J_p) = L^+(p) + L^-(p).$$

Proof. First we consider the case that p is the *-norm $\|x\|$. Then the defining operator J of the *-norm satisfies $J = J^* = J^{-1}$, and hence is represented in the form : $J = J^+ - J^-$, where J^+ and J^- are *-projections satisfying $1 = J^+ + J^-$. We set $L^+ = L^+(*)$, $L^- = L^-(*)$. L^+ is the elements x of L satisfying $\|x\|^2 = (Jx\ x)$, and is identical to the range of J^+. Similarly, L^- is identical to the range of J^-, these are the eigenspaces for the eigenvalues 1 and -1 of J, respectively, and $L = L^+ + L^-$ is a canonical partition of L.

Next, we return to a general unitary norm p, and let K be the *-positive #-unitary operator in L which induces p. Then

$$J_p = JK^2 = K^{\#}JK.$$

An element x of L belongs to $L^+(p)$ iff

$$\|Kx\|^2 = \langle x,x\rangle = (JKx,Kx),$$

ie., $Kx = JKx$ or $x = J_p x$ holds. In other words x is an eigenvector of J_p for the eigenvalue 1. Then $L^+(p)$ is the eigenspace of J_p for the eigenvalue 1. Similarly, $L^-(p)$ is the eigenspace of J_p for the eigenvalue -1. Now we set

$$J_p^+ = \frac{1}{2}(1 + J_p) = K^{\#}J^+K,$$

$$J_p^- = \frac{1}{2}(1 - J_p) = K^{\#}J^-K.$$

Then $L^+(p)$ and $L^-(p)$ are the ranges of J_p^+ and J_p^-, and

$$\mathcal{D}(J_p) = \mathcal{D}(K^2) = L^+(p) + L^-(p).$$

$L^+(p)$ and $L^-(p)$ are mutually #-orthogonal to each other. Then it determines a canonical quasi-partition of L.

2.4.4. (Theorem). Let $L = [L_1 + L_2]$ be a canonical quasipartition of L. Then it is identical to a certain quasi-partition of L defined by a certain unitary norm of L.

Proof. Define an operator S with the domain $\mathcal{D}$ in L by

$$S : x_1+x_2 \to x_1-x_2 \quad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2.$$

We show that S is a #-positive #-unitary operator in L. S satisfies $S = S^{-1}$, and

$$\langle Sx,x\rangle = \langle x_1,x_1\rangle - \langle x_2,x_2\rangle \geq 0$$

for $x = x_1+x_2 \in \mathcal{D}$. Then $S^{\#} \supseteq S$ holds. We show $S^{\#} \subseteq S$. Let y and z be elements of L satisfying

$$\langle Sx,y\rangle = \langle x,z\rangle \quad \text{for} \quad x \in \mathcal{U}.$$

Let

$$u = \frac{1}{2}(y+z), \quad v = \frac{1}{2}(y-z).$$

Then

$$\langle x_1,v\rangle = \langle x_2,u\rangle \quad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2.$$

Since L_1 and L_2 contains 0, we have

$$\langle x_1,v\rangle = \langle x_2,u\rangle = 0 \quad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2,$$

and v belongs to the #-orthogonal complement L_2 of L. Similarly, u belongs to L_1. Then remarking

$$y = u+v, \quad z = u-v,$$

we find that $z = Sy$, and $S^{\#} = S$ holds. Hence S is a #-positive #-unitary operator in L and identical to the defining operator J_p of a certain unitary norm. Hence $L_1 = L^+(p)$ and $L_2 = L^-(p)$ holds, and the quasipartition is defined by p.

§2.5. Reduction by #-selforthogonal subspace.

2.5.1. (Definition). Let $\mathcal{M}$ be a certain closed #-self-orthogonal subspace of L. Then setting $L_1 = \mathcal{M}$, $L_3 = J\mathcal{M}$, and L_2 the *-orthogonal complement of L_1+L_3 in L, L is represented as a linear sum $L = L_1+L_2+L_3$ of mutually *-orthogonal spaces L_1, L_2 and L_3. We call it the reduction of L by the space $\mathcal{M} = L_1$. We immediately obtain that :

2.5.2. (Lemma). Let $L = L_1+L_2+L_3$ be a reduction of L by a certain closed #-selforthogonal subspace $:= L_1$ of L. Then L_2 is a Krein space whose #-innerproduct and *-norm are the restrictions of those of L on L_2. Each element x of L has a unique representation

$$x = x_1 + x_2 + Jx_3,$$

where $x_1, x_3 \in \mathfrak{M}$, $x_2 \in L_2$. The #-innerproduct and the *-norm of L has the following representation :

$$\langle x,y\rangle = (x_1|y_3) + \langle x_2,y_2\rangle + (x_3|y_1),$$

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2.$$

If A is a continuous linear operator in L, A and $A^\#$ have the following matrix representation.

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad A^\# = \begin{pmatrix} A_{33}^* & A_{23}^\# & A_{13}^* \\ A_{32}^\# & A_{22}^\# & A_{12}^\# \\ A_{31}^* & A_{21}^\# & A_{11}^* \end{pmatrix}$$

Chapter 3. Spectral analysis in Pontrjagin space.

§3.1. Pontrjagin space.

3.1.1. (Definition). A Krein space L is called a Pontrjagin space if C has a certain canonical partition $L = L_1+L_2$ such that either L_1 or L_2 is of finite dimensional.

In this chapter we always denote by L a Pontrjagin space with a canonical partition $L = L_1 + L_2$ whose L_2 is of dimension $n < +\infty$. The partition determines a certain finite unitary norm $\| \ \|$ which we call the *-norm of L. The operator J which defines the *-norm of L is represented as $J = J^+ - J^-$. L_2 is the range of J^-, and hence dim J = holds.

3.1.2. (Lemma). If $\mathfrak{M}$ is a #-selforthogonal linear subspace of L, then the dimension of $\mathfrak{M}$ is $\leq n$.

Proof. Let E be the *-projection in L whose range is the closure $[\mathfrak{M}]$ of the range of $\mathfrak{M}$. $[\mathfrak{M}]$ is also #-selforthogonal. Then

$$(JEx|Ey) = \langle Ex, Ey \rangle = 0$$

and $EJE = 0$. Remarking $J = I - 2J^-$, we have

$$E = 2EJ^-E \quad \text{and} \quad \dim E \quad \dim J^- = .$$

3.1.3. (Theorem). Every #-unitary operator in L is continuous.

Proof. Let U be a #-unitary operator, and take its Cartan decomposition $U = KV$. V is biunitary, and continuous on L. Hence it is sufficient to see that K is continuous on L. Consider the spectral decomposition $K = \int_0^+ \lambda dE(\lambda)$ of K. Since $JKJ = K^{-1}$, we have $J\psi(K)J = \psi(K^{-1})$ for any bounded Borel function ψ on $0 \leq \lambda \leq +\infty$. In particular, we consider the projections E_1, E_2, E_3 which are spectral measures on the sets $(0,1)$, $\{1\}$ and $(1,+\infty)$, respectively, since the

mapping $\lambda \to \lambda^{-1}$ carries $(0,1)$ onto $(1,+\infty)$, we find that $JE_1J = E_3$, $JE_2J = E_2$, and the ranges of E_1 and E_3 are #-selforthogonal. By Lemma 3.1.2 we have $\dim E_3 = \dim E_1 < n$. Then the spectrum of K consists of at most $2n+1$ points, and K [illegible] become continuous in L and we have the Theorem.

3.1.4. (Corollary). Every unitary norm of L is finite.

Proof. Every unitary norm is induced from a certain *-positive #-unitary operator in L, and by Theorem 4.1.2 it is finite-valued on L.

3.1.5. (Theorem). Let $\mathcal{L} = M_1+M_2$ be a canonical quasi-partition of L. Then it is a canonical partition of L and $\dim M_2 = n$ holds.

Proof. The partition determines a unitary norm P of L, and $\mathcal{L}$ is identical to $\mathcal{L}(K^2)$, where K is a *-positive #-unitary operator which induces the norm P. Since K is continuous, we have $L = \mathcal{L}$ and $L= M_1+M_2$ is a partition of L. M_2 is the range of the operator J_p. Since $J_p^- = K^{\#}J^-K$, $\dim M_2 = \eta$ holds, and the Theorem is proved.

3.2. #-spectral and #-prespectral operators.

We now introduce operators in a Krein space with certain nice properties.

3.2.1. (Definition). An operator A in a Krein space L_0 is called #-spectral if there is a continuous #-unitary operator X and a biselfadjoint operator B in L_0 satisfying $A = X^{\#}BX$.

3.2.2. We return again to the Pontrjagin space L. The discussion of spectral analysis of #-selfadjoint operator would finish if every #-selfadjoint operator in L would have been #-spectral, because such an operator A have the spectral representations $A = \int \lambda dE(\lambda)$ by continuous #-projections $\{E(\lambda) : -\infty < \lambda < +\infty\}$. But even in Minkowsky spaces, #-selfadjoint operators generally do not have such a wishful property.

3.2.3. So we introduce the concept of #-prespectrality, which is a property of operators in L slightly weaker than the #-spectrality. On the other hand it preserves the following two conditions.

3.2.4. Every #-selfadjoint operator in Minkowsky space is #-prespectral.

3.2.5. #-prespectrality of operators in L is preserved under the strong convergences in bounded nets of operators in L.

3.2.6. We start from modifying the representation of a #-spectral operator A in L. A is written as $A = X^{\#}CX$, by a (continuous) #-unitary operator X and a biselfadjoint operator C in L. Consider the Cartan decomposition $X = UK$, where U is biunitary and K is *-positive #-unitary in L. We set $B = U^{\#}CU$, and represent A again by $A = K^{\#}BK$. Then B is also biselfadjoint. We define a *-positive operator T in L by

$$T = K^2(I+K^2)^{-1}.$$

From $JKJ = K^{-1}$, we have $JTJ = I-T$, and

$$TA(I-T) = (TA(I-T))^*.$$

Indeed,

$$TA(I-T) = (I+K^2)^{-1}KBK(I+K^2)^{-1}.$$

Summarizing these properties of A we define :

3.2.7. (Definition). An operator A in L is called #-prespectral if A is #-selfadjoint and has a certain *-positive operator T in L satisfying $JTJ = I-T$, and

$$TA(I-T) = (TA(I-T))^*.$$

3.2.8. (Lemma). A continuous operator A in L is #-spectral iff A is #-selfadjoint, and satisfies the condition of Definition 3.2.7 whose operator T in L is injective.

Proof. Indeed, if there is such an injective T, we can take a *-positive operator K satisfying $T = K^2(I+K^2)^{-1}$. K is #-unitary, and $B := KAK^{\#}$ is biselfadjoint in L. Then $A := K^{\#}BK$ is #-spectral.

§3.3. Canonical represntations of #-selfadjoint operators.

Now we generalize the Jordan's canonical representation of matrices to the cases of #-selfadjoint operators in L.

3.3.1. (Theorem). Let A be a continuous #-selfadjoint operator in a Pontrjagin space L with $\dim L^- = n$. Then A has the following matrix representation :

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix},$$

where $A_{11}^* = A_{33}$, $A_{23}^* = A_{12}$, $A_{13}^* = A_{13}$. The representation is determined by the reduction $L = L_1+L_2+L_3$ of L by a certain #-selforthogonal closed subspace L_1 of L satisfying $\dim L_1 = \dim L_3 \quad n$, and A_{22} becomes a #-spectral operator in L_2.

3.3.2. (Definition). The matrix representation of an operator A in L satisfying the requirement of Theorem 3.3.1 is called the canonical representation of A.

3.3.3. (Lemma). A continuous operator A in L is #-prespectral iff A has the canonical representation.

Indeed, let A be a continuous operator with the canonical representation. Then A is #-selfadjoint. A_{22} is #-spectral and has a representation $A_{22} = K_2^{\#}B_2K_2$, where B_2 is biself-adjoint and K is #-unitary, and further we can suppose that K is *-positive in L_2. Let I_K denote the identity in L_K, and define the operators T_2 in L_2 and T in L by

$$T_2 = K_2^2(I_2+K_2^2)^{-1}, \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0^2 & I_3 \end{pmatrix}$$

Then clearly, T is *-positive and satisfies $JTJ = I-T$ and $TA(I-T) = (TA(I-T))^*$. Hence A is #-prespectral.

Conversely, suppose that A is #-prespectral. Then there is a *-positive operator T in L such that $JTJ = I-T$ and $TA(I-T)$ is *-selfadjoint. Let L_1 be the kernel of T. Then L_1 is a closed subspace of L. $L_3 := JL_1$ is the kernel of JTJ (= $I-T$), and hence L_1 is *-orthogonal to L_3, and L_1 and L_3 are #-selforthogonal. Let $A := (A_{ij})$ be the matrix representation of A defined by the reduction $L = L_1+L_2+L_3$. We show that L_1 is invariant under A. Indeed, if x belongs to L_1, then $Tx = 0$, and hence $(I-T)x = x$ holds. Now

$$TAx = TA(I-T)x$$

$$= (TA(I-T))^*x = (I-T)A^*Tx = 0.$$

Therefore Ax belongs to L_1, and L_1 is invariant under A. From this we find that $A_{21} = A_{31} = 0$, and $A_{12} = (A_{21})^{\#} = 0$. T has the matrix representation

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0^2 & I_3 \end{pmatrix} .$$

T_2 is *-positive in L_2, and satisfies $J_2T_2J_2 = I_2-T_2$, and $T_2A_{22}(I_2-T_2)$ is *-selfadjoint. Further, T_2 is injective on L_2. Then by Lemma 3.2. A_{22} becomes #-spectral. Thus has the canonical representation.

§3.4. Proof of the canonical representation theorem.

The assertion 3.2.4 is now equivalent to the following 3.4.1.

3.4.1. (Lemma). If A is a #-selfadjoint operator in a Minkowsky space L, then A has the canonical representation.

To prove 3.4.1, let L_1 be a maximal subspace of L which is invariant under A, and consider the reduction $L = L_1+L_2+L_3$ of L by the space L_1. Then L has the representation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix}.$$

Suffice it to show that A_{22} is #-spectral in L_2. Notice that L_2 does not contain any nontrivial #-selforthogonal subspace $\mathcal{M}$ which is invariant under A_{22}. Indeed, if $\mathcal{M}$ is such a space, then $L_2+\mathcal{M}$ becomes a #-selforthogonal subspace of L which is invariant under A.

Then 3.4.1 is verified by showing the next lemma.

3.4.2. (Lemma). If a #-selfadjoint operator A in a Minkowsky space L does not have any nontrivial invariant #-selforthogonal subspace, then A is #-spectral.

Proof. Let S be the set of eigenvalues of A. For each $\alpha \in S$ consider the operator

$$E(\alpha) := (2\pi i)^{-1}\int_C (A-z)^{-1}dz,$$

where C is a small circular surrounding α to the positive direction. It is very well-known that $I = \Sigma_{\alpha \in S} E(\alpha)$, $E(\alpha)^2 = E(\alpha)$ and $E(\alpha)E(\beta) = 0$ for $\alpha \neq \beta$. The range of $E(\alpha)$ is the eigenspace of A associated to the eigenvalue α. Remarking that $A^{\#} = A$, if α belongs to S, then $\overline{\alpha}$ also belongs to S. $E(\alpha)$ commutes to A, and

$$E(\alpha)^{\#} = E(\overline{\alpha}) \neq 0.$$

S is contained in the real line. Indeed, if α is a non-real element of S, then $E(\alpha)^{\#}E(\alpha) = 0$, and $E(\alpha)$ is #-selforthogonal. Then the range of $E(\alpha)$ is a nontrivial #-selforthogonal, invariant subspace of A, and it leads a contradiction. If α is real, then $AE(\alpha) = \alpha E(\alpha)$ holds. Indeed, let n be the least number such that $(A-\alpha)^n E(\alpha) = 0$. If $n \geq 2$, we can take a number m such that $2m \geq n$ and $m \leq n-1$. The operator $(A-\alpha)^m E(\alpha)$ is $\neq 0$ and #-selforthogonal. Indeed,

$$((A-\alpha)^m E(\alpha))^{\#}((A-\alpha)^m E(\alpha))$$

$$= (A-\alpha)^{2m} E(\alpha) = 0.$$

This is a contradiciton.

The signs of $\langle x,x \rangle$ for elements $x \neq 0$ of the range of $E(\alpha)$ is constant, and we denote it by sign $E(\alpha)$.

Indeed, if there are elements $x \neq 0$ and $y \neq 0$ such that $\langle x,x \rangle \geq 0$ and $\langle y,y \rangle \leq 0$, we can take a certain element $z = \alpha x + (1-\alpha)y \neq 0$ satisfying $\langle z,z \rangle = 0$, and $\{\alpha z : \alpha \in C\}$ is a nontrivial invariant subspace of A.

Now we set

$$E^{+}(A) = \Sigma_{\text{sign } E(\alpha)=+} E(\alpha),$$

$$E^{-}(A) = \Sigma_{\text{sign } E(\alpha)=-} E(\alpha).$$

and let M_1 and M_2 be the ranges of $E^{+}(A)$ and $E^{-}(A)$. Then $L = M_1+M_2$ is a canonical partition of L and defines a certain finite unitary norm p of L. Consider the innerproduct $(x|y)_p$ corresponding to p. Then A is p-selfadjoint on L. Let K be the *-positive #-unitary operator in L which induces p. Then $K^{\#}AK$ is biselfadjoint, and A is #-spectral in L. Thus 3.4.2 and hence 3.4.1 are verified.

We now show 3.2.5 as the next lemma 3.4.3.

3.4.3. (Lemma). If $\{A_\alpha\}$ is a bounded net of #-prespectral operators in L which converges strongly to an operator A in L, then A is #-prespectral.

Proof. For each A_α we take a certain *-positive operator T_α satisfying $JT_\alpha J = I-T_\alpha$ and

$$T_\alpha A_\alpha (I-T_\alpha) = (T_\alpha A(I-T_\alpha))^{*}.$$

Then in the Hilbert space L we have $0 \leqq T_\alpha \leqq I$, and we can take a certain subnet $\{T_\beta\}$ of $\{T_\alpha\}$ converging weakly to a certain operator T in L_1. T is also *-positive and satisfies $JTJ = I-T$.

We show that

$$TA(I-T) = (TA(I-T))^*.$$

The identity $T_\beta A_\beta(I-T_\beta) = (T_\beta A_\beta(I-T_\beta))^*$ is equivalent to

$$T_\beta A_\beta - A_\beta^* T_\beta = T_\beta(A_\beta - A_\beta^*)T_\beta,$$

Where $\{A_\beta\}$ is norm-bounded, and $T_\beta A_\beta - A_\beta^* T_\beta$ converges weakly to $TA-A^*T$. Since A_β are #-selfadjoint we have

$$A_\beta^* = JA_\beta J = (1-2J^-)A_\beta(1-2J^-),$$

and

$$A_\beta - A_\beta^* = 2J^-A_\beta + 2A_\beta J^- - 4J^-A_\beta J^-.$$

J^- is of finite dimensional, and the right hand of the above identity converges uniformly to $A-A^*$.

Remark that

$$T_\beta(A_\beta - A^*)T_\beta = T_\beta(A-A^*)T_\beta + T_\beta R_\beta T_\beta,$$

where $|R_\beta| \to 0$. Then to see that $T_\beta(A_\beta - A_\beta^*)T_\beta$ converges weakly to $T(A-A^*)T$, it is sufficient to find that if X is an operator of finite dimensional, $T_\beta X T_\beta$ converges weakly to TXT. This also turns to verify the case that X is an operator of one-dimensional range, say $X = ab^*$, which is an operator in $L : ab^* : x \to (x|b)a$. Notice that

$$T_\beta ab^* T_\beta = (T_\beta a)(T_\beta b)^*.$$

Then $T_\beta ab^* T_\beta$ converges weakly to Tab^*T. Thus we obtain the identity

$$TA(I-T) = ((TA(I-T))^*,$$

and A is #-prespectral. The Lemma is proved.

3.4.4. (Proof of Theorem 3.3.1). The discussion of this section insures the validity of Theorem 3.3.1, but for completeness we summalize how our argument goes to this conclusion.
If L is a Minkowsky space, the Theorem is valid (3.4.1).
So, even if L is a general Pontrjagin space, every #-selfadjoint operator of finite rank has the canonical representation.

If A is any continuous #-selfadjoint operator in L, we can take a bounded net of #-selfadjoint operators of finite ranks converging strongly to L. Now operators in this net are #-prespectral (by Lemma 3.4.1). Then by Lemma 3.4.3 A is #-prespectral. Then by Lemma 3.4.1 A has the canonical representation.